THE BOOK WAS DRENCHED

A HISTORY

OF

GREEK MATHEMATICS

VOLUME II

OF

GREEK MATHEMATICS

BY

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WORDSWORTH.

VOLUME II

FROM ARISTARCHUS TO DIOPHANTUS

OXFORD

AT THE CLARENDON PRESS

1921

OXFORD UNIVERSITY PRESS
London Edinburgh Glasgow Copenhagen
New York Toronto Melbourne Cape Town
Bombay Calcutta Madras Shanghai
HUMPHREY MILFORD
Publisher to the University

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XII

ARISTARCHUS OF SAMOS

HISTORIANS of mathematics have, as a rule, given too little attention to Aristarchus of Samos. The reason is no doubt that he was an astronomer, and therefore it might be supposed that his work would have no sufficient interest for the mathematician. The Greeks knew better; they called him Aristarchus 'the mathematician', to distinguish him from the host of other Aristarchuses; he is also included by Vitruvius among the few great men who possessed an equally profound knowledge of all branches of science, geometry, astronomy, music, &c.

'Men of this type are rare, men such as were, in times past, Aristarchus of Samos, Philolaus and Archytas of Tarentum, Apollonius of Perga, Eratosthenes of Cyrene, Archimedes and Scopinas of Syracuse, who left to posterity many mechanical and gnomonic appliances which they invented and explained on mathematical (lit. 'numerical') principles.'[1]

That Aristarchus was a very capable geometer is proved by his extant work *On the sizes and distances of the Sun and Moon* which will be noticed later in this chapter: in the mechanical line he is credited with the discovery of an improved sun-dial, the so-called σκάφη, which had, not a plane, but a concave hemispherical surface, with a pointer erected vertically in the middle throwing shadows and so enabling the direction and the height of the sun to be read off by means of lines marked on the surface of the hemisphere. He also wrote on vision, light and colours. His views on the latter subjects were no doubt largely influenced by his master, Strato of Lampsacus; thus Strato held that colours were emanations from bodies, material molecules, as it were, which imparted to the intervening air the same colour as that possessed by the body, while Aristarchus said that colours are 'shapes or forms

[1] Vitruvius, *De architectura*, i. 1. 16.

stamping the air with impressions like themselves, as it were', that 'colours in darkness have no colouring', and that 'light is the colour impinging on a substratum'.

Two facts enable us to fix Aristarchus's date approximately. In 281/280 B.C. he made an observation of the summer solstice; and a book of his, presently to be mentioned, was published before the date of Archimedes's *Psammites* or *Sand-reckoner*, a work written before 216 B.C. Aristarchus, therefore, probably lived *circa* 310–230 B.C., that is, he was older than Archimedes by about 25 years.

To Aristarchus belongs the high honour of having been the first to formulate the Copernican hypothesis, which was then abandoned again until it was revived by Copernicus himself. His claim to the title of 'the ancient Copernicus' is still, in my opinion, quite unshaken, notwithstanding the ingenious and elaborate arguments brought forward by Schiaparelli to prove that it was Heraclides of Pontus who first conceived the heliocentric idea. Heraclides is (along with one Ecphantus, a Pythagorean) credited with having been the first to hold that the earth revolves about its own axis every 24 hours, and he was the first to discover that Mercury and Venus revolve, like satellites, about the sun. But though this proves that Heraclides came near, if he did not actually reach, the hypothesis of Tycho Brahe, according to which the earth was in the centre and the rest of the system, the sun with the planets revolving round it, revolved round the earth, it does not suggest that he moved the earth away from the centre. The contrary is indeed stated by Aëtius, who says that 'Heraclides and Ecphantus make the earth move, *not in the sense of translation*, but by way of turning on an axle, like a wheel, from west to east, about its own centre'.[1] None of the champions of Heraclides have been able to meet this positive statement. But we have conclusive evidence in favour of the claim of Aristarchus; indeed, ancient testimony is unanimous on the point. Not only does Plutarch tell us that Cleanthes held that Aristarchus ought to be indicted for the impiety of 'putting the Hearth of the Universe in motion'[2]; we have the best possible testimony in the precise statement of a great

[1] Aët. iii. 13. 3, *Vors.* i³, p. 341. 8.
[2] Plutarch, *De facie in orbe lunae*, c. 6, pp. 922 F–923 A.

contemporary, Archimedes. In the *Sand-reckoner* Archimedes has this passage.

'You [King Gelon] are aware that "universe" is the name given by most astronomers to the sphere the centre of which is the centre of the earth, while its radius is equal to the straight line between the centre of the sun and the centre of the earth. This is the common account, as you have heard from astronomers. But Aristarchus brought out *a book consisting of certain hypotheses*, wherein it appears, as a consequence of the assumptions made, that the universe is many times greater than the "universe" just mentioned. His hypotheses are that *the fixed stars and the sun remain unmoved, that the earth revolves about the sun in the circumference of a circle, the sun lying in the middle of the orbit*, and that the sphere of the fixed stars, situated about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface.'

(The last statement is a variation of a traditional phrase, for which there are many parallels (cf. Aristarchus's Hypothesis 2 'that the earth is in the relation of a point and centre to the sphere in which the moon moves'), and is a method of saying that the 'universe' is infinitely great in relation not merely to the size of the sun but even to the orbit of the earth in its revolution about it; the assumption was necessary to Aristarchus in order that he might not have to take account of parallax.)

Plutarch, in the passage referred to above, also makes it clear that Aristarchus followed Heraclides in attributing to the earth the daily rotation about its axis. The bold hypothesis of Aristarchus found few adherents. Seleucus, of Seleucia on the Tigris, is the only convinced supporter of it of whom we hear, and it was speedily abandoned altogether, mainly owing to the great authority of Hipparchus. Nor do we find any trace of the heliocentric hypothesis in Aristarchus's extant work *On the sizes and distances of the Sun and Moon*. This is presumably because that work was written before the hypothesis was formulated in the book referred to by Archimedes. The geometry of the treatise is, however, unaffected by the difference between the hypotheses.

Archimedes also says that it was Aristarchus who discovered that the apparent angular diameter of the sun is about 1/720th part of the zodiac circle, that is to say, half a degree. We do not know how he arrived at this pretty accurate figure: but, as he is credited with the invention of the σκάφη, he may have used this instrument for the purpose. But here again the discovery must apparently have been later than the treatise *On sizes and distances*, for the value of the subtended angle is there assumed to be 2° (Hypothesis 6). How Aristarchus came to assume a value so excessive is uncertain. As the mathematics of his treatise is not dependent on the actual value taken, 2° may have been assumed merely by way of illustration; or it may have been a guess at the apparent diameter made before he had thought of attempting to measure it. Aristarchus assumed that the angular diameters of the sun and moon at the centre of the earth are equal.

The method of the treatise depends on the just observation, which is Aristarchus's third 'hypothesis', that 'when the moon appears to us halved, the great circle which divides the dark and the bright portions of the moon is in the direction of our eye'; the effect of this (since the moon receives its light from the sun), is that at the time of the dichotomy the centres of the sun, moon and earth form a triangle right-angled at the centre of the moon. Two other assumptions were necessary: first, an estimate of the size of the angle of the latter triangle at the centre of the earth at the moment of dichotomy: this Aristarchus assumed (Hypothesis 4) to be 'less than a quadrant by one-thirtieth of a quadrant', i.e. 87°, again an inaccurate estimate, the true value being 89° 50′; secondly, an estimate of the breadth of the earth's shadow where the moon traverses it: this he assumed to be 'the breadth of two moons' (Hypothesis 5).

The inaccuracy of the assumptions does not, however, detract from the mathematical interest of the succeeding investigation. Here we find the logical sequence of propositions and the absolute rigour of demonstration characteristic of Greek geometry; the only remaining drawback would be the practical difficulty of determining the exact moment when the moon 'appears to us halved'. The form and style of the book are thoroughly classical, as befits the period between Euclid and Archimedes;

the Greek is even remarkably attractive. The content from the mathematical point of view is no less interesting, for we have here the first specimen extant of pure geometry used with a *trigonometrical* object, in which respect it is a sort of forerunner of Archimedes's *Measurement of a Circle*. Aristarchus does not actually evaluate the trigonometrical ratios on which the ratios of the sizes and distances to be obtained depend; he finds limits between which they lie, and that by means of certain propositions which he assumes without proof, and which therefore must have been generally known to mathematicians of his day. These propositions are the equivalents of the statements that,

(1) if α is what we call the circular measure of an angle and α is less than $\frac{1}{2}\pi$, then the ratio $\sin\alpha/\alpha$ *decreases*, and the ratio $\tan\alpha/\alpha$ *increases*, as α increases from 0 to $\frac{1}{2}\pi$;

(2) if β be the circular measure of another angle less than $\frac{1}{2}\pi$, and $\alpha > \beta$, then

$$\frac{\sin\alpha}{\sin\beta} < \frac{\alpha}{\beta} < \frac{\tan\alpha}{\tan\beta}.$$

Aristarchus of course deals, not with actual circular measures, sines and tangents, but with angles (expressed not in degrees but as fractions of right angles), arcs of circles and their chords. Particular results obtained by Aristarchus are the equivalent of the following:

$$\tfrac{1}{18} > \sin 3^\circ > \tfrac{1}{20}, \qquad \text{[Prop. 7]}$$

$$\tfrac{1}{45} > \sin 1^\circ > \tfrac{1}{60}, \qquad \text{[Prop. 11]}$$

$$1 > \cos 1^\circ > \tfrac{89}{90}, \qquad \text{[Prop. 12]}$$

$$1 > \cos^2 1^\circ > \tfrac{44}{45}. \qquad \text{[Prop. 13]}$$

The book consists of eighteen propositions. Beginning with six hypotheses to the effect already indicated, Aristarchus declares that he is now in a position to prove

(1) that the distance of the sun from the earth is greater than eighteen times, but less than twenty times, the distance of the moon from the earth;

(2) that the diameter of the sun has the same ratio as aforesaid to the diameter of the moon;

(3) that the diameter of the sun has to the diameter of the earth a ratio greater than 19 : 3, but less than 43 : 6.

The propositions containing these results are Props. 7, 9 and 15.

Prop. 1 is preliminary, proving that two equal spheres are comprehended by one cylinder, and two unequal spheres by one cone with its vertex in the direction of the lesser sphere, and the cylinder or cone touches the spheres in circles at right angles to the line of centres. Prop. 2 proves that, if a sphere be illuminated by another sphere larger than itself, the illuminated portion is greater than a hemisphere. Prop. 3 shows that the circle in the moon which divides the dark from the bright portion is least when the cone comprehending the sun and the moon has its vertex at our eye. The 'dividing circle', as we shall call it for short, which was in Hypothesis 3 spoken of as a great circle, is proved in Prop. 4 to be, not a great circle, but a small circle not perceptibly different from a great circle. The proof is typical and is worth giving along with that of some connected propositions (11 and 12).

B is the centre of the moon, A that of the earth, CD the diameter of the 'dividing circle in the moon', EF the parallel diameter in the moon. BA meets the circular section of the moon through A and EF in G, and CD in L. GH, GK are arcs each of which is equal to half the arc CE. By Hypothesis 6 the angle CAD is 'one-fifteenth of a sign' $= 2°$, and the angle $BAC = 1°$.

Now, says Aristarchus,

$$1° : 45° \; [> \tan 1° : \tan 45°]$$
$$> BC : CA,$$

and, *a fortiori*,

$$BC : BA \text{ or } BG : BA$$
$$< 1 : 45;$$

that is,

$$BG < \tfrac{1}{45} BA$$
$$< \tfrac{1}{44} GA;$$

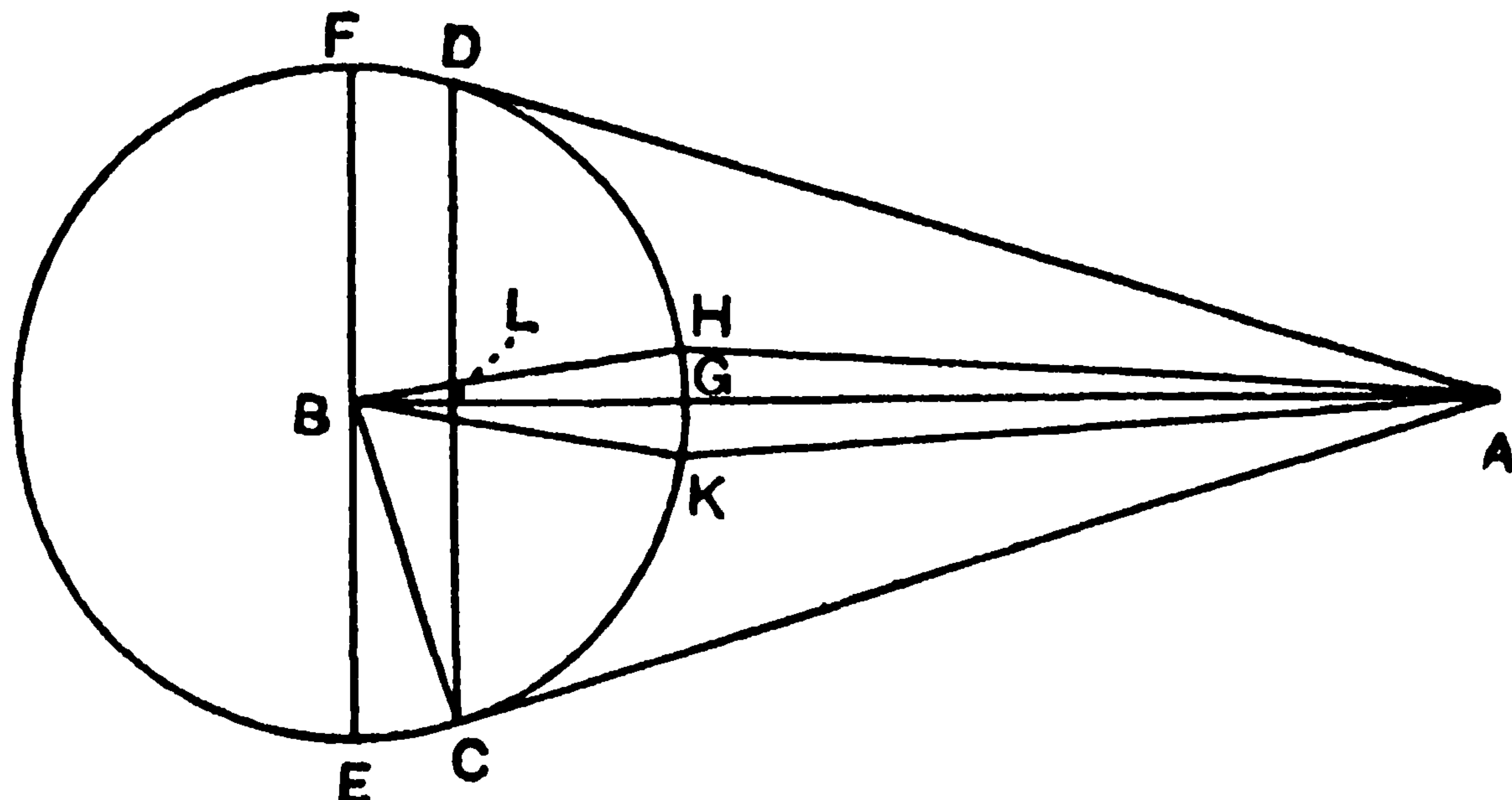
therefore, *a fortiori*,

$$BH < \tfrac{1}{44} HA.$$

Now $BH : HA \; [= \sin HAB : \sin HBA]$

$$> \angle HAB : \angle HBA,$$

whence $\angle HAB < \frac{1}{44} \angle HBA,$



and (taking the doubles) $\angle HAK < \frac{1}{44} \angle HBK.$

But $\angle HBK = \angle EBC = \frac{1}{90} R$ (where R is a right angle);

therefore $\angle HAK < \frac{1}{3960} R.$

But 'a magnitude (arc HK) seen under such an angle is imperceptible to our eye';

therefore, *a fortiori*, the arcs CE, DF are severally imperceptible to our eye. Q. E. D.

An easy deduction from the same figure is Prop. 12, which shows that the ratio of CD, the diameter of the 'dividing circle', to EF, the diameter of the moon, is < 1 but $> \frac{89}{90}$.

We have $\angle EBC = \angle BAC = 1°$;

therefore (arc EC) $= \frac{1}{90}$ (arc EG),

and accordingly (arc CG) : (arc GE) $= 89 : 90$.

Doubling the arcs, we have

$$(\text{arc } CGD) : (\text{arc } EGF) = 89 : 90.$$

But $CD : EF >$ (arc CGD) : (arc EGF)

[equivalent to $\sin\alpha / \sin\beta > \alpha/\beta$, where $\angle CBD = 2\alpha$, and $2\beta = \pi$];

therefore $CD : EF \; [= \cos 1°] > 89 : 90$,

while obviously $CD : EF < 1$.

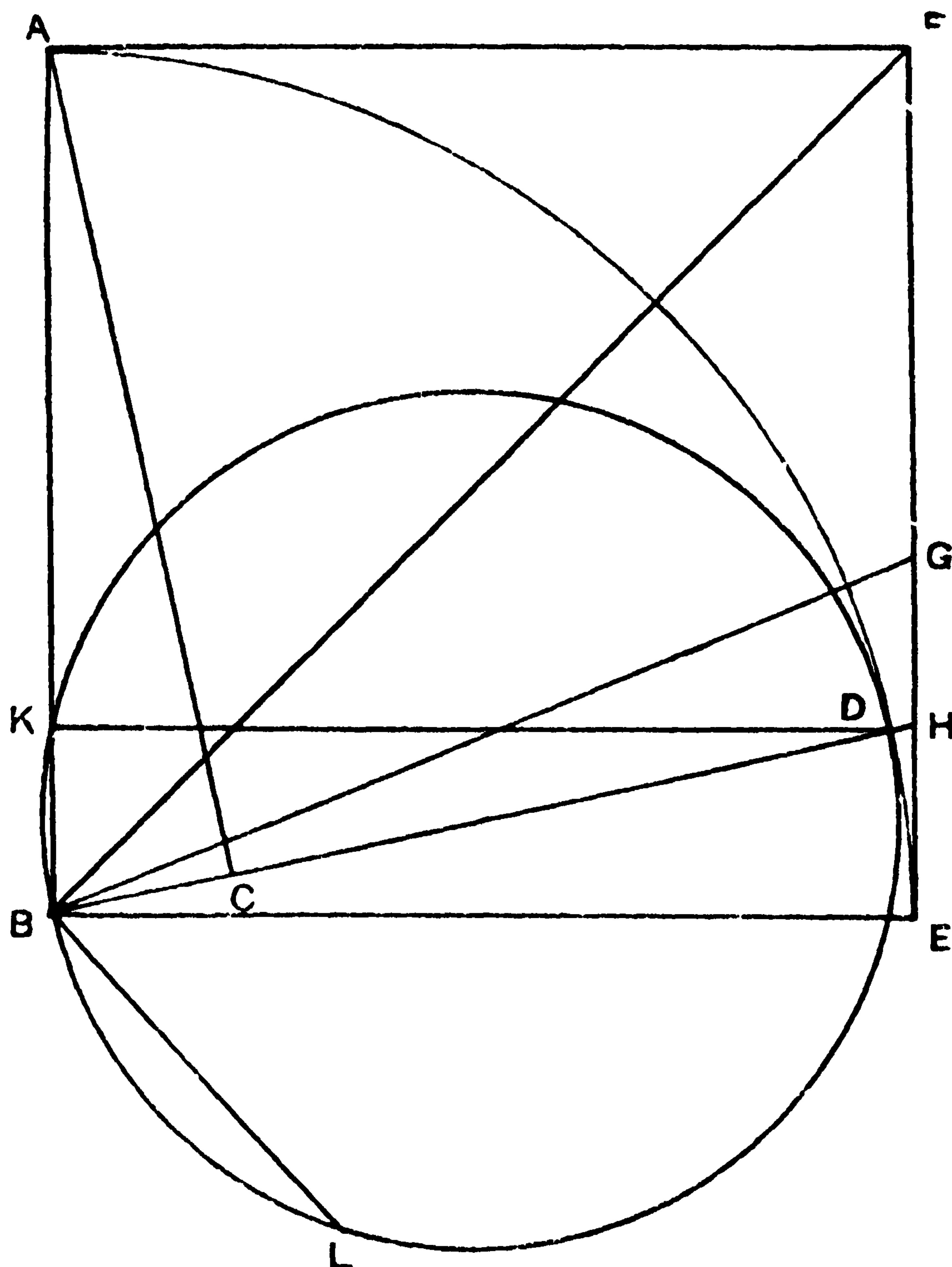
Prop. 11 finds limits to the ratio $EF : BA$ (the ratio of the diameter of the moon to the distance of its centre from the centre of the earth); the ratio is $< 2 : 45$ but $> 1 : 30$.

I. Now, by Hypothesis 4, $\angle ABC = 87°$,

so that $\angle HBE = \angle BAC = 3°$;

therefore $\angle GBE : \angle HBE = \frac{1}{4} R : \frac{1}{30} R$

$= 15 : 2$,



so that $GE : HE \, [= \tan GBE : \tan HBE] > \angle GBE : \angle HBE$

$> 15 : 2$. (1)

The ratio which has to be proved $> 18 : 1$ is $AB : BC$ or $FE : EH$.

Now $FG : GE = FB : BE$,

whence $FG^2 : GE^2 = FB^2 : BE^2 = 2 : 1$,

and $FG : GE = \sqrt{2} : 1$

$> 7 : 5$

(this is the approximation to $\sqrt{2}$ mentioned by Plato and known to the Pythagoreans).

Therefore $FE:EG > 12:5$ or $36:15$.

Compounding this with (1) above, we have

$$FE:EH > 36:2 \text{ or } 18:1.$$

II. To prove $BA < 20\,BC$.

Let BH meet the circle AE in D, and draw DK parallel to EB. Circumscribe a circle about the triangle BKD, and let the chord BL be equal to the radius (r) of the circle.

Now $\angle BDK = \angle DBE = \frac{1}{30}R$,

so that arc $BK = \frac{1}{60}$ (circumference of circle).

Thus (arc BK) : (arc BL) $= \frac{1}{60} : \frac{1}{6}$,

$= 1:10$.

And (arc BK) : (arc BL) $< BK:r$

[this is equivalent to $\alpha/\beta < \sin\alpha/\sin\beta$, where $\alpha < \beta < \frac{1}{2}\pi$],

so that $r < 10\,BK$,

and $BD < 20\,BK$.

But $BD:BK = AB:BC$;

therefore $AB < 20\,BC$. Q. E. D.

The remaining results obtained in the treatise can be visualized by means of the three figures annexed, which have reference to the positions of the sun (centre A), the earth (centre B) and the moon (centre C) during an eclipse. Fig. 1 shows the middle position of the moon relatively to the earth's shadow which is bounded by the cone comprehending the sun and the earth. ON is the arc with centre B along which move the extremities of the diameter of the dividing circle in the moon. Fig. 3 shows the same position of the moon in the middle of the shadow, but on a larger scale. Fig. 2 shows the moon at the moment when it has just entered the shadow; and, as the breadth of the earth's shadow is that of two moons (Hypothesis 5), the moon in the position shown touches BN at N and BL at L, where L is the middle point of the arc ON. It should be added that, in Fig. 1, UV is the diameter of the circle in which the sun is touched by the double cone with B as vertex, which comprehends both the sun and the moon,

while Y, Z are the points in which the perpendicular through A, the centre of the sun, to BA meets the cone enveloping the sun and the earth.

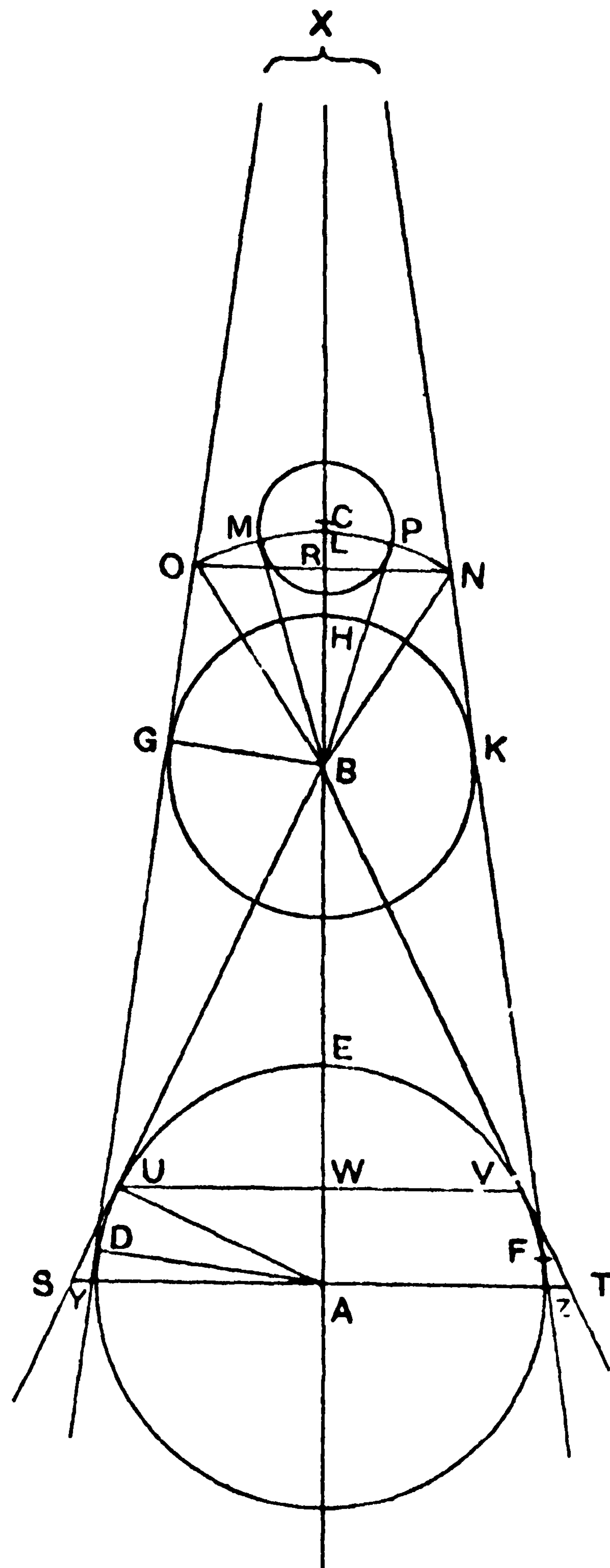


Fig. 1.

This being premised, the main results obtained are as follows:

Prop. 13.

(1) ON : (diam. of moon) $< 2:1$

but $> 88:45$.

(2) ON : (diam. of sun) $< 1:9$

but $> 22:225$.

(3) $ON:YZ > 979:10125$.

Prop. 14 (Fig. 3).

$BC:CS > 675:1$.

Prop. 15.

(Diam. of sun) : (diam. of earth) $> 19:3$

but $< 43:6$.

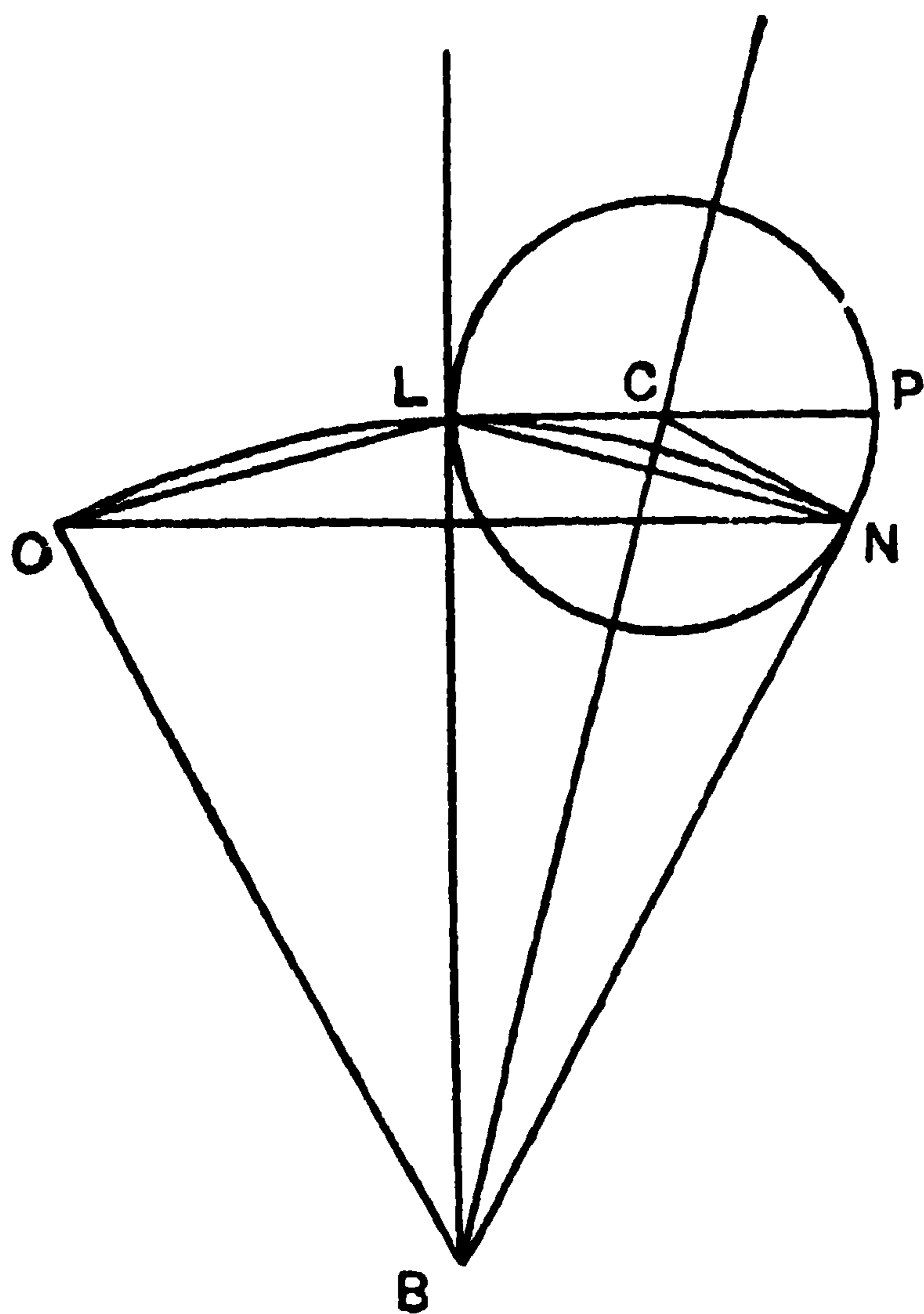


Fig. 2.

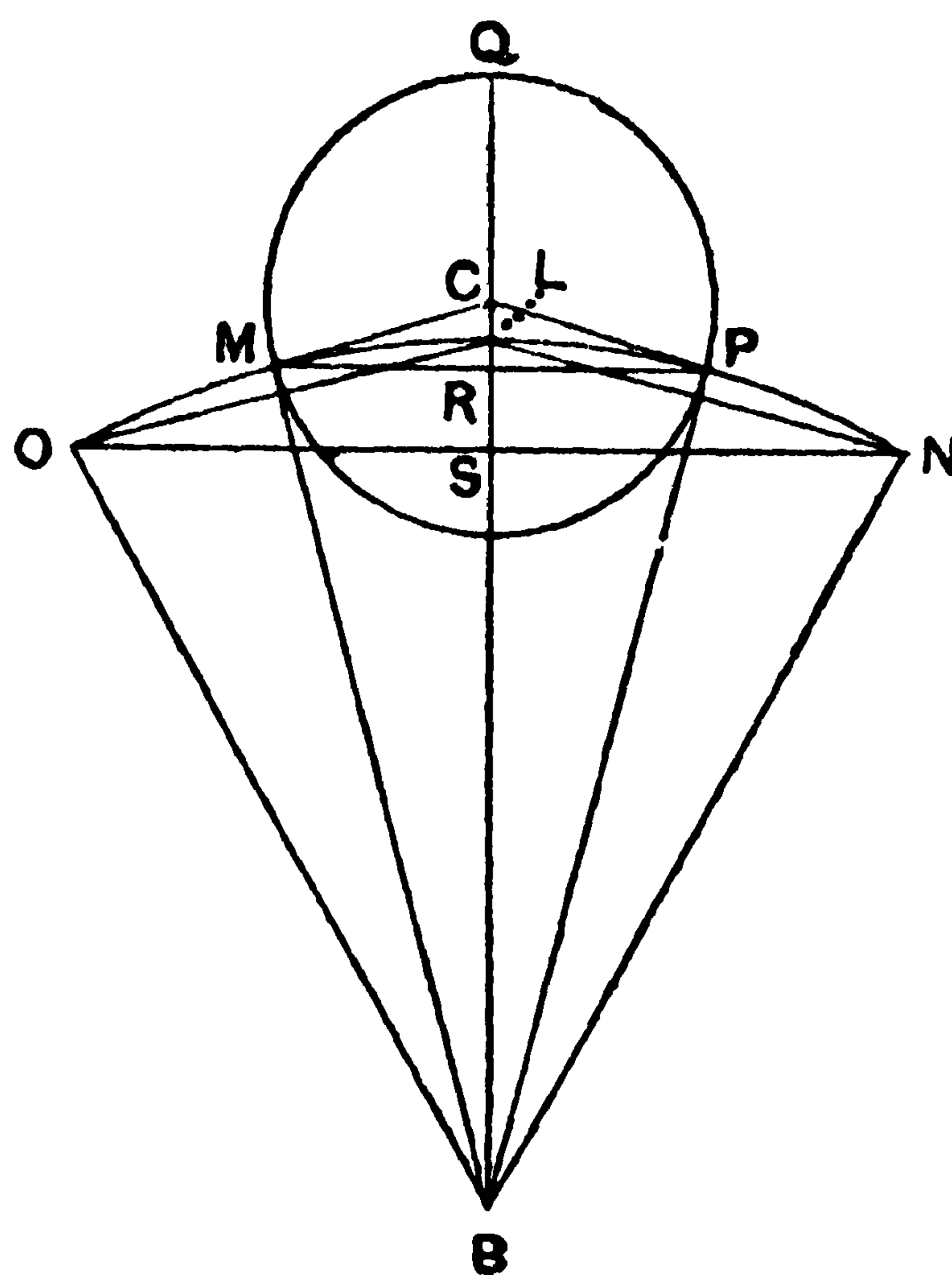


Fig. 3.

Prop. 17.

(Diam. of earth) : (diam. of moon) $> 108:43$

but $< 60:19$.

It is worth while to show how these results are proved.

Prop. 13.

(1) In Fig. 2 it is clear that

$$ON < 2\,LN \text{ and, } a \textit{ fortiori}, < 2\,LP.$$

The triangles LON, CLN being similar,

$$ON:NL = NL:LC;$$

therefore $ON:NL = NL:\tfrac{1}{2}LP$

$> 89:45$. (by Prop. 12)

Hence $ON : LC = ON^2 : NL^2$

$> 89^2 : 45^2$;

therefore $ON : LP > 7921 : 4050$

$> 88 : 45$, says Aristarchus.

[If $\frac{7921}{4050}$ be developed as a continued fraction, we easily obtain $1 + \frac{1}{1+} \frac{1}{21+} \frac{1}{2}$, which is in fact $\frac{88}{45}$.]

(2) $ON < 2$ (diam. of moon).

But (diam. of moon) $< \frac{1}{18}$ (diam. of sun); (Prop. 7)

therefore $ON < \frac{1}{9}$ (diam. of sun).

Again ON : (diam. of moon) $> 88 : 45$, from above,

and (diam. of moon) : (diam. of sun) $> 1 : 20$; (Prop. 7)

therefore, *ex aequali*,

ON : (diam. of sun) $> 88 : 900$

$> 22 : 225$.

(3) Since the same cone comprehends the sun and the moon, the triangle BUV (Fig. 1) and the triangle BLN (Fig. 2) are similar, and

$LN : LP = UV$: (diam. of sun)

$= WU : UA$

$= UA : AS$

$< UA : AY$.

But $LN : LP > 89 : 90$; (Prop. 12)

therefore, *a fortiori*, $UA : AY > 89 : 90$.

And $UA : AY = 2UA : YZ$

$=$ (diam. of sun) : YZ.

But ON : (diam. of sun) $> 22 : 225$; (Prop. 13)

therefore, *ex aequali*,

$ON : YZ > 89 \times 22 : 90 \times 225$

$> 979 : 10125$.

Prop. 14 (Fig. 3).

The arcs OM, ML, LP, PN are all equal; therefore so are the chords. BM, BP are tangents to the circle MQP, so that CM is perpendicular to BM, while BM is perpendicular to OL. Therefore the triangles LOS, CMR are similar.

Therefore $SO : MR = SL : RC$.

But $SO < 2\,MR$, since $ON < 2\,MP$; (Prop. 13)

therefore $SL < 2\,RC$,

and, *a fortiori*, $SR < 2\,RC$, or $SC < 3\,RC$,

that is, $CR : CS > 1 : 3$.

Again, $MC : CR = BC : CM$

$> 45 : 1$; (see Prop. 11)

therefore, *ex aequali*,

$CM : CS > 15 : 1$.

And $BC : CM > 45 : 1$;

therefore $BC : CS > 675 : 1$.

Prop. 15 (Fig. 1).

We have $NO :$ (diam. of sun) $< 1 : 9$, (Prop. 13)

and, *a fortiori*, $YZ : NO > 9 : 1$;

therefore, by similar triangles, if YO, ZN meet in X,

$AX : XR > 9 : 1$,

and *convertendo*, $XA : AR < 9 : 8$.

But $AB > 18\,BC$, and, *a fortiori*, $> 18\,BR$,

whence $AB > 18\,(AR - AB)$, or $19\,AB > 18\,AR$;

that is, $AR : AB < 19 : 18$.

Therefore, *ex aequali*,

$XA : AB < 19 : 16$,

and, *convertendo*, $AX : XB > 19 : 3$;

therefore (diam. of sun) : (diam. of earth) $> 19 : 3$.

Lastly, since $CB : CR > 675 : 1$, (Prop. 14)

$CB : BR < 675 : 674$.

But $$AB : BC < 20 : 1;$$

therefore, *ex aequali*,

$$AB : BR < 13500 : 674$$
$$< 6750 : 337,$$

whence, by inversion and *componendo*,

$$RA : AB > 7087 : 6750. \qquad (1)$$

But $$AX : XR = YZ : NO$$
$$< 10125 : 979; \qquad \text{(Prop. 13)}$$

therefore, *convertendo*,

$$XA : AR > 10125 : 9146.$$

From this and (1) we have, *ex aequali*,

$$XA : AB > 10125 \times 7087 : 9146 \times 6750$$
$$> 71755875 : 61735500$$
$$> 43 : 37, \textit{ a fortiori}.$$

[It is difficult not to see in $43 : 37$ the expression $1 + \frac{1}{6+}\frac{1}{6}$, which suggests that $43 : 37$ was obtained by developing the ratio as a continued fraction.]

Therefore, *convertendo*,

$$XA : XB < 43 : 6,$$

whence (diam. of sun) : (diam. of earth) $< 43 : 6$. Q. E. D.

XIII

ARCHIMEDES

THE siege and capture of Syracuse by Marcellus during the second Punic war furnished the occasion for the appearance of Archimedes as a personage in history; it is with this historical event that most of the detailed stories of him are connected; and the fact that he was killed in the sack of the city in 212 B.C., when he is supposed to have been 75 years of age, enables us to fix his date at about 287–212 B.C. He was the son of Phidias, the astronomer, and was on intimate terms with, if not related to, King Hieron and his son Gelon. It appears from a passage of Diodorus that he spent some time in Egypt, which visit was the occasion of his discovery of the so-called Archimedean screw as a means of pumping water.[1] It may be inferred that he studied at Alexandria with the successors of Euclid. It was probably at Alexandria that he made the acquaintance of Conon of Samos (for whom he had the highest regard both as a mathematician and a friend) and of Eratosthenes of Cyrene. To the former he was in the habit of communicating his discourses before their publication; while it was to Eratosthenes that he sent *The Method*, with an introductory letter which is of the highest interest, as well as (if we may judge by its heading) the famous Cattle-Problem.

Traditions.

It is natural that history or legend should say more of his mechanical inventions than of his mathematical achievements, which would appeal less to the average mind. His machines were used with great effect against the Romans in the siege of Syracuse. Thus he contrived (so we are told) catapults so ingeniously constructed as to be equally serviceable at long or

[1] Diodorus, v. 37. 3.

short range, machines for discharging showers of missiles through holes made in the walls, and others consisting of long movable poles projecting beyond the walls which either dropped heavy weights on the enemy's ships, or grappled their prows by means of an iron hand or a beak like that of a crane, then lifted them into the air and let them fall again.[1] Marcellus is said to have derided his own engineers with the words, 'Shall we not make an end of fighting against this geometrical Briareus who uses our ships like cups to ladle water from the sea, drives off our *sambuca* ignominiously with cudgel-blows, and by the multitude of missiles that he hurls at us all at once outdoes the hundred-handed giants of mythology?'; but all to no purpose, for the Romans were in such abject terror that, 'if they did but see a piece of rope or wood projecting above the wall, they would cry "there it is", declaring that Archimedes was setting some engine in motion against them, and would turn their backs and run away'.[2] These things, however, were merely the 'diversions of geometry at play',[3] and Archimedes himself attached no importance to them. According to Plutarch,

'though these inventions had obtained for him the renown of more than human sagacity, he yet would not even deign to leave behind him any written work on such subjects, but, regarding as ignoble and sordid the business of mechanics and every sort of art which is directed to use and profit, he placed his whole ambition in those speculations the beauty and subtlety of which is untainted by any admixture of the common needs of life.'[4]

(*a*) *Astronomy.*

Archimedes did indeed write one mechanical book, *On Sphere-making*, which is lost; this described the construction of a sphere to imitate the motions of the sun, moon and planets.[5] Cicero saw this contrivance and gives a description of it; he says that it represented the periods of the moon and the apparent motion of the sun with such accuracy that it would even (over a short period) show the eclipses of the sun and moon.[6] As Pappus speaks of 'those who understand

[1] Polybius, *Hist.* viii. 7, 8; Livy xxiv. 34; Plutarch, *Marcellus*, cc. 15–17.
[2] *Ib.*, c. 17. [3] *Ib.*, c. 14. [4] *Ib.*, c. 17.
[5] Carpus in Pappus, viii, p. 1026. 9; Proclus on Eucl. I, p. 41. 16.
[6] Cicero, *De rep.* i. 21, 22, *Tusc.* i. 63, *De nat. deor.* ii. 88.

the making of spheres and produce a model of the heavens by means of the circular motion of water', it is possible that Archimedes's sphere was moved by water. In any case Archimedes was much occupied with astronomy. Livy calls him 'unicus spectator caeli siderumque'.[1] Hipparchus says, 'From these observations it is clear that the differences in the years are altogether small, but, as to the solstices, I almost think that Archimedes and I have both erred to the extent of a quarter of a day both in the observation and in the deduction therefrom'.[2] Archimedes then had evidently considered the length of the year. Macrobius says he discovered the distances of the planets,[3] and he himself describes in his *Sand-reckoner* the apparatus by which he measured the apparent angular diameter of the sun.

(β) *Mechanics.*

Archimedes wrote, as we shall see, on theoretical mechanics, and it was by theory that he solved the problem *To move a given weight by a given force*, for it was in reliance 'on the irresistible cogency of his proof' that he declared to Hieron that any given weight could be moved by any given force (however small), and boasted that, 'if he were given a place to stand on, he could move the earth' (πᾶ βῶ, καὶ κινῶ τὰν γᾶν, as he said in his Doric dialect). The story, told by Plutarch, is that, 'when Hieron was struck with amazement and asked Archimedes to reduce the problem to practice and to give an illustration of some great weight moved by a small force, he fixed upon a ship of burden with three masts from the king's arsenal which had only been drawn up with great labour by many men, and loading her with many passengers and a full freight, himself the while sitting far off, with no great effort but only holding the end of a compound pulley (πολύσπαστος) quietly in his hand and pulling at it, he drew the ship along smoothly and safely as if she were moving through the sea.'[4]

The story that Archimedes set the Roman ships on fire by an arrangement of burning-glasses or concave mirrors is not found in any authority earlier than Lucian; but it is quite

[1] Livy xxiv. 34. 2. [2] Ptolemy, *Syntaxis*, III. 1, vol. i, p. 194. 23.
[3] Macrobius, *In Somn. Scip.* ii. 3; cf. the figures in Hippolytus, *Refut.*, p. 66. 52 sq., ed. Duncker.
[4] Plutarch, *Marcellus*, c. 14.

likely that he discovered some form of burning-mirror, e.g. a paraboloid of revolution, which would reflect to one point all rays falling on its concave surface in a direction parallel to its axis.

Archimedes's own view of the relative importance of his many discoveries is well shown by his request to his friends and relatives that they should place upon his tomb a representation of a cylinder circumscribing a sphere, with an inscription giving the ratio which the cylinder bears to the sphere; from which we may infer that he regarded the discovery of this ratio as his greatest achievement. Cicero, when quaestor in Sicily, found the tomb in a neglected state and repaired it[1]; but it has now disappeared, and no one knows where he was buried.

Archimedes's entire preoccupation by his abstract studies is illustrated by a number of stories. We are told that he would forget all about his food and such necessities of life, and would be drawing geometrical figures in the ashes of the fire or, when anointing himself, in the oil on his body.[2] Of the same sort is the tale that, when he discovered in a bath the solution of the question referred to him by Hieron, as to whether a certain crown supposed to have been made of gold did not in fact contain a certain proportion of silver, he ran naked through the street to his home shouting εὕρηκα, εὕρηκα.[3] He was killed in the sack of Syracuse by a Roman soldier. The story is told in various forms: the most picturesque is that found in Tzetzes, which represents him as saying to a Roman soldier who found him intent on some diagrams which he had drawn in the dust and came too close, 'Stand away, fellow, from my diagram', whereat the man was so enraged that he killed him.[4]

Summary of main achievements.

In geometry Archimedes's work consists in the main of original investigations into the quadrature of curvilinear plane figures and the quadrature and cubature of curved surfaces. These investigations, beginning where Euclid's Book XII left off, actually (in the words of Chasles) 'gave

[1] Cicero, *Tusc.* v. 64 sq.
[2] Plutarch, *Marcellus*, c. 17.
[3] Vitruvius, *De architectura*, ix. 1. 9, 10.
[4] Tzetzes, *Chiliad.* ii. 35. 135.

birth to the calculus of the infinite conceived and brought to perfection successively by Kepler, Cavalieri, Fermat, Leibniz and Newton'. He performed in fact what is equivalent to *integration* in finding the area of a parabolic segment, and of a spiral, the surface and volume of a sphere and a segment of a sphere, and the volumes of any segments of the solids of revolution of the second degree. In arithmetic he calculated approximations to the value of π, in the course of which calculation he shows that he could approximate to the value of square roots of large or small non-square numbers; he further invented a system of arithmetical terminology by which he could express in language any number up to that which we should write down with 1 followed by 80,000 million million ciphers. In mechanics he not only worked out the principles of the subject but advanced so far as to find the centre of gravity of a segment of a parabola, a semicircle, a cone, a hemisphere, a segment of a sphere, a right segment of a paraboloid and a spheroid of revolution. His mechanics, as we shall see, has become more important in relation to his geometry since the discovery of the treatise called *The Method* which was formerly supposed to be lost. Lastly, he invented the whole science of hydrostatics, which again he carried so far as to give a most complete investigation of the positions of rest and stability of a right segment of a paraboloid of revolution floating in a fluid with its base either upwards or downwards, but so that the base is either wholly above or wholly below the surface of the fluid. This represents a sum of mathematical achievement unsurpassed by any one man in the world's history.

Character of treatises.

The treatises are, without exception, monuments of mathematical exposition; the gradual revelation of the plan of attack, the masterly ordering of the propositions, the stern elimination of everything not immediately relevant to the purpose, the finish of the whole, are so impressive in their perfection as to create a feeling akin to awe in the mind of the reader. As Plutarch said, 'It is not possible to find in geometry more difficult and troublesome questions or proofs set out in simpler and clearer propositions'.[1] There is at the

[1] Plutarch, *Marcellus*, c. 17.

same time a certain mystery veiling the way in which he arrived at his results. For it is clear that they were not *discovered* by the steps which lead up to them in the finished treatises. If the geometrical treatises stood alone, Archimedes might seem, as Wallis said, 'as it were of set purpose to have covered up the traces of his investigation, as if he had grudged posterity the secret of his method of inquiry, while he wished to extort from them assent to his results'. And indeed (again in the words of Wallis) 'not only Archimedes but nearly all the ancients so hid from posterity their method of Analysis (though it is clear that they had one) that more modern mathematicians found it easier to invent a new Analysis than to seek out the old'. A partial exception is now furnished by *The Method* of Archimedes, so happily discovered by Heiberg. In this book Archimedes tells us how he discovered certain theorems in quadrature and cubature, namely by the use of mechanics, weighing elements of a figure against elements of another simpler figure the mensuration of which was already known. At the same time he is careful to insist on the difference between (1) the means which may be sufficient to suggest the truth of theorems, although not furnishing scientific proofs of them, and (2) the rigorous demonstrations of them by orthodox geometrical methods which must follow before they can be finally accepted as established:

'certain things', he says, 'first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge.' 'This', he adds, 'is a reason why, in the case of the theorems that the volumes of a cone and a pyramid are one-third of the volumes of the cylinder and prism respectively having the same base and equal height, the proofs of which Eudoxus was the first to discover, no small share of the credit should be given to Democritus who was the first to state the fact, though without proof.'

Finally, he says that the very first theorem which he found out by means of mechanics was that of the separate treatise

on the *Quadrature of the parabola*, namely that *the area of any segment of a section of a right-angled cone (i.e. a parabola) is four-thirds of that of the triangle which has the same base and height.* The mechanical proof, however, of this theorem in the *Quadrature of the Parabola* is different from that in the *Method*, and is more complete in that the argument is clinched by formally applying the method of exhaustion.

List of works still extant.

The extant works of Archimedes in the order in which they appear in Heiberg's second edition, following the order of the manuscripts so far as the first seven treatises are concerned, are as follows:

(5) *On the Sphere and Cylinder*: two Books.
(9) *Measurement of a Circle.*
(7) *On Conoids and Spheroids.*
(6) *On Spirals.*
(1) *On Plane Equilibriums*, Book I.
(3) „ „ „ Book II.
(10) *The Sand-reckoner (Psammites).*
(2) *Quadrature of the Parabola.*
(8) *On Floating Bodies*: two Books.
? *Stomachion* (a fragment).
(4) *The Method.*

This, however, was not the order of composition; and, judging (*a*) by statements in Archimedes's own prefaces to certain of the treatises and (*b*) by the use in certain treatises of results obtained in others, we can make out an approximate chronological order, which I have indicated in the above list by figures in brackets. The treatise *On Floating Bodies* was formerly only known in the Latin translation by William of Moerbeke, but the Greek text of it has now been in great part restored by Heiberg from the Constantinople manuscript which also contains *The Method* and the fragment of the *Stomachion*.

Besides these works we have a collection of propositions (*Liber assumptorum*) which has reached us through the Arabic. Although in the title of the translation by Thābit b.

Qurra the book is attributed to Archimedes, the propositions cannot be his in their present form, since his name is several times mentioned in them; but it is quite likely that some of them are of Archimedean origin, notably those about the geometrical figures called ἄρβηλος ('shoemaker's knife') and σάλινον (probably 'salt-cellar') respectively and Prop. 8 bearing on the trisection of an angle.

There is also the *Cattle-Problem* in epigrammatic form, which purports by its heading to have been communicated by Archimedes to the mathematicians at Alexandria in a letter to Eratosthenes. Whether the epigrammatic form is due to Archimedes himself or not, there is no sufficient reason for doubting the possibility that the substance of it was set as a problem by Archimedes.

Traces of lost works.

Of works which are lost we have the following traces.

1. Investigations relating to *polyhedra* are referred to by Pappus who, after alluding to the five regular polyhedra, describes thirteen others discovered by Archimedes which are semi-regular, being contained by polygons equilateral and equiangular but not all similar.[1]

2. There was a book of arithmetical content dedicated to Zeuxippus. We learn from Archimedes himself that it dealt with the *naming of numbers* (κατονόμαξις τῶν ἀριθμῶν)[2] and expounded the system, which we find in the *Sand-reckoner*, of expressing numbers higher than those which could be written in the ordinary Greek notation, numbers in fact (as we have said) up to the enormous figure represented by 1 followed by 80,000 million million ciphers.

3. One or more works on mechanics are alluded to containing propositions not included in the extant treatise *On Plane Equilibriums*. Pappus mentions a work *On Balances* or *Levers* (περὶ ζυγῶν) in which it was proved (as it also was in Philon's and Heron's *Mechanics*) that 'greater circles overpower lesser circles when they revolve about the same centre'.[3] Heron, too, speaks of writings of Archimedes 'which bear the title of

[1] Pappus, v, pp. 352–8.
[2] Archimedes, vol. ii, pp. 216. 18, 236. 17–22; cf. p. 220. 4.
[3] Pappus, viii, p. 1068.

"works on the lever"'.[1] Simplicius refers to *problems on the centre of gravity*, *κεντροβαρικά*, such as the many elegant problems solved by Archimedes and others, the object of which is to show how to find the centre of gravity, that is, the point in a body such that if the body is hung up from it, the body will remain at rest in any position.[2] This recalls the assumption in the *Quadratūre of the Parabola* (6) that, if a body hangs at rest from a point, the centre of gravity of the body and the point of suspension are in the same vertical line. Pappus has a similar remark with reference to a point of *support*, adding that the centre of gravity is determined as the intersection of two straight lines in the body, through two points of support, which straight lines are vertical when the body is in equilibrium so supported. Pappus also gives the characteristic of the centre of gravity mentioned by Simplicius, observing that this is the most fundamental principle of the theory of the centre of gravity, the elementary propositions of which are found in Archimedes's *On Equilibriums* (*περὶ ἰσορροπιῶν*) and Heron's *Mechanics*. Archimedes himself cites propositions which must have been proved elsewhere, e.g. that the centre of gravity of a cone divides the axis in the ratio 3 : 1, the longer segment being that adjacent to the vertex[3]; he also says that 'it is proved in the *Equilibriums*' that the centre of gravity of any segment of a right-angled conoid (i.e. paraboloid of revolution) divides the axis in such a way that the portion towards the vertex is double of the remainder.[4] It is possible that there was originally a larger work by Archimedes *On Equilibriums* of which the surviving books *On Plane Equilibriums* formed only a part; in that case *περὶ ζυγῶν* and *κεντροβαρικά* may only be alternative titles. Finally, Heron says that Archimedes laid down a certain procedure in a book bearing the title 'Book on Supports'.[5]

4. Theon of Alexandria quotes a proposition from a work of Archimedes called *Catoptrica* (properties of mirrors) to the effect that things thrown into water look larger and still larger the farther they sink.[6] Olympiodorus, too, mentions

[1] Heron, *Mechanics*, i. 32.
[2] Simpl. on Arist. *De caelo*, ii, p. 508 a 30, Brandis; p. 543. 24, Heib.
[3] *Method*, Lemma 10.
[4] *On Floating Bodies*, ii. 2.
[5] Heron, *Mechanics*, i. 25.
[6] Theon on Ptolemy's *Syntaxis*, i, p. 29, Halma.

that Archimedes proved the phenomenon of refraction 'by means of the ring placed in the vessel (of water)'.[1] A scholiast to the Pseudo-Euclid's *Catoptrica* quotes a proof, which he attributes to Archimedes, of the equality of the angles of incidence and of reflection in a mirror.

The text of Archimedes.

Heron, Pappus and Theon all cite works of Archimedes which no longer survive, a fact which shows that such works were still extant at Alexandria as late as the third and fourth centuries A.D. But it is evident that attention came to be concentrated on two works only, the *Measurement of a Circle* and *On the Sphere and Cylinder*. Eutocius (*fl.* about A.D. 500) only wrote commentaries on these works and on the *Plane Equilibriums*, and he does not seem even to have been acquainted with the *Quadrature of the Parabola* or the work *On Spirals*, although these have survived. Isidorus of Miletus revised the commentaries of Eutocius on the *Measurement of a Circle* and the two Books *On the Sphere and Cylinder*, and it would seem to have been in the school of Isidorus that these treatises were turned from their original Doric into the ordinary language, with alterations designed to make them more intelligible to elementary pupils. But neither in Isidorus's time nor earlier was there any collected edition of Archimedes's works, so that those which were less read tended to disappear.

In the ninth century Leon, who restored the University of Constantinople, collected together all the works that he could find at Constantinople, and had the manuscript written (the archetype, Heiberg's A) which, through its derivatives, was, up to the discovery of the Constantinople manuscript (C) containing *The Method*, the only source for the Greek text. Leon's manuscript came, in the twelfth century, to the Norman Court at Palermo, and thence passed to the House of Hohenstaufen. Then, with all the library of Manfred, it was given to the Pope by Charles of Anjou after the battle of Benevento in 1266. It was in the Papal Library in the years 1269 and 1311, but, some time after 1368, passed into

[1] Olympiodorus on Arist. *Meteorologica*, ii, p. 94, Ideler; p. 211. 18, Busse.

private hands. In 1491 it belonged to Georgius Valla, who translated from it the portions published in his posthumous work *De expetendis et fugiendis rebus* (1501), and intended to publish the whole of Archimedes with Eutocius's commentaries. On Valla's death in 1500 it was bought by Albertus Pius, Prince of Carpi, passing in 1530 to his nephew, Rodolphus Pius, in whose possession it remained till 1544. At some time between 1544 and 1564 it disappeared, leaving no trace.

The greater part of A was translated into Latin in 1269 by William of Moerbeke at the Papal Court at Viterbo. This translation, in William's own hand, exists at Rome (Cod. Ottobon. lat. 1850, Heiberg's B), and is one of our prime sources, for, although the translation was hastily done and the translator sometimes misunderstood the Greek, he followed its wording so closely that his version is, for purposes of collation, as good as a Greek manuscript. William used also, for his translation, another manuscript from the same library which contained works not included in A. This manuscript was a collection of works on mechanics and optics; William translated from it the two Books *On Floating Bodies*, and it also contained the *Plane Equilibriums* and the *Quadrature of the Parabola*, for which books William used both manuscripts.

The four most important extant Greek manuscripts (except C, the Constantinople manuscript discovered in 1906) were copied from A. The earliest is E, the Venice manuscript (Marcianus 305), which was written between the years 1449 and 1472. The next is D, the Florence manuscript (Laurent. XXVIII. 4), which was copied in 1491 for Angelo Poliziano, permission having been obtained with some difficulty in consequence of the jealousy with which Valla guarded his treasure. The other two are G (Paris. 2360) copied from A after it had passed to Albertus Pius, and H (Paris. 2361) copied in 1544 by Christopherus Auverus for Georges d'Armagnac, Bishop of Rodez. These four manuscripts, with the translation of William of Moerbeke (B), enable the readings of A to be inferred.

A Latin translation was made at the instance of Pope Nicholas V about the year 1450 by Jacobus Cremonensis.

It was made from A, which was therefore accessible to Pope Nicholas though it does not seem to have belonged to him. Regiomontanus made a copy of this translation about 1468 and revised it with the help of E (the Venice manuscript of the Greek text) and a copy of the same translation belonging to Cardinal Bessarion, as well as another 'old copy' which seems to have been B.

The *editio princeps* was published at Basel (*apud Hervagium*) by Thomas Gechauff Venatorius in 1544. The Greek text was based on a Nürnberg MS. (Norimberg. Cent. V, app. 12) which was copied in the sixteenth century from A but with interpolations derived from B; the Latin translation was Regiomontanus's revision of Jacobus Cremonensis (Norimb. Cent. V, 15).

A translation by F. Commandinus published at Venice in 1558 contained the *Measurement of a Circle, On Spirals*, the *Quadrature of the Parabola, On Conoids and Spheroids*, and the *Sand-reckoner*. This translation was based on the Basel edition, but Commandinus also consulted E and other Greek manuscripts.

Torelli's edition (Oxford, 1792) also followed the *editio princeps* in the main, but Torelli also collated E. The book was brought out after Torelli's death by Abram Robertson, who also collated five more manuscripts, including D, G and H. The collation, however, was not well done, and the edition was not properly corrected when in the press.

The second edition of Heiberg's text of all the works of Archimedes with Eutocius's commentaries, Latin translation, apparatus criticus, &c., is now available (1910–15) and, of course, supersedes the first edition (1880–1) and all others. It naturally includes *The Method*, the fragment of the *Stomachion*, and so much of the Greek text of the two Books *On Floating Bodies* as could be restored from the newly discovered Constantinople manuscript.[1]

Contents of *The Method*.

Our description of the extant works of Archimedes may suitably begin with *The Method* (the full title is *On*

[1] *The Works of Archimedes*, edited in modern notation by the present writer in 1897, was based on Heiberg's first edition, and the Supplement

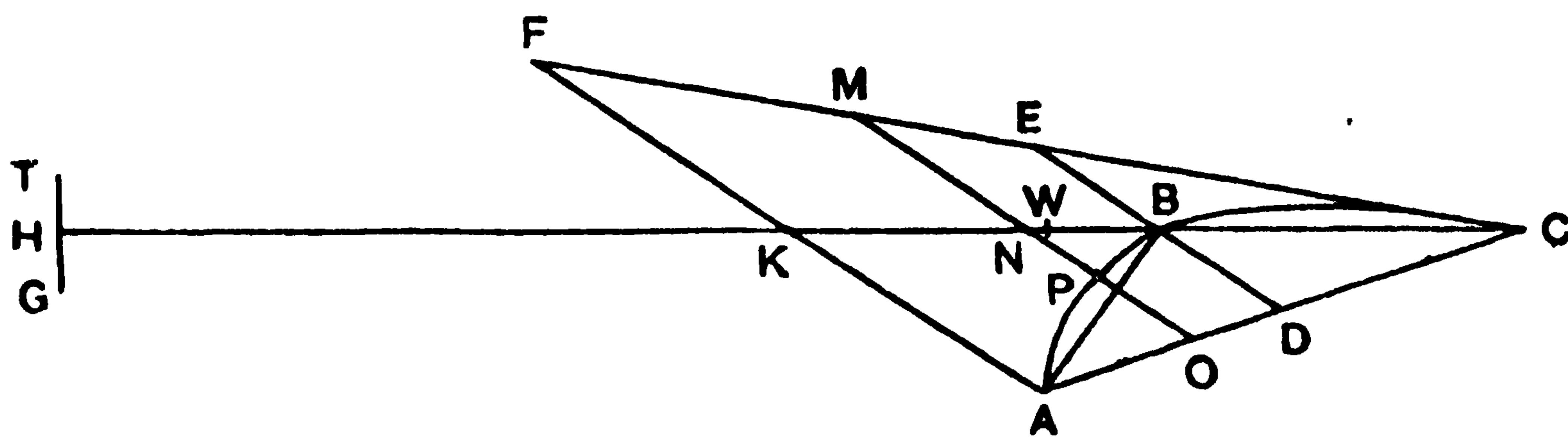
Mechanical Theorems, Method (communicated) *to Eratosthenes*). Premising certain propositions in mechanics mostly taken from the *Plane Equilibriums*, and a lemma which forms Prop. 1 of *On Conoids and Spheroids*, Archimedes obtains by his mechanical method the following results. The area of any segment of a section of a right-angled cone (parabola) is $\frac{4}{3}$ of the triangle with the same base and height (Prop. 1). The right cylinder circumscribing a sphere or a spheroid of revolution and with axis equal to the diameter or axis of revolution of the sphere or spheroid is $1\frac{1}{2}$ times the sphere or spheroid respectively (Props. 2, 3). Props. 4, 7, 8, 11 find the volume of any segment cut off, by a plane at right angles to the axis, from any right-angled conoid (paraboloid of revolution), sphere, spheroid, and obtuse-angled conoid (hyperboloid) in terms of the cone which has the same base as the segment and equal height. In Props. 5, 6, 9, 10 Archimedes uses his method to find the centre of gravity of a segment of a paraboloid of revolution, a sphere, and a spheroid respectively. Props. 12–15 and Prop. 16 are concerned with the cubature of two special solid figures. (1) Suppose a prism with a square base to have a cylinder inscribed in it, the circular bases of the cylinder being circles inscribed in the squares which are the bases of the prism, and suppose a plane drawn through one side of one base of the prism and through that diameter of the circle in the opposite base which is parallel to the said side. This plane cuts off a solid bounded by two planes and by part of the curved surface of the cylinder (a solid shaped like a hoof cut off by a plane); and Props. 12–15 prove that its volume is one-sixth of the volume of the prism. (2) Suppose a cylinder inscribed in a cube, so that the circular bases of the cylinder are circles inscribed in two opposite faces of the cube, and suppose another cylinder similarly inscribed with reference to two other opposite faces. The two cylinders enclose a certain solid which is actually made up of eight 'hoofs' like that of Prop. 12. Prop. 16 proves that the volume of this solid is two-thirds of that of the cube. Archimedes observes in his preface that a remarkable fact about

(1912) containing *The Method*, on the original edition of Heiberg (in

these solids respectively is that each of them is equal to a solid enclosed by *planes*, whereas the volume of curvilinear solids (spheres, spheroids, &c.) is generally only expressible in terms of other curvilinear solids (cones and cylinders). In accordance with his dictum that the results obtained by the mechanical method are merely indicated, but not actually proved, unless confirmed by the rigorous methods of pure geometry, Archimedes proved the facts about the two last-named solids by the orthodox method of exhaustion as regularly used by him in his other geometrical treatises; the proofs, partly lost, were given in Props. 15 and 16.

We will first illustrate the method by giving the argument of Prop. 1 about the area of a parabolic segment.

Let ABC be the segment, BD its diameter, CF the tangent at C. Let P be any point on the segment, and let AKF,



$OPNM$ be drawn parallel to BD. Join CB and produce it to meet MO in N and FA in K, and let KH be made equal to KC.

Now, by a proposition 'proved in a lemma' (cf. *Quadrature of the Parabola*, Prop. 5)

$$\begin{aligned} MO : OP &= CA : AO \\ &= CK : KN \\ &= HK : KN. \end{aligned}$$

Also, by the property of the parabola, $EB = BD$, so that $MN = NO$ and $FK = KA$.

It follows that, if HC be regarded as the bar of a balance, a line TG equal to PO and placed with its middle point at H balances, about K, the straight line MO placed where it is, i.e. with its middle point at N.

Similarly with *all* lines, as MO, PO, in the triangle CFA and the segment CBA respectively.

And there are the same number of these lines. Therefore

the whole segment of the parabola acting at H balances the triangle CFA placed where it is.

But the centre of gravity of the triangle CFA is at W, where $CW = 2\,WK$ [and the whole triangle may be taken as acting at W].

$$\text{Therefore}\quad (\text{segment } ABC) : \Delta CFA = WK : KH$$

$$= 1 : 3,$$

$$\text{so that}\quad (\text{segment } ABC) = \tfrac{1}{3}\Delta CFA$$

$$= \tfrac{4}{3}\Delta ABC. \qquad \text{Q. E. D.}$$

It will be observed that Archimedes takes the segment and the triangle to be *made up* of parallel lines indefinitely close together. In reality they are made up of indefinitely narrow strips, but the width (dx, as we might say) being the same for the elements of the triangle and segment respectively, divides out. And of course the weight of each element in both is proportional to the area. Archimedes also, without mentioning *moments*, in effect assumes that the sum of the moments of each particle of a figure, acting where it is, is equal to the moment of the whole figure applied as one mass at its centre of gravity.

We will now take the case of any segment of a spheroid of revolution, because that will cover several propositions of Archimedes as particular cases.

The ellipse with axes AA', BB' is a section made by the plane of the paper in a spheroid with axis AA'. It is required to find the volume of any right segment ADC of the spheroid in terms of the right cone with the same base and height.

Let DC be the diameter of the circular base of the segment. Join AB, AB', and produce them to meet the tangent at A' to the ellipse in K, K', and DC produced in E, F.

Conceive a cylinder described with axis AA' and base the circle on KK' as diameter, and cones described with AG as axis and bases the circles on EF, DC as diameters.

Let N be any point on AG, and let $MOPQNQ'P'O'M'$ be drawn through N parallel to BB' or DC as shown in the figure.

Produce $A'A$ to H so that $HA = AA'$.

Now $$HA : AN = A'A : AN$$
$$= KA : AQ$$
$$= MN : NQ$$
$$= MN^2 : MN . NQ.$$

It is now necessary to prove that $MN . NQ = NP^2 + NQ^2$.

By the property of the ellipse,

$$AN . NA' : NP^2 = (\tfrac{1}{2} AA')^2 : (\tfrac{1}{2} BB')^2$$
$$= AN^2 : NQ^2;$$

therefore $$NQ^2 : NP^2 = AN^2 : AN . NA'$$
$$= NQ^2 : NQ . QM,$$

whence $$NP^2 = MQ . QN.$$

Add NQ^2 to each side, and we have

$$NP^2 + NQ^2 = MN . NQ.$$

Therefore, from above,

$$HA : AN = MN^2 : (NP^2 + NQ^2). \qquad (1)$$

But MN^2, NP^2, NQ^2 are to one another as the areas of the circles with MM', PP', QQ' respectively as diameters, and these

circles are sections made by the plane though N at right angles to AA' in the cylinder, the spheroid and the cone AEF respectively.

Therefore, if HAA' be a lever, and the sections of the spheroid and cone be both placed with their centres of gravity at H, these sections placed at H balance, about A, the section MM' of the cylinder where it is.

Treating all the corresponding sections of the segment of the spheroid, the cone and the cylinder in the same way, we find that the cylinder with axis AG, where it is, balances, about A, the cone AEF and the segment ADC together, when both are placed with their centres of gravity at H; and, if W be the centre of gravity of the cylinder, i.e. the middle point of AG,

$$HA : AW = (\text{cylinder, axis } AG) : (\text{cone } AEF + \text{segmt. } ADC).$$

If we call V the volume of the cone AEF, and S that of the segment of the spheroid, we have

$$(\text{cylinder}) : (V+S) = 3V . \frac{AA'^2}{AG^2} : (V+S),$$

while $$HA : AW = AA' : \tfrac{1}{2}AG.$$

Therefore $$AA' : \tfrac{1}{2}AG = 3V . \frac{AA'^2}{AG^2} : (V+S),$$

and $$(V+S) = \tfrac{3}{2}V . \frac{AA'}{AG},$$

whence $$S = V\left(\frac{3AA'}{2AG} - 1\right).$$

Again, let V' be the volume of the cone ADC.

Then $$V : V' = EG^2 : DG^2$$

$$= \frac{BB'^2}{AA'^2} . AG^2 : DG^2.$$

But $$DG^2 : AG . GA' = BB'^2 : AA'^2.$$

Therefore $$V : V' = AG^2 : AG . GA'$$

$$= AG : GA'.$$

It follows that $S = V' . \frac{AG}{GA'}\left(\frac{3AA'}{2AG} - 1\right)$

$$= V' . \frac{\frac{3}{2}AA' - AG}{A'G}$$

$$= V' . \frac{\frac{1}{2}AA' + A'G}{A'G},$$

which is the result stated by Archimedes in Prop. 8.

The result is the same for the segment of a sphere. The proof, of course slightly simpler, is given in Prop. 7.

In the particular case where the segment is half the sphere or spheroid, the relation becomes

$$S = 2V', \qquad \text{(Props. 2, 3)}$$

and it follows that the volume of the whole sphere or spheroid is $4V'$, where V' is the volume of the cone ABB'; i.e. the volume of the sphere or spheroid is two-thirds of that of the circumscribing cylinder.

In order now to find the centre of gravity of the segment of a spheroid, we must have the segment acting *where it is*, not at H.

Therefore formula (1) above will not serve. But we found

that $\qquad MN . NQ = (NP^2 + NQ^2),$

whence $\qquad MN^2 : (NP^2 + NQ^2) = (NP^2 + NQ^2) : NQ^2;$

therefore $\qquad HA : AN = (NP^2 + NQ^2) : NQ^2.$

(This is separately proved by Archimedes for the sphere in Prop. 9.)

From this we derive, as usual, that the cone AEF and the segment ADC both acting *where they are* balance a volume equal to the cone AEF placed with its centre of gravity at H.

Now the centre of gravity of the cone AEF is on the line AG at a distance $\frac{3}{4}AG$ from A. Let X be the required centre of gravity of the segment. Then, taking moments about A,

we have $\qquad V . HA = S . AX + V . \frac{3}{4}AG,$

or $\qquad V(AA' - \frac{3}{4}AG) = S . AX$

$$= V\left(\frac{\frac{3}{2}AA'}{AG} - 1\right) AX, \text{ from above.}$$

Therefore $AX:AG = (AA' - \frac{3}{4}AG):(\frac{3}{2}AA' - AG)$

$= (4AA' - 3AG):(6AA' - 4AG)$;

whence $AX:XG = (4AA' - 3AG):(2AA' - AG)$

$= (AG + 4A'G):(AG + 2A'G)$,

which is the result obtained by Archimedes in Prop. 9 for the sphere and in Prop. 10 for the spheroid.

In the case of the hemi-spheroid or hemisphere the ratio $AX:XG$ becomes $5:3$, a result obtained for the hemisphere in Prop. 6.

The cases of the paraboloid of revolution (Props. 4, 5) and the hyperboloid of revolution (Prop. 11) follow the same course, and it is unnecessary to reproduce them.

For the cases of the two solids dealt with at the end of the treatise the reader must be referred to the propositions themselves. Incidentally, in Prop. 13, Archimedes finds the centre of gravity of the half of a cylinder cut by a plane through the axis, or, in other words, the centre of gravity of a semicircle.

We will now take the other treatises in the order in which they appear in the editions.

On the Sphere and Cylinder, I, II.

The main results obtained in Book I are shortly stated in a prefatory letter to Dositheus. Archimedes tells us that they are new, and that he is now publishing them for the first time, in order that mathematicians may be able to examine the proofs and judge of their value. The results are (1) that the surface of a sphere is four times that of a great circle of the sphere, (2) that the surface of any segment of a sphere is equal to a circle the radius of which is equal to the straight line drawn from the vertex of the segment to a point on the circumference of the base, (3) that the volume of a cylinder circumscribing a sphere and with height equal to the diameter of the sphere is $\frac{3}{2}$ of the volume of the sphere, (4) that the surface of the circumscribing cylinder including its bases is also $\frac{3}{2}$ of the surface of the sphere. It is worthy of note that, while the first and third of these propositions appear in the book in this order (Props. 33 and 34 respec-

tively), this was not the order of their discovery; for Archimedes tells us in *The Method* that

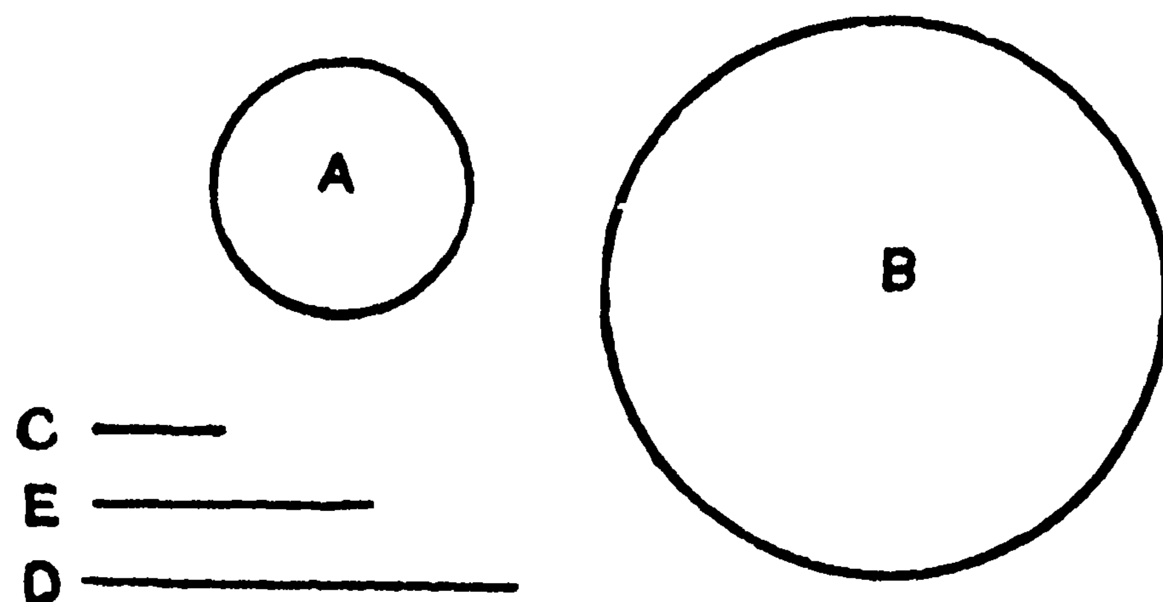
'from the theorem that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere I conceived the notion that the surface of any sphere is four times as great as a great circle in it; for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius'.

Book I begins with definitions (of 'concave in the same direction' as applied to curves or broken lines and surfaces, of a 'solid sector' and a 'solid rhombus') followed by five Assumptions, all of importance. *Of all lines which have the same extremities the straight line is the least*, and, if there are two curved or bent lines in a plane having the same extremities and concave in the same direction, but one is wholly included by, or partly included by and partly common with, the other, then that which is included is the lesser of the two. Similarly with plane surfaces and surfaces concave in the same direction. Lastly, Assumption 5 is the famous 'Axiom of Archimedes', which however was, according to Archimedes himself, used by earlier geometers (Eudoxus in particular), to the effect that *Of unequal magnitudes the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude of the same kind*; the axiom is of course practically equivalent to Eucl. V, Def. 4, and is closely connected with the theorem of Eucl. X. 1.

As, in applying the method of exhaustion, Archimedes uses both circumscribed and inscribed figures with a view to *compressing* them into coalescence with the curvilinear figure to be measured, he has to begin with propositions showing that, given two unequal magnitudes, then, however near the ratio of the greater to the less is to 1, it is possible to find two straight lines such that the greater is to the less in a still less ratio (>1), and to circumscribe and inscribe similar polygons to a circle or sector such that the perimeter or the area of the circumscribed polygon is to that of the inner in a ratio less than the given ratio (Props. 2 6): also, just as Euclid proves

that, if we continually double the number of the sides of the regular polygon inscribed in a circle, segments will ultimately be left which are together less than any assigned area, Archimedes has to supplement this (Prop. 6) by proving that, if we increase the number of the sides of a *circumscribed* regular polygon sufficiently, we can make the excess of the area of the polygon over that of the circle less than any given area. Archimedes then addresses himself to the problems of finding the *surface* of any right cone or cylinder, problems finally solved in Props. 13 (the cylinder) and 14 (the cone). Circumscribing and inscribing regular polygons to the bases of the cone and cylinder, he erects pyramids and prisms respectively on the polygons as bases and circumscribed or inscribed to the cone and cylinder respectively. In Props. 7 and 8 he finds the surface of the pyramids inscribed and circumscribed to the cone, and in Props. 9 and 10 he proves that the surfaces of the inscribed and circumscribed pyramids respectively (excluding the base) are less and greater than the surface of the cone (excluding the base). Props. 11 and 12 prove the same thing of the prisms inscribed and circumscribed to the cylinder, and finally Props. 13 and 14 prove, by the method of exhaustion, that the surface of the cone or cylinder (excluding the bases) is equal to the circle the radius of which is a mean proportional between the 'side' (i.e. generator) of the cone or cylinder and the radius or diameter of the base (i.e. is equal to πrs in the case of the cone and $2\pi rs$ in the case of the cylinder, where r is the radius of the base and s a generator). As Archimedes here applies the method of exhaustion for the first time, we will illustrate by the case of the cone (Prop. 14).

Let A be the base of the cone, C a straight line equal to its



radius, D a line equal to a generator of the cone, E a mean proportional to C, D, and B a circle with radius equal to E.

If S is the surface of the cone, we have to prove that $S = B$.

For, if S is not equal to B, it must be either greater or less.

I. Suppose $B < S$.

Circumscribe a regular polygon about B, and inscribe a similar polygon in it, such that the former has to the latter a ratio less than $S:B$ (Prop. 5). Describe about A a similar polygon and set up from it a pyramid circumscribing the cone.

Then (polygon about A) : (polygon about B)

$= C^2 : E^2$

$= C : D$

$=$ (polygon about A) : (surface of pyramid).

Therefore (surface of pyramid) $=$ (polygon about B).

But (polygon about B) : (polygon in B) $< S:B$;

therefore (surface of pyramid) : (polygon in B) $< S:B$.

But this is impossible, since (surface of pyramid) $> S$, while (polygon in B) $< B$;
therefore B is not less than S.

II. Suppose $B > S$.

Circumscribe and inscribe similar regular polygons to B such that the former has to the latter a ratio $< B:S$. Inscribe in A a similar polygon, and erect on A the inscribed pyramid.

Then (polygon in A) : (polygon in B) $= C^2 : E^2$

$= C : D$

$>$ (polygon in A) : (surface of pyramid).

(The latter inference is clear, because the ratio of $C:D$ is greater than the ratio of the perpendiculars from the centre of A and from the vertex of the pyramid respectively on any side of the polygon in A; in other words, if $\beta < \alpha < \frac{1}{2}\pi$, $\sin\alpha > \sin\beta$.)

Therefore (surface of pyramid) $>$ (polygon in B).

But (polygon about B) : (polygon in B) $< B:S$,

whence (*a fortiori*)

(polygon about B) : (surface of pyramid) $< B:S$,

which is impossible, for (polygon about B) $> B$, while (surface of pyramid) $< S$.

Therefore B is not greater than S.

Hence S, being neither greater nor less than B, is equal to B.

Archimedes next addresses himself to the problem of finding the surface and volume of a sphere or a segment thereof, but has to interpolate some propositions about 'solid rhombi' (figures made up of two right cones, unequal or equal, with bases coincident and vertices in opposite directions) the necessity of which will shortly appear.

Taking a great circle of the sphere or a segment of it, he inscribes a regular polygon of an even number of sides bisected

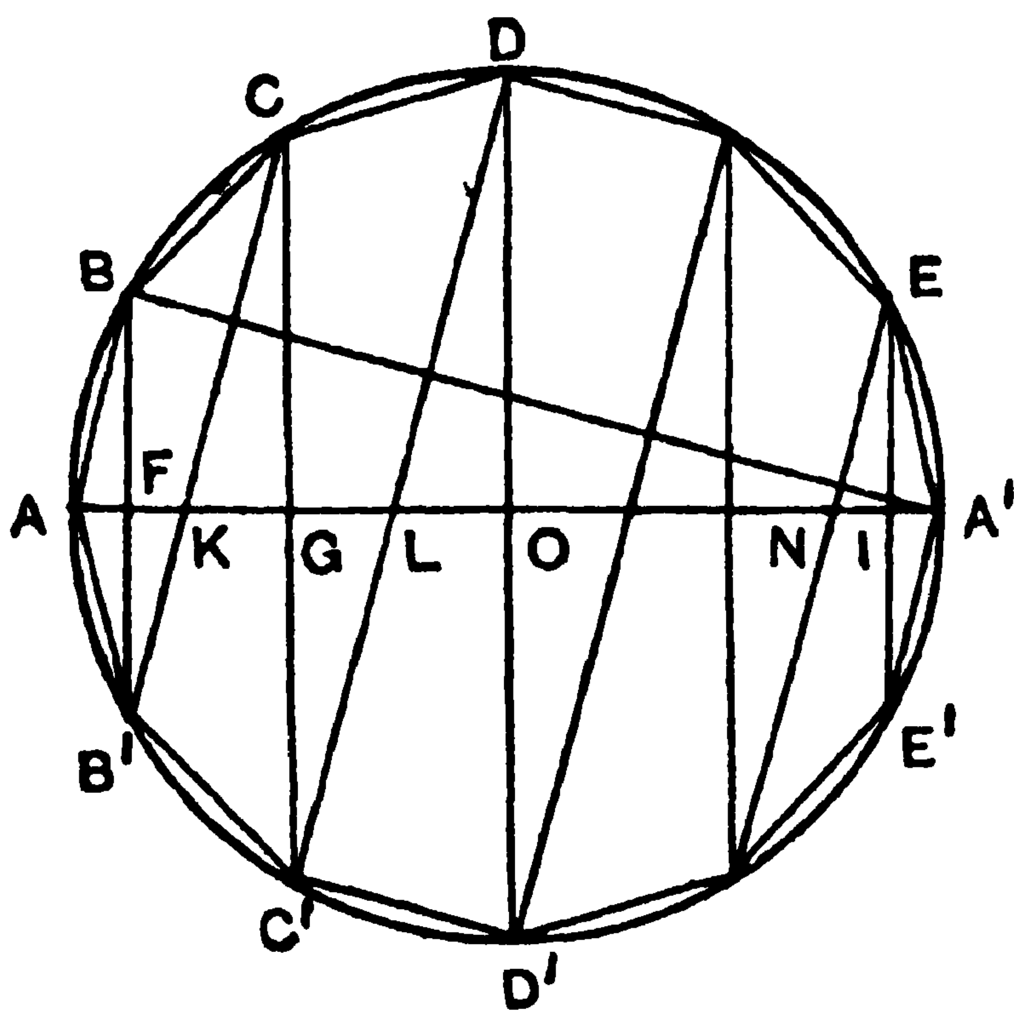


FIG. 1.

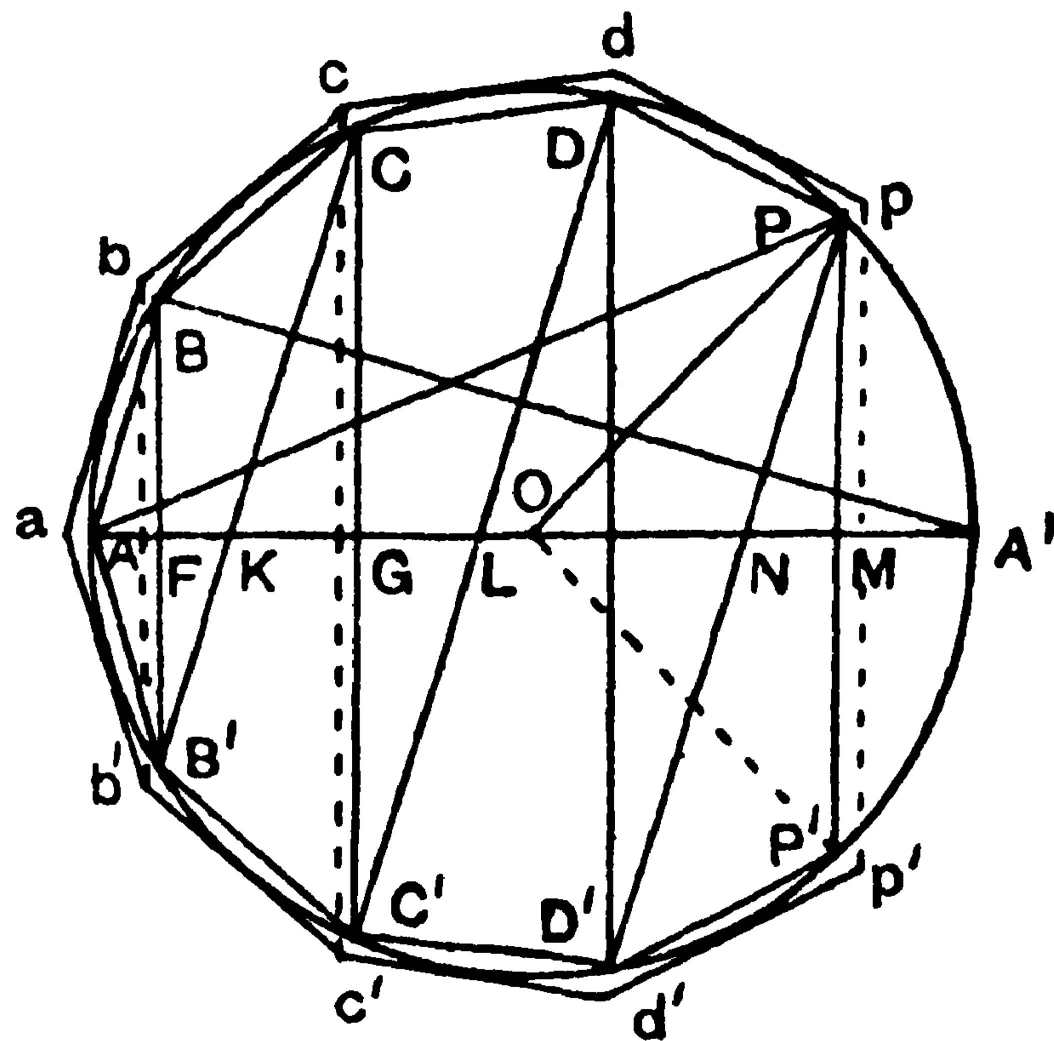


FIG. 2.

by the diameter AA', and approximates to the surface and volume of the sphere or segment by making the polygon revolve about AA' and measuring the surface and volume of solid so inscribed (Props. 21–7). He then does the same for the a circumscribed solid (Props. 28–32). Construct the inscribed polygons as shown in the above figures. Joining BB', CC', ..., CB', DC' ... we see that BB', CC' ... are all parallel, and so are AB, CB', DC'....

Therefore, by similar triangles, $BF:FA = A'B:BA$, and

$$BF:FA = B'F:FK$$
$$= CG:GK$$
$$= C'G:GL$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$= E'I:IA' \text{ in Fig. 1}$$
$$(= PM:MN \text{ in Fig. 2}),$$

whence, adding antecedents and consequents, we have

(Fig. 1) $(BB'+CC'+\ldots+EE'):AA' = A'B:BA$, (Prop. 21)

(Fig. 2) $(BB'+CC'+\ldots+\frac{1}{2}PP'):AM = A'B:BA$. (Prop. 22)

When we make the polygon revolve about AA', the surface of the inscribed figure so obtained is made up of the surfaces of cones and frusta of cones; Prop. 14 has proved that the surface of the cone ABB' is what we should write $\pi . AB . BF$, and Prop. 16 has proved that the surface of the frustum $BCC'B'$ is $\pi . BC(BF+CG)$. It follows that, since $AB = BC = \ldots$, the surface of the inscribed solid is

$$\pi . AB\{\tfrac{1}{2}BB'+\tfrac{1}{2}(BB'+CC')+\ldots\},$$

that is, $\pi . AB(BB'+CC'+\ldots+EE')$ (Fig. 1), (Prop. 24)

or $\pi . AB(BB'+CC'+\ldots+\frac{1}{2}PP')$ (Fig. 2). (Prop. 35)

Hence, from above, the surface of the inscribed solid is $\pi . A'B . AA'$ or $\pi . A'B . AM$, and is therefore less than $\pi . AA'^2$ (Prop. 25) or $\pi . A'A . AM$, that is, $\pi . AP^2$ (Prop. 37).

Similar propositions with regard to surfaces formed by the revolution about AA' of regular circumscribed solids prove that their surfaces are greater than $\pi . AA'^2$ and $\pi . AP^2$ respectively (Props. 28–30 and Props. 39–40). The case of the segment is more complicated because the circumscribed polygon with its sides parallel to AB, BC ... DP circumscribes the *sector* POP'. Consequently, if the segment is less than a semicircle, as CAC', the base of the circumscribed polygon (cc') is on the side of CC' towards A, and therefore the circumscribed polygon leaves over a small strip of the inscribed. This complication is dealt with in Props. 39–40. Having then arrived at circumscribed and inscribed figures with surfaces greater and less than $\pi . AA'^2$ and $\pi . AP^2$ respectively, and having proved (Props. 32, 41) that the surfaces of the circumscribed and inscribed figures are to one another in the duplicate ratio of their sides, Archimedes proceeds to prove formally, by the method of exhaustion, that the surfaces of the sphere and segment are equal to these circles respectively (Props. 33 and 42); $\pi . AA'^2$ is of course equal to four times the great circle of the sphere. The segment is, for convenience, taken to be

less than a hemisphere, and Prop. 43 proves that the same formula applies also to a segment greater than a hemisphere.

As regards the volumes different considerations involving 'solid rhombi' come in. For convenience Archimedes takes, in the case of the whole sphere, an inscribed polygon of $4n$ sides (Fig. 1). It is easily seen that the solid figure formed by its revolution is made up of the following: first, the solid rhombus formed by the revolution of the quadrilateral $ABOB'$ (the volume of this is shown to be equal to the cone with base equal to the surface of the cone ABB' and height equal to p, the perpendicular from O on AB, Prop. 18); secondly, the extinguisher-shaped figure formed by the revolution of the triangle BOC about AA' (this figure is equal to the difference between two solid rhombi formed by the revolution of $TBOB'$ and $TCOC'$ respectively about AA', where T is the point of intersection of CB, $C'B'$ produced with $A'A$ produced, and this difference is proved to be equal to a cone with base equal to the surface of the frustum of a cone described by BC in its revolution and height equal to p the perpendicular from O on BC, Prop. 20); and so on; finally, the figure formed by the revolution of the triangle COD about AA' is the difference between a cone and a solid rhombus, which is proved equal to a cone with base equal to the surface of the frustum of a cone described by CD in its revolution and height p (Prop. 19). Consequently, by addition, the volume of the whole solid of revolution is equal to the cone with base equal to its whole surface and height p (Prop. 26). But the whole of the surface of the solid is less than $4\pi r^2$, and $p < r$; therefore the volume of the inscribed solid is less than four times the cone with base πr^2 and height r (Prop. 27).

It is then proved in a similar way that the revolution of the similar circumscribed polygon of $4n$ sides gives a solid the volume of which is *greater* than four times the same cone (Props. 28–31 Cor.). Lastly, the volumes of the circumscribed and inscribed figures are to one another in the triplicate ratio of their sides (Prop. 32); and Archimedes is now in a position to apply the method of exhaustion to prove that the volume of the sphere is 4 times the cone with base πr^2 and height r (Prop. 34).

Dealing with the segment of a sphere, Archimedes takes, for

convenience, a segment less than a hemisphere and, by the same chain of argument (Props. 38, 40 Corr., 41 and 42), proves (Prop. 44) that the volume of the *sector* of the sphere bounded by the surface of the segment is equal to a cone with base equal to the surface of the segment and height equal to the radius, i.e. the cone with base $\pi . AP^2$ and height r (Fig. 2).

It is noteworthy that the proportions obtained in Props. 21, 22 (see p. 39 above) can be expressed in trigonometrical form. If $4n$ is the number of the sides of the polygon inscribed in the circle, and $2n$ the number of the sides of the polygon inscribed in the segment, and if the angle AOP is denoted by α, the trigonometrical equivalents of the proportions are respectively

(1) $$\sin\frac{\pi}{2n}+\sin\frac{2\pi}{2n}+\ldots+\sin(2n-1)\frac{\pi}{2n}=\cot\frac{\pi}{4n};$$

(2) $$2\left\{\sin\frac{\alpha}{n}+\sin\frac{2\alpha}{n}+\ldots+\sin(n-1)\frac{\alpha}{n}\right\}+\sin\alpha$$
$$=(1-\cos\alpha)\cot\frac{\alpha}{2n}.$$

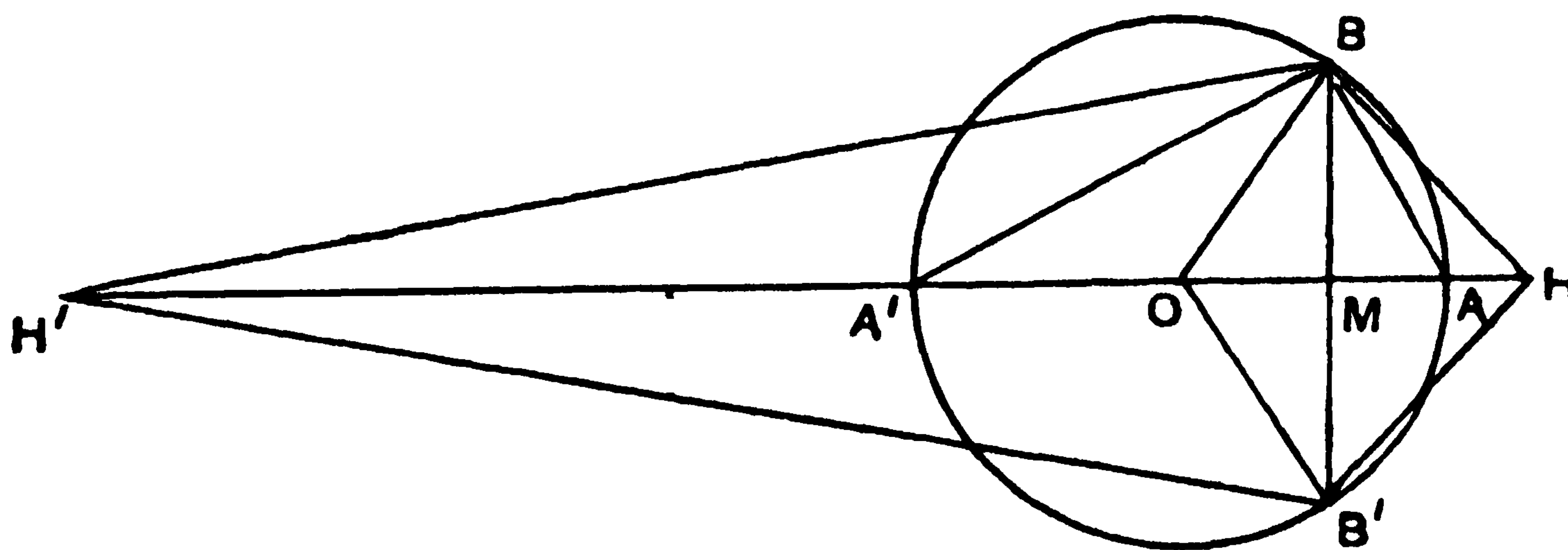
Thus the two proportions give in effect a summation of the series
$$\sin\theta+\sin 2\theta+\ldots+\sin(n-1)\theta,$$
both generally where $n\theta$ is equal to any angle α less than π and in the particular case where n is even and $\theta=\pi/n$. Props. 24 and 35 prove that the areas of the circles equal to the surfaces of the solids of revolution described by the polygons inscribed in the sphere and segment are the above series multiplied by $4\pi r^2\sin\frac{\pi}{4n}$ and $\pi r^2 . 2\sin\frac{\alpha}{2n}$ respectively and are therefore $4\pi r^2\cos\frac{\pi}{4n}$ and $\pi r^2 . 2\cos\frac{\alpha}{2n}(1-\cos\alpha)$ respectively. Archimedes's results for the surfaces of the sphere and segment, $4\pi r^2$ and $2\pi r^2(1-\cos\alpha)$, are the limiting values of these expressions when n is indefinitely increased and when therefore $\cos\frac{\pi}{4n}$ and $\cos\frac{\alpha}{2n}$ become unity. And the two series multiplied by $4\pi r^2\sin\frac{\pi}{4n}$ and

$\pi r^2 . 2 \sin \frac{\alpha}{2n}$ respectively are (when n is indefinitely increased) precisely what we should represent by the integrals

$$4\pi r^2 . \tfrac{1}{2}\int_0^\pi \sin\theta\, d\theta, \text{ or } 4\pi r^2,$$

and $$\pi r^2 . \int_0^\alpha 2\sin\theta\, d\theta, \text{ or } 2\pi r^2(1-\cos\alpha).$$

Book II contains six problems and three theorems. Of the theorems Prop. 2 completes the investigation of the volume of any segment of a sphere, Prop. 44 of Book I having only brought us to the volume of the corresponding sector. If ABB' be a segment of a sphere cut off by a plane at right angles to AA', we learnt in I. 44 that the volume of the *sector*



$OBAB'$ is equal to the cone with base equal to the surface of the segment and height equal to the radius, i.e. $\frac{1}{3}\pi . AB^2 . r$, where r is the radius. The volume of the segment is therefore

$$\tfrac{1}{3}\pi . AB^2 . r - \tfrac{1}{3}\pi . BM^2 . OM.$$

Archimedes wishes to express this as a cone with base the same as that of the segment. Let AM, the height of the segment, $= h$.

Now $$AB^2 : BM^2 = A'A : A'M = 2r : (2r-h).$$

Therefore

$$\tfrac{1}{3}\pi(AB^2 . r - BM^2 . OM) = \tfrac{1}{3}\pi . BM^2\left\{\frac{2r^2}{2r-h} - (r-h)\right\}$$

$$= \tfrac{1}{3}\pi . BM^2 . h\left(\frac{3r-h}{2r-h}\right).$$

That is, the segment is equal to the cone with the same base as that of the segment and height $h(3r-h)/(2r-h)$.

This is expressed by Archimedes thus. If HM is the height of the required cone,

$$HM : AM = (OA' + A'M) : A'M, \qquad (1)$$

and similarly the cone equal to the segment $A'BB'$ has the height $H'M$, where

$$H'M : A'M = (OA + AM) : AM. \qquad (2)$$

His proof is, of course, not in the above form but purely geometrical.

This proposition leads to the most important proposition in the Book, Prop. 4, which solves the problem *To cut a given sphere by a plane in such a way that the volumes of the segments are to one another in a given ratio.*

Cubic equation arising out of II. 4.

If $m : n$ be the given ratio of the cones which are equal to the segments and the heights of which are h, h', we have

$$h\left(\frac{3r-h}{2r-h}\right) = \frac{m}{n} h'\left(\frac{3r-h'}{2r-h'}\right),$$

and, if we eliminate h' by means of the relation $h + h' = 2r$, we easily obtain the following cubic equation in h,

$$h^3 - 3h^2 r + \frac{4m}{m+n} r^3 = 0.$$

Archimedes in effect reduces the problem to this equation, which, however, he treats as a particular case of the more general problem corresponding to the equation

$$(r+h) : b = c^2 : (2r-h)^2,$$

where b is a given length and c^2 any given area,

or $\qquad x^2(a-x) = bc^2$, where $x = 2r-h$ and $3r = a$.

Archimedes obtains his cubic equation with one unknown by means of a *geometrical* elimination of H, H' from the equation $HM = \frac{m}{n} . H'M$, where HM, $H'M$ have the values determined by the proportions (1) and (2) above, after which the one variable point M remaining corresponds to the one unknown of the cubic equation. His method is, first, to find

values for each of the ratios $A'H' : H'M$ and $H'H : A'H'$ which are alike independent of H, H' and then, secondly, to equate the ratio compounded of these two to the known value of the ratio $HH' : H'M$.

(α) We have, from (2),

$$A'H' : H'M = OA : (OA + AM). \qquad (3)$$

(β) From (1) and (2), *separando*,

$$AH : AM = OA' : A'M, \qquad (4)$$

$$A'H' : A'M = OA : AM. \qquad (5)$$

Equating the values of the ratio $A'M : AM$ given by (4), (5), we have

$$OA' : AH = A'H' : OA$$
$$= OH' : OH,$$

whence $HH' : OH' = OH' : A'H'$, (since $OA = OA'$)

or $HH' . A'H' = OH'^2$,

so that $HH' : A'H' = OH'^2 : A'H'^2$. (6)

But, by (5), $OA' : A'H' = AM : A'M$,

and, *componendo*, $OH' : A'H' = AA' : A'M$.

By substitution in (6),

$$HH' : A'H' = AA'^2 : A'M^2. \qquad (7)$$

Compounding with (3), we obtain

$$HH' : H'M = (AA'^2 : A'M^2) . (OA : OA + AM). \qquad (8)$$

[The algebraical equivalent of this is

$$\frac{m+n}{n} = \frac{4r^3}{(2r-h)^2 (r+h)},$$

which reduces to $\dfrac{m+n}{m} = \dfrac{4r^3}{3h^2 r - h^3}$,

or $h^3 - 3h^2 r + \dfrac{4m}{m+n} r^3 = 0$, as above.]

Archimedes expresses the result (8) more simply by producing OA to D so that $OA = AD$, and then dividing AD at

E so that $AD:DE = HH':H'M$ or $(m+n):n$. We have then $OA = AD$ and $OA+AM = MD$, so that (8) reduces to

$$AD:DE = (AA'^2:A'M^2).(AD:MD),$$

or
$$MD:DE = AA'^2:A'M^2.$$

Now, says Archimedes, D is given, since $AD = OA$. Also, $AD:DE$ being a given ratio, DE is given. Hence the problem reduces itself to that of dividing $A'D$ into two parts at M such that

$$MD : (\text{a given length}) = (\text{a given area}) : A'M^2.$$

That is, the generalized equation is of the form

$$x^2\ (a-x) = bc^2, \text{ as above.}$$

(i) Archimedes's own solution of the cubic.

Archimedes adds that, 'if the problem is propounded in this general form, it requires a διορισμός [i.e. it is necessary to investigate the limits of possibility], but if the conditions are added which exist in the present case [i.e. in the actual problem of Prop. 4], it does not require a διορισμός' (in other words, a solution is always possible). He then promises to give 'at the end' an analysis and synthesis of both problems [i.e. the διορισμός and the problem itself]. The promised solutions do not appear in the treatise as we have it, but Eutocius gives solutions taken from 'an old book' which he managed to discover after laborious search, and which, since it was partly written in Archimedes's favourite Doric, he with fair reason assumed to contain the missing *addendum* by Archimedes.

In the Archimedean fragment preserved by Eutocius the above equation, $x^2(a-x) = bc^2$, is solved by means of the intersection of a parabola and a rectangular hyperbola, the equations of which may be written thus

$$x^2 = \frac{c^2}{a}y, \quad (a-x)\,y = ab.$$

The διορισμός takes the form of investigating the maximum possible value of $x^2(a-x)$, and it is proved that this maximum value for a real solution is that corresponding to the value $x = \frac{2}{3}a$. This is established by showing that, if $bc^2 = \frac{4}{27}a^3$,

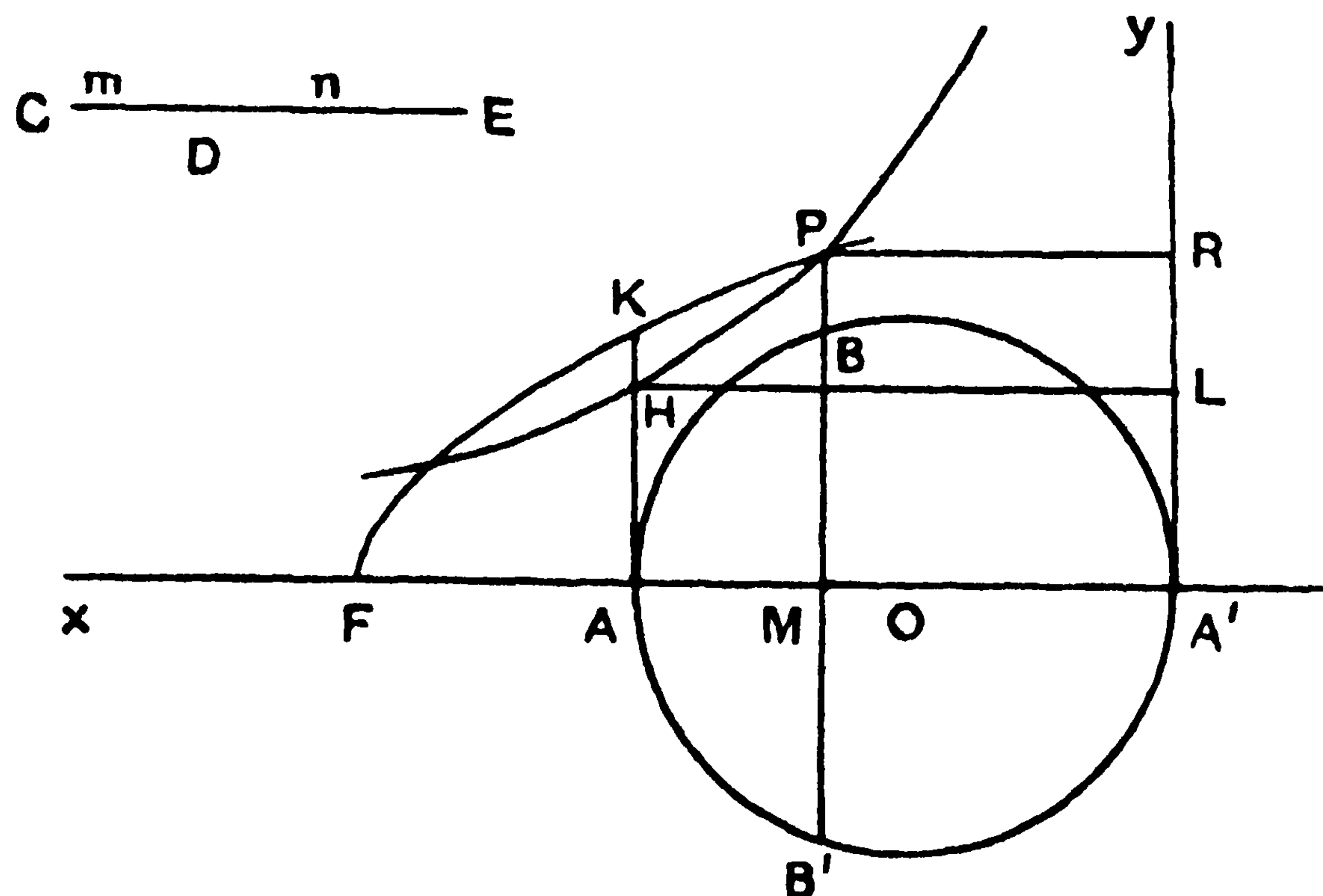
the curves touch at the point for which $x = \frac{2}{3}a$. If on the other hand $bc^2 < \frac{4}{27}a^3$, it is proved that there are two real solutions. In the particular case arising in Prop. 4 it is clear that the condition for a real solution is satisfied, for the expression corresponding to bc^2 is $\frac{m}{m+n} 4r^3$, and it is only necessary that $\frac{m}{m+n} 4r^3$ should be not greater than $\frac{4}{27}a^3$ or $4r^3$, which is obviously the case.

(ii) Solution of the cubic by Dionysodorus.

It is convenient to add here that Eutocius gives, in addition to the solution by Archimedes, two other solutions of our problem. One, by Dionysodorus, solves the cubic equation in the less general form in which it is required for Archimedes's proposition. This form, obtained from (8) above, by putting $A'M = x$, is

$$4r^2 : x^2 = (3r - x) : \frac{n}{m+n} r,$$

and the solution is obtained by drawing the parabola and



the rectangular hyperbola which we should represent by the equations

$$\frac{n}{m+n} r(3r - x) = y^2 \text{ and } \frac{n}{m+n} 2r^2 = xy,$$

referred to $A'A$ and the perpendicular to it through A as axes of x, y respectively.

(We make FA equal to OA, and draw the perpendicular AH of such a length that

$$FA : AH = CE : ED = (m+n) : n.)$$

(iii). Solution of the original problem of II. 4 by Diocles.

Diocles proceeded in a different manner, satisfying, by a geometrical construction, not the derivative cubic equation, but the three simultaneous relations which hold in Archimedes's proposition, namely

$$\left.\begin{aligned} HM : H'M &= m : n \\ HA : h &= r : h' \\ H'A' : h' &= r : h \end{aligned}\right\},$$

with the slight generalization that he substitutes for r in these equations another length a.

The problem is, given a straight line AA', a ratio $m:n$, and another straight line AK $(=a)$, to divide AA' at a point M and at the same time to find two points H, H' on AA' produced such that the above relations (with a in place of r) hold.

The analysis leading to the construction is very ingenious. Place AK $(=a)$ at right angles to AA', and draw $A'K'$ equal and parallel to it.

Suppose the problem solved, and the points M, H, H' all found.

Join KM, produce it, and complete the rectangle $KGEK'$.

Draw QMN through M parallel to AK. Produce $K'M$ to meet KG produced in F.

By similar triangles,

$$FA : AM = K'A' : A'M, \text{ or } FA : h = a : h',$$

whence $FA = AH$ (k, suppose).

Similarly $A'E = A'H'$ (k', suppose).

Again, by similar triangles,

$$(FA + AM) : (A'K' + A'M) = AM : A'M$$
$$= (AK + AM) : (EA' + A'M),$$

or $$(k+h) : (a+h') = (a+h) : (k'+h'),$$

i.e. $$(k+h)(k'+h') = (a+h)(a+h'). \quad (1)$$

Now, by hypothesis,

$$m : n = (k+h) : (k'+h')$$
$$= (k+h)(k'+h') : (k'+h')^2$$
$$= (a+h)(a+h') : (k'+h')^2 \text{ [by (1)]}. \quad (2)$$

Measure AR, $A'R'$ on AA' produced both ways equal to a. Draw RP, $R'P'$ at right angles to RR' as shown in the figure. Measure along MN the length MV equal to MA' or h', and draw PP' through V, A' to meet RP, $R'P'$.

Then $$QV = k'+h', \quad P'V = \sqrt{2}(a+h'),$$
$$PV = \sqrt{2}(a+h),$$

whence $$PV . P'V = 2(a+h)(a+h');$$

and, from (2) above,

$$2m : n = 2(a+h)(a+h') : (k'+h')^2$$
$$= PV . P'V : QV^2. \quad (3)$$

Therefore Q is on an ellipse in which PP' is a diameter, and QV is an ordinate to it.

Again, $\square\, GQNK$ is equal to $\square\, AA'K'K$, whence

$$GQ . QN = AA' . A'K' = (h+h')a = 2ra, \quad (4)$$

and therefore Q is on the rectangular hyperbola with KF, KK' as asymptotes and passing through A'.

How this ingenious analysis was suggested it is not possible to say. It is the equivalent of reducing the four unknowns h, h', k, k' to two, by putting $h = r+x$, $h' = r-x$ and $k' = y$, and then reducing the given relations to two equations in x, y, which are coordinates of a point in relation to Ox, Oy as axes, where O is the middle point of AA', and Ox lies along OA', while Oy is perpendicular to it.

Our original relations (p. 47) give

$$y = k' = \frac{ah'}{h} = a\frac{r-x}{r+x}, \quad k = \frac{ah}{h'} = a\frac{r+x}{r-x}, \quad \text{and} \quad \frac{m}{n} = \frac{h+k}{h'+k'}.$$

We have at once, from the first two equations,

$$ky = a\frac{r+x}{r-x}y = a^2,$$

whence $$(r+x)\,y = a\,(r-x),$$

and $$(x+r)\,(y+a) = 2\,ra,$$

which is the rectangular hyperbola (4) above.

Again, $$\frac{m}{n} = \frac{h+k}{h'+k'} = \frac{(r+x)\left(1+\dfrac{a}{r-x}\right)}{(r-x)\left(1+\dfrac{a}{r+x}\right)},$$

whence we obtain a cubic equation in x,

$$(r+x)^2\,(r+a-x) = \frac{m}{n}\,(r-x)^2\,(r+a+x),$$

which gives

$$\frac{m}{n}\,(r-x)^2\left(\frac{r+a+x}{r+x}\right)^2 = (r+a)^2 - x^2.$$

But $\dfrac{y}{r-x} = \dfrac{a}{r+x}$, whence $\dfrac{y+r-x}{r-x} = \dfrac{r+a+x}{r+x}$,

and the equation becomes

$$\frac{m}{n}\,(y+r-x)^2 = (r+a)^2 - x^2,$$

which is the ellipse (3) above.

To return to Archimedes. Book II of our treatise contains further problems: To find a sphere equal to a given cone or cylinder (Prop. 1), solved by reduction to the finding of two mean proportionals; to cut a sphere by a plane into two segments having their surfaces in a given ratio (Prop. 3), which is easy (by means of I. 42, 43); given two segments of spheres, to find a third segment of a sphere similar to one of the given segments and having its surface equal to that of the other (Prop. 6); the same problem with volume substituted for surface (Prop. 5), which is again reduced to the finding of two mean proportionals; from a given sphere to cut off a segment having a given ratio to the cone with the same base and equal height (Prop. 7). The Book concludes with two interesting theorems. If a sphere be cut by a plane into two segments, the greater of which has its surface equal to S and its volume equal to V, while S', V' are the surface and volume of the lesser, then $V:V' < S^2:S'^2$ but $> S^{\frac{3}{2}}:S'^{\frac{3}{2}}$ (Prop. 8): and, of all segments of spheres which have their surfaces equal, the hemisphere is the greatest in volume (Prop. 9).

Measurement of a Circle.

The book on the *Measurement of a Circle* consists of three propositions only, and is not in its original form, having lost (as the treatise *On the Sphere and Cylinder* also has) practically all trace of the Doric dialect in which Archimedes wrote; it may be only a fragment of a larger treatise. The three propositions which survive prove (1) that the area of a circle is equal to that of a right-angled triangle in which the perpendicular is equal to the radius, and the base to the circumference, of the circle, (2) that the area of a circle is to the square on its diameter as 11 to 14 (the text of this proposition is, however, unsatisfactory, and it cannot have been placed by Archimedes before Prop. 3, on which it depends), (3) *that the ratio of the circumference of any circle to its diameter* (i.e. π) *is* $< 3\frac{1}{7}$ *but* $> 3\frac{10}{71}$. Prop. 1 is proved by the method of exhaustion in Archimedes's usual form: he approximates to the area of the circle in both directions (*a*) by inscribing successive regular polygons with a number of

sides continually doubled, beginning from a square, (*b*) by circumscribing a similar set of regular polygons beginning from a square, it being shown that, if the number of the sides of these polygons be continually doubled, more than half of the portion of the polygon outside the circle will be taken away each time, so that we shall ultimately arrive at a circumscribed polygon greater than the circle by a space less than any assigned area.

Prop. 3, containing the arithmetical approximation to π, is the most interesting. The method amounts to calculating approximately the perimeter of two regular polygons of 96 sides, one of which is circumscribed, and the other inscribed, to the circle: and the calculation starts from a greater and a lesser limit to the value of $\sqrt{3}$, which Archimedes assumes without remark as known, namely

$$\tfrac{265}{153} < \sqrt{3} < \tfrac{1351}{780}.$$

How did Archimedes arrive at these particular approximations? No puzzle has exercised more fascination upon writers interested in the history of mathematics. De Lagny, Mollweide, Buzengeiger, Hauber, Zeuthen, P. Tannery, Heilermann, Hultsch, Hunrath, Wertheim, Bobynin: these are the names of some of the authors of different conjectures. The simplest supposition is certainly that of Hunrath and Hultsch, who suggested that the formula used was

$$a \pm \frac{b}{2a} > \sqrt{(a^2 \pm b)} > a \pm \frac{b}{2a \pm 1},$$

where a^2 is the nearest square number above or below $a^2 \pm b$, as the case may be. The use of the first part of this formula by Heron, who made a number of such approximations, is proved by a passage in his *Metrica*[1], where a rule equivalent to this is applied to $\sqrt{720}$; the second part of the formula is used by the Arabian Alkarkhī (eleventh century) who drew from Greek sources, and one approximation in Heron may be obtained in this way.[2] Another suggestion (that of Tannery

[1] Heron, *Metrica*, i. 8.

[2] *Stereom.* ii, p. 184. 19, Hultsch; p. 154. 19, Heib. $\sqrt{54} = 7\frac{1}{3} = 7\frac{5}{15}$ instead of $7\frac{5}{14}$.

E 2

and Zeuthen) is that the successive solutions in integers of the equations

$$\left.\begin{aligned} x^2-3y^2 &= 1 \\ x^2-3y^2 &= -2 \end{aligned}\right\}$$

may have been found in a similar way to those of the equations $x^2-2y^2 = \pm 1$ given by Theon of Smyrna after the Pythagoreans. The rest of the suggestions amount for the most part to the use of the method of continued fractions more or less disguised.

Applying the above formula, we easily find

$$2-\tfrac{1}{4} > \sqrt{3} > 2-\tfrac{1}{3},$$

or

$$\tfrac{7}{4} > \sqrt{3} > \tfrac{5}{3}.$$

Next, clearing of fractions, we consider 5 as an approximation to $\sqrt{3 \,.\, 3^2}$ or $\sqrt{27}$, and we have

$$5+\tfrac{2}{10} > 3\sqrt{3} > 5+\tfrac{2}{11},$$

whence

$$\tfrac{26}{15} > \sqrt{3} > \tfrac{19}{11}.$$

Clearing of fractions again, and taking 26 as an approximation to $\sqrt{3 \,.\, 15^2}$ or $\sqrt{675}$, we have

$$26-\tfrac{1}{52} > 15\sqrt{3} > 26-\tfrac{1}{51},$$

which reduces to

$$\tfrac{1351}{780} > \sqrt{3} > \tfrac{265}{153}.$$

Archimedes first takes the case of the circumscribed polygon. Let CA be the tangent at A to a circular arc with centre O. Make the angle AOC equal to one-third of a right angle. Bisect the angle AOC by OD, the angle AOD by OE, the angle AOE by OF, and the angle AOF by OG. Produce GA to AH, making AH equal to AG. The angle GOH is then equal to the angle FOA which is $\tfrac{1}{24}$th of a right angle, so that GH is the side of a circumscribed regular polygon with 96 sides.

Now $\qquad OA : AC\,[= \sqrt{3} : 1] > 265 : 153, \qquad (1)$

and $\qquad OC : CA = 2 : 1 = 306 : 153. \qquad (2)$

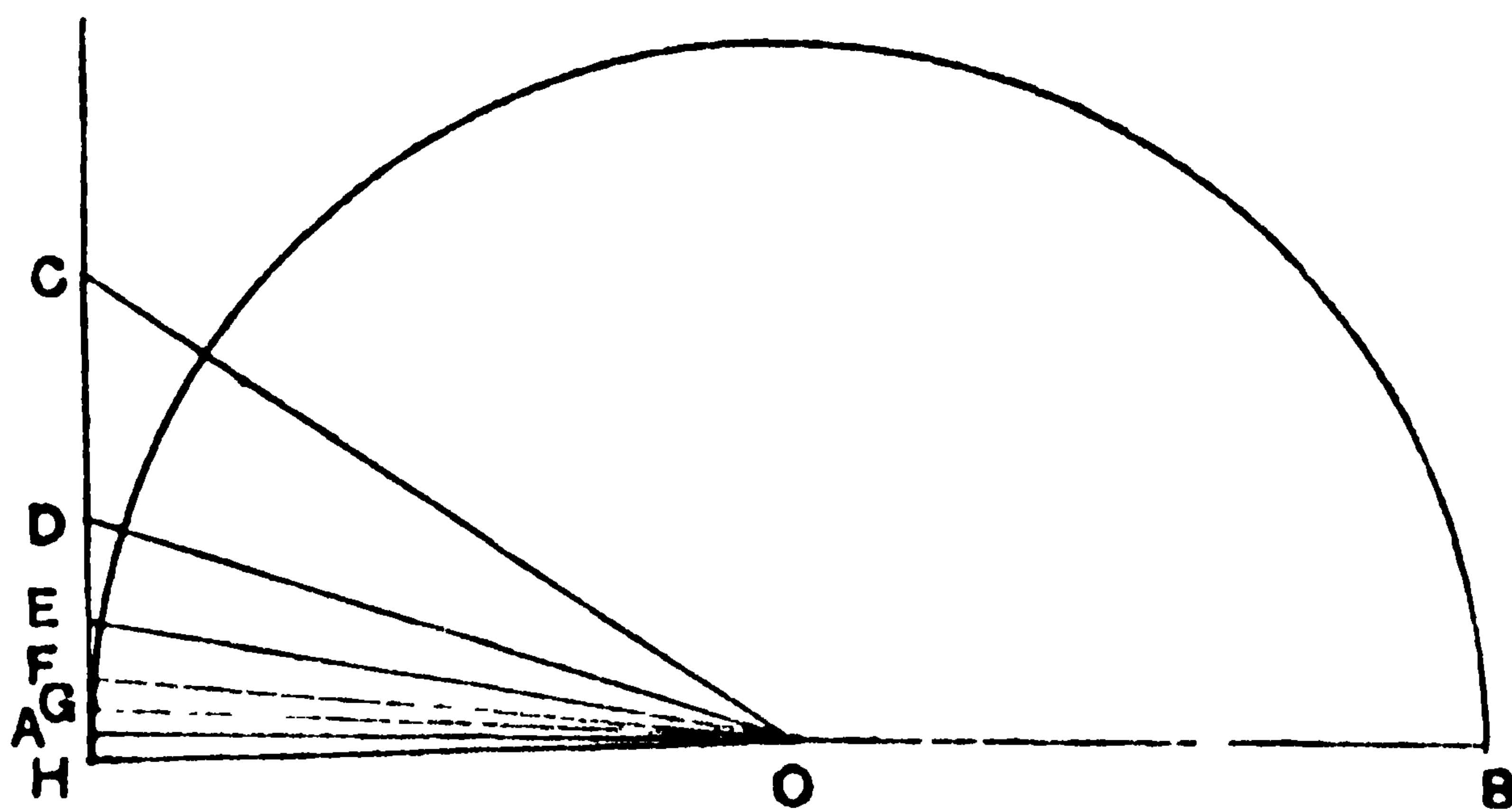
And, since OD bisects the angle COA,

$$CO : OA = CD : DA,$$

so that $$(CO + OA) : OA = CA : DA,$$

or $$(CO + OA) : CA = OA : AD.$$

Hence $$OA : AD > 571 : 153, \quad \text{by (1) and (2).}$$



And $$OD^2 : AD^2 = (OA^2 + AD^2) : AD^2$$
$$> (571^2 + 153^2) : 153^2$$
$$> 349450 : 23409.$$

Therefore, says Archimedes,

$$OD : DA > 591\tfrac{1}{8} : 153.$$

Next, just as we have found the limit of $OD : AD$ from $OC : CA$ and the limit of $OA : AC$, we find the limits of $OA : AE$ and $OE : AE$ from the limits of $OD : DA$ and $OA : AD$, and so on. This gives ultimately the limit of $OA : AG$.

Dealing with the inscribed polygon, Archimedes gets a similar series of approximations. ABC being a semicircle, the angle BAC is made equal to one-third of a right angle. Then, if the angle BAC is bisected by AD, the angle BAD by AE, the angle BAE by AF, and the angle BAF by AG, the straight line BG is the side of an inscribed polygon with 96 sides.

Now the triangles ADB, BDd, ACd are similar;

therefore $AD : DB = BD : Dd = AC : Cd$

$= AB : Bd$, since AD bisects $\angle BAC$,

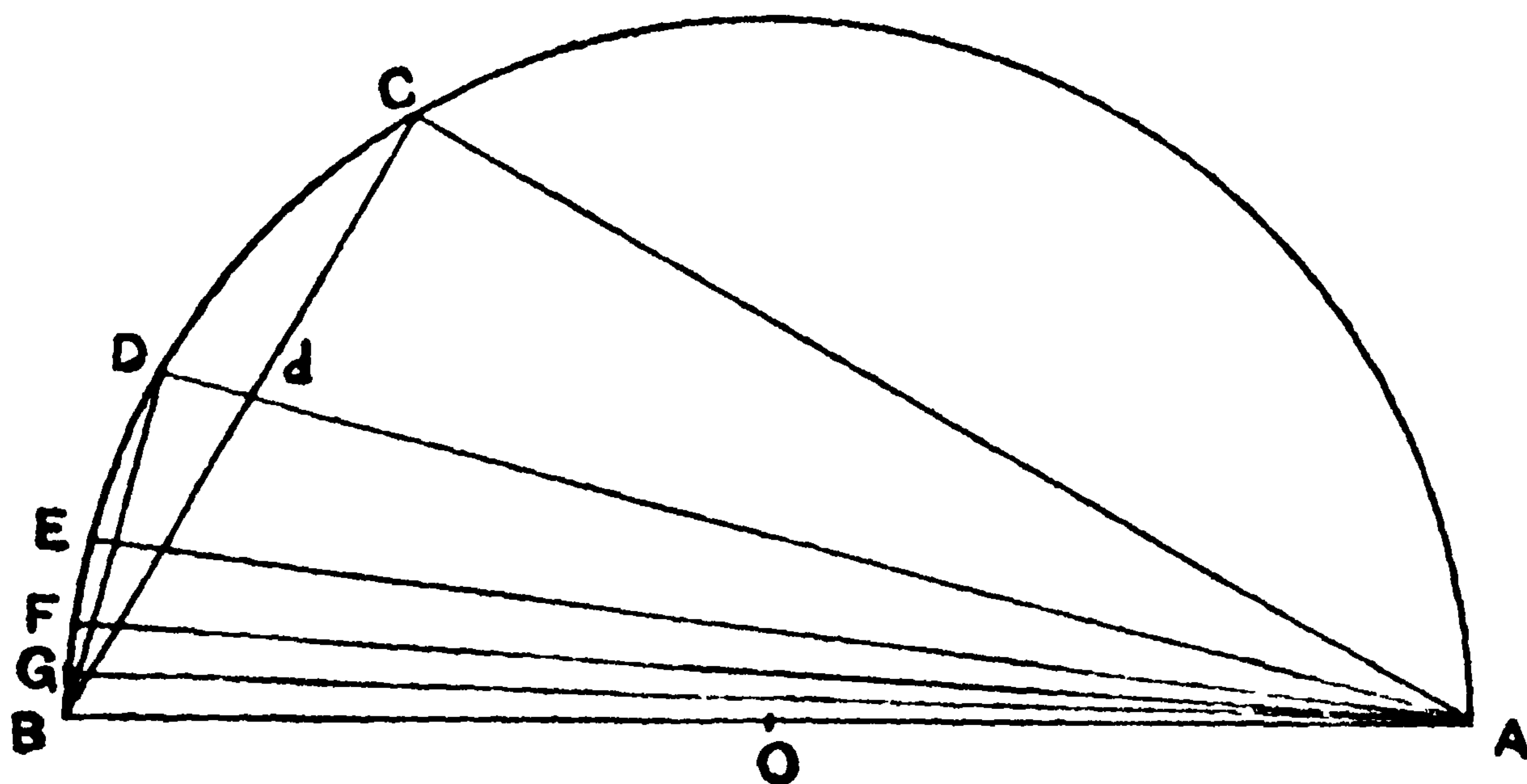
$= (AB + AC) : (Bd + Cd)$

$= (AB + AC) : BC$.

But $AC : CB < 1351 : 780$,

while $BA : BC = 2 : 1 = 1560 : 780$.

Therefore $AD : DB < 2911 : 780$.



Hence $AB^2 : BD^2 < (2911^2 + 780^2) : 780^2$

$< 9082321 : 608400$,

and, says Archimedes,

$AB : BD < 3013\frac{3}{4} : 780$.

Next, just as a limit is found for $AD : DB$ and $AB : BD$ from $AB : BC$ and the limit of $AC : CB$, so we find limits for $AE : EB$ and $AB : BE$ from the limits of $AB : BD$ and $AD : DB$, and so on, and finally we obtain the limit of $AB : BG$.

We have therefore in both cases two series of terms a_0, a_1, $a_2 \dots a_n$ and b_0, b_1, $b_2 \dots b_n$, for which the rule of formation is

$$a_1 = a_0 + b_0, \quad a_2 = a_1 + b_1, \dots,$$

where $b_1 = \sqrt{(a_1^2 + c^2)}$, $b_2 = \sqrt{(a_2^2 + c^2)} \dots$;

and in the first case

$$a_0 = 265, \quad b_0 = 306, \quad c = 153,$$

while in the second case

$$a_0 = 1351, \quad b_0 = 1560, \quad c = 780.$$

The series of values found by Archimedes are shown in the following table:

a	b	c	n	a	b	c
265	306	153	0	1351	1560	780
571	$> [\sqrt{(571^2+153^2)}]$	153	1	2911	$< \sqrt{(2911^2+780^2)}$	780
	$> 591\frac{1}{8}$				$< 3013\frac{1}{2}\frac{1}{4}$	
$1162\frac{1}{8}$	$> [\sqrt{\{(1162\frac{1}{8})^2+153^2\}}]$	153	2	$5924\frac{1}{2}\frac{1}{4}$	...	780
	$> 1172\frac{1}{8}$			1823	$< \sqrt{(1823^2+240^2)}$	240
					$< 1838\frac{9}{11}$	
$2334\frac{1}{4}$	$> [\sqrt{\{(2334\frac{1}{4})^2+153^2\}}]$	153	3	$3661\frac{9}{11}$	...	240 †
	$> 2339\frac{1}{4}$			1007	$< \sqrt{(1007^2+66^2)}$	66
					$< 1009\frac{1}{6}$	
$4673\frac{1}{2}$		153	4	$2016\frac{1}{6}$	$< \sqrt{\{(2016\frac{1}{8})^2+66^2\}}$	66
					$< 2017\frac{1}{4}$	

and, bearing in mind that in the first case the final ratio $a_4 : c$ is the ratio $OA : AG = 2\,OA : GH$, and in the second case the final ratio $b_4 : c$ is the ratio $AB : BG$, while GH in the first figure and BG in the second are the sides of regular polygons of 96 sides circumscribed and inscribed respectively, we have finally

$$\frac{96 \times 153}{4673\frac{1}{2}} > \pi > \frac{96 \times 66}{2017\frac{1}{4}}.$$

Archimedes simply infers from this that

$$3\tfrac{1}{7} > \pi > 3\tfrac{10}{71}.$$

As a matter of fact $\dfrac{96 \times 153}{4673\frac{1}{2}} = 3\dfrac{667\frac{1}{2}}{4673\frac{1}{2}}$, and $\dfrac{667\frac{1}{2}}{4672\frac{1}{2}} = \frac{1}{7}$.

It is also to be observed that $3\frac{10}{71} = 3 + \cfrac{1}{7+\cfrac{1}{10}}$, and it may have been arrived at by a method equivalent to developing the fraction $\dfrac{6336}{2017\frac{1}{4}}$ in the form of a continued fraction.

It should be noted that, in the text as we have it, the values of b_1, b_2, b_3, b_4 are simply stated in their final form without the intermediate step containing the radical except in the first

* † Here the ratios of a to c are in the first instance reduced to lower terms.

case of all, where we are told that $OD^2 : AD^2 > 349450 : 23409$ and then that $OD : DA > 591\frac{1}{8} : 153$. At the points marked * and † in the table Archimedes simplifies the ratio $a_2 : c$ and $a_3 : c$ before calculating b_2, b_3 respectively, by multiplying each term in the first case by $\frac{4}{13}$ and in the second case by $\frac{11}{40}$. He gives no explanation of the exact figure taken as the approximation to the square root in each case, or of the method by which he obtained it. We may, however, be sure that the method amounted to the use of the formula $(a \pm b)^2 = a^2 \pm 2ab + b^2$, much as our method of extracting the square root also depends upon it.

We have already seen (vol. i, p. 232) that, according to Heron, Archimedes made a still closer approximation to the value of π.

On Conoids and Spheroids.

The main problems attacked in this treatise are, in Archimedes's manner, stated in his preface addressed to Dositheus, which also sets out the premisses with regard to the solid figures in question. These premisses consist of definitions and obvious inferences from them. The figures are (1) the *right-angled conoid* (paraboloid of revolution), (2) the *obtuse-angled conoid* (hyperboloid of revolution), and (3) the *spheroids* (*a*) the *oblong*, described by the revolution of an ellipse about its 'greater diameter' (major axis), (*b*) the *flat*, described by the revolution of an ellipse about its 'lesser diameter' (minor axis). Other definitions are those of the *vertex* and *axis* of the figures or segments thereof, the vertex of a segment being the point of contact of the tangent plane to the solid which is parallel to the base of the segment. The *centre* is only recognized in the case of the spheroid; what corresponds to the centre in the case of the hyperboloid is the 'vertex of the enveloping cone' (described by the revolution of what Archimedes calls the 'nearest lines to the section of the obtuse-angled cone', i.e. the asymptotes of the hyperbola), and the line between this point and the vertex of the hyperboloid or segment is called, not the axis or diameter, but (the line) 'adjacent to the axis'. The axis of the segment is in the case of the paraboloid the line through the vertex of the segment parallel to the axis of the paraboloid, in the case

of the hyperboloid the portion within the solid of the line joining the vertex of the enveloping cone to the vertex of the segment and produced, and in the case of the spheroids the line joining the points of contact of the two tangent planes parallel to the base of the segment. Definitions are added of a 'segment of a cone' (the figure cut off towards the vertex by an elliptical, not circular, section of the cone) and a 'frustum of a cylinder' (cut off by two parallel elliptical sections).

Props. 1 to 18 with a Lemma at the beginning are preliminary to the main subject of the treatise. The Lemma and Props. 1, 2 are general propositions needed afterwards. They include propositions in summation,

$$2\{a+2a+3a+\ldots+na\} > n \cdot na > 2\{a+2a+\ldots+(n-1)a\}$$

(Lemma)

(this is clear from $S_n = \frac{1}{2}n(n+1)a$);

$$(n+1)(na)^2 + a(a+2a+3a+\ldots+na)$$
$$= 3\{a^2+(2a)^2+(3a)^2+\ldots+(na)^2\};$$

(Lemma to Prop. 2)

whence (Cor.)

$$3\{a^2+(2a)^2+(3a)^2+\ldots+(na)^2\} > n(na)^2$$
$$> 3\{a^2+(2a)^2+\ldots+(\overline{n-1}\,a)^2\};$$

lastly, Prop. 2 gives limits for the sum of n terms of the series $ax+x^2$, $a \cdot 2x+(2x)^2$, $a \cdot 3x+(3x)^2, \ldots$, in the form of inequalities of ratios, thus:

$$n\{a \cdot nx+(nx)^2\} : \Sigma_1^{n-1}\{a \cdot rx+(rx)^2\}$$
$$> (a+nx) : (\tfrac{1}{2}a+\tfrac{1}{3}nx)$$
$$> n\{a \cdot nx+(nx)^2\} : \Sigma_1^{n}\{a \cdot rx+(rx)^2\}.$$

Prop. 3 proves that, if QQ' be a chord of a parabola bisected at V by the diameter PV, then, if PV be of constant length, the areas of the triangle PQQ' and of the segment PQQ' are also constant, whatever be the direction of QQ'; to prove it Archimedes assumes a proposition 'proved in the conics' and by no means easy, namely that, if QD be perpendicular to PV, and if p, p_a be the parameters corresponding to the ordinates parallel to QQ' and the principal ordinates respectively, then

$$QV^2 : QD^2 = p : p_a.$$

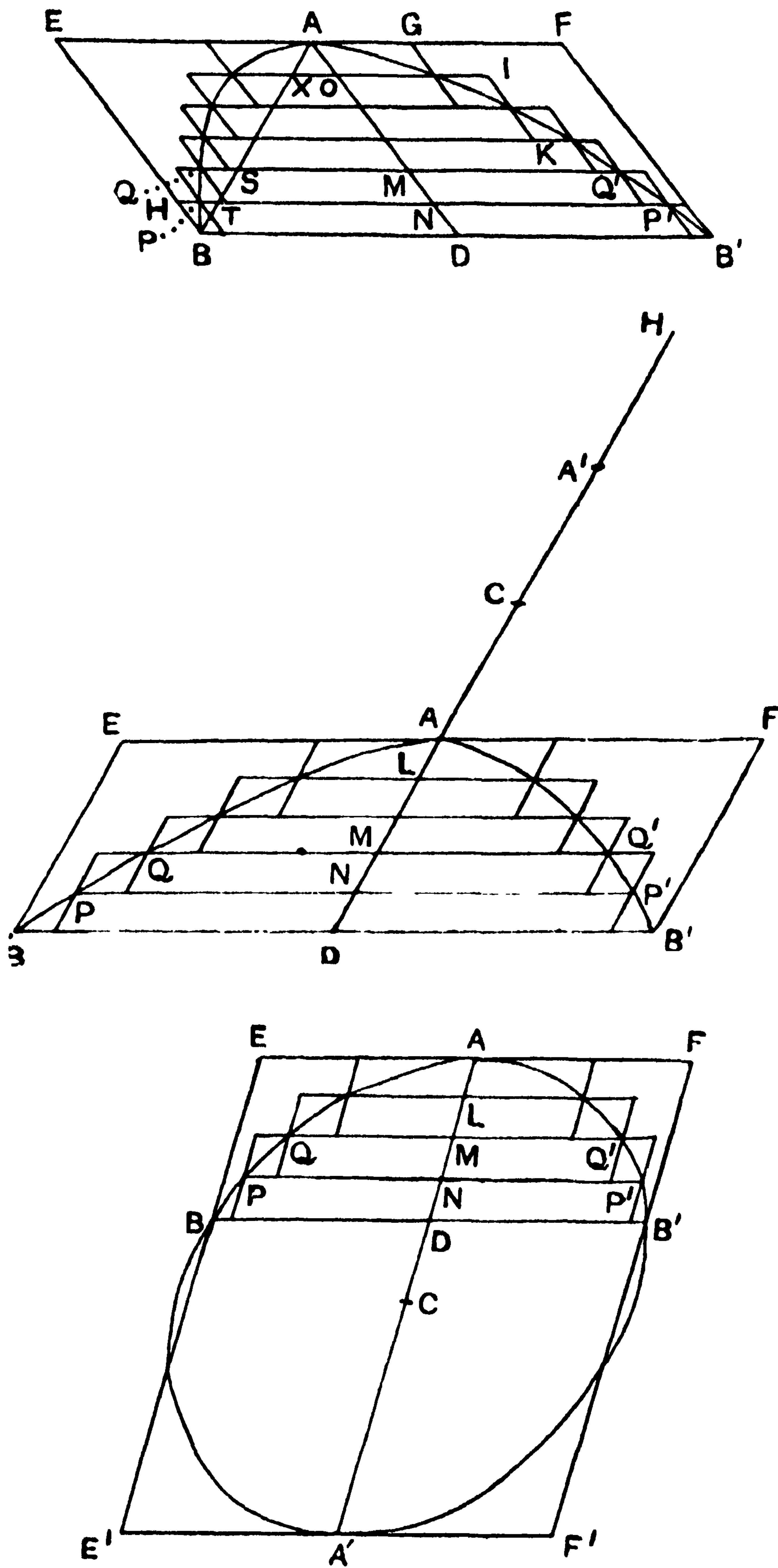
Props. 4–6 deal with the area of an ellipse, which is, in the

first of the three propositions, proved to be to the area of the auxiliary circle as the minor axis to the major; equilateral polygons of $4n$ sides are inscribed in the circle and compared with corresponding polygons inscribed in the ellipse, which are determined by the intersections with the ellipse of the double ordinates passing through the angular points of the polygons inscribed in the circle, and the method of exhaustion is then applied in the usual way. Props. 7, 8 show how, given an ellipse with centre C and a straight line CO in a plane perpendicular to that of the ellipse and passing through an axis of it, (1) in the case where OC is perpendicular to that axis, (2) in the case where it is not, we can find an (in general oblique) circular cone with vertex O such that the given ellipse is a section of it, or, in other words, how we can find the circular sections of the cone with vertex O which passes through the circumference of the ellipse; similarly Prop. 9 shows how to find the circular sections of a cylinder with CO as axis and with surface passing through the circumference of an ellipse with centre C, where CO is in the plane through an axis of the ellipse and perpendicular to its plane, but is not itself perpendicular to that axis. Props. 11–18 give simple properties of the conoids and spheroids, easily derivable from the properties of the respective conics; they explain the nature and relation of the sections made by planes cutting the solids respectively in different ways (planes through the axis, parallel to the axis, through the centre or the vertex of the enveloping cone, perpendicular to the axis, or cutting it obliquely, respectively), with especial reference to the elliptical sections of each solid, the similarity of parallel elliptical sections, &c. Then with Prop. 19 the real business of the treatise begins, namely the investigation of the volume of segments (right or oblique) of the two conoids and the spheroids respectively.

The method is, in all cases, to circumscribe and inscribe to the segment solid figures made up of cylinders or 'frusta of cylinders', which can be made to differ as little as we please from one another, so that the circumscribed and inscribed figures are, as it were, compressed together and into coincidence with the segment which is intermediate between them.

In each diagram the plane of the paper is a plane through the axis of the conoid or spheroid at right angles to the plane

of the section which is the base of the segment, and which is a circle or an ellipse according as the said base is or is not at right angles to the axis; the plane of the paper cuts the base in a diameter of the circle or an axis of the ellipse as the case may be.



The nature of the inscribed and circumscribed figures will be seen from the above figures showing segments of a paraboloid, a hyperboloid and a spheroid respectively, cut off

by planes obliquely inclined to the axis. The base of the segment is an ellipse in which BB' is an axis, and its plane is at right angles to the plane of the paper, which passes through the axis of the solid and cuts it in a parabola, a hyperbola, or an ellipse respectively. The axis of the segment is cut into a number of equal parts in each case, and planes are drawn through each point of section parallel to the base, cutting the solid in ellipses, similar to the base, in which PP', QQ', &c., are axes. Describing frusta of cylinders with axis AD and passing through these elliptical sections respectively, we draw the circumscribed and inscribed solids consisting of these frusta. It is evident that, beginning from A, the first inscribed frustum is equal to the first circumscribed frustum, the second to the second, and so on, but there is one more circumscribed frustum than inscribed, and the difference between the circumscribed and inscribed solids is equal to the *last frustum* of which BB' is the base, and ND is the axis. Since ND can be made as small as we please, the difference between the circumscribed and inscribed solids can be made less than any assigned solid whatever. Hence we have the requirements for applying the method of exhaustion.

Consider now separately the cases of the paraboloid, the hyperboloid and the spheroid.

I. The *paraboloid* (Props. 20–22).

The frustum the base of which is the ellipse in which PP' is an axis is proportional to PP'^2 or PN^2, i.e. proportional to AN. Suppose that the axis AD $(= c)$ is divided into n equal parts. Archimedes compares each frustum in the inscribed and circumscribed figure with the frustum of the whole cylinder BF cut off by the same planes. Thus

$$\begin{aligned}(\text{first frustum in } BF) : (\text{first frustum in inscribed figure}) &= BD^2 : PN^2 \\ &= AD : AN \\ &= BD : TN.\end{aligned}$$

Similarly

$$\begin{aligned}(\text{second frustum in } BF) : (\text{second in inscribed figure}) &= HN : SM,\end{aligned}$$

and so on. The last frustum in the cylinder BF has none to

correspond to it in the inscribed figure, and we should write the ratio as $(BD : \text{zero})$.

Archimedes concludes, by means of a lemma in proportions forming Prop. 1, that

$$\begin{aligned}(\text{frustum } BF) &: (\text{inscribed figure}) \\ &= (BD+HN+\ldots):(TN+SM+\ldots+XO) \\ &= n^2k:(k+2k+3k+\ldots+\overline{n-1}\,k),\end{aligned}$$

where $XO = k$, so that $BD = nk$.

In like manner, he concludes that

$$\begin{aligned}(\text{frustum } BF) &: (\text{circumscribed figure}) \\ &= n^2k:(k+2k+3k+\ldots+nk).\end{aligned}$$

But, by the Lemma preceding Prop. 1,

$$k+2k+3k+\ldots+\overline{n-1}\,k < \tfrac{1}{2}n^2k < k+2k+3k+\ldots+nk,$$

whence

$$(\text{frustum } BF):(\text{inscr. fig.}) > 2 > (\text{frustum } BF):(\text{circumscr. fig.}).$$

This indicates the desired result, which is then confirmed by the method of exhaustion, namely that

$$(\text{frustum } BF) = 2\,(\text{segment of paraboloid}),$$

or, if V be the volume of the 'segment of a cone', with vertex A and base the same as that of the segment,

$$(\text{volume of segment}) = \tfrac{3}{2}V.$$

Archimedes, it will be seen, proves in effect that, if k be indefinitely diminished, and n indefinitely increased, while nk remains equal to c, then

$$\text{limit of } k\,\{k+2k+3k+\ldots+(n-1)k\} = \tfrac{1}{2}c^2,$$

that is, in our notation,

$$\int_0^c x\,dx = \tfrac{1}{2}c^2.$$

Prop. 23 proves that the volume is constant for a given length of axis AD, whether the segment is cut off by a plane perpendicular or not perpendicular to the axis, and Prop. 24 shows that the volumes of two segments are as the squares on their axes.

II. In the case of the *hyperboloid* (Props. 25, 26) let the axis AD be divided into n parts, each of length h, and let $AA'=a$. Then the ratio of the volume of the frustum of a cylinder on the ellipse of which any double ordinate QQ' is an axis to the volume of the corresponding portion of the whole frustum BF takes a different form; for, if $AM = rh$, we have

$$\begin{aligned}(\text{frustum in } BF) : (\text{frustum on base } QQ') &= BD^2 : QM^2\\ &= AD \,.\, A'D : AM \,.\, A'M\\ &= \{a \,.\, nh + (nh)^2\} : \{a \,.\, rh + (rh)^2\}.\end{aligned}$$

By means of this relation Archimedes proves that

$$(\text{frustum } BF) : (\text{inscribed figure}) = n\{a \,.\, nh + (nh)^2\} : \Sigma_1^{n-1}\{a \,.\, rh + (rh)^2\},$$

and

$$(\text{frustum } BF) : (\text{circumscribed figure}) = n\{a \,.\, nh + (nh)^2\} : \Sigma_1^{n}\{a \,.\, rh + (rh)^2\}.$$

But, by Prop. 2,

$$\begin{aligned}n\{a \,.\, nh + (nh)^2\} : \Sigma_1^{n-1}\{a \,.\, rh + (rh)^2\} &> (a + nh) : (\tfrac{1}{2}a + \tfrac{1}{3}nh)\\ &> n\{a \,.\, nh + (nh)^2\} : \Sigma_1^{n}\{a \,.\, rh + (rh)^2\}.\end{aligned}$$

From these relations it is inferred that

$$(\text{frustum } BF) : (\text{volume of segment}) = (a + nh) : (\tfrac{1}{2}a + \tfrac{1}{3}nh),$$

or $\quad$ (volume of segment) : (volume of cone ABB')

$$= (AD + 3CA) : (AD + 2CA);$$

and this is confirmed by the method of exhaustion.

The result obtained by Archimedes is equivalent to proving that, if h be indefinitely diminished while n is indefinitely increased but nh remains always equal to b, then

$$\text{limit of } n(ab + b^2) / S_n = (a + b) / (\tfrac{1}{2}a + \tfrac{1}{3}b),$$

or

$$\text{limit of } \frac{b}{n} S_n = b^2 (\tfrac{1}{2}a + \tfrac{1}{3}b),$$

where

$$S_n = a(h + 2h + 3h + \ldots + nh) + \{h^2 + (2h)^2 + (3h)^2 + \ldots + (nh)^2\}$$

so that

$$hS_n = ah\,(h+2h+\ldots+nh)+h\,\{h^2+(2h)^2+\ldots+(nh)^2\}.$$

The limit of this latter expression is what we should write

$$\int_0^b (ax+x^2)\,dx = b^2\,(\tfrac{1}{2}a+\tfrac{1}{3}b),$$

and Archimedes's procedure is the equivalent of this integration.

III. In the case of the *spheroid* (Props. 29, 30) we take a segment less than half the spheroid.

As in the case of the hyperboloid,

$$\begin{aligned}(\text{frustum in } BF) &: (\text{frustum on base } QQ') \\ &= BD^2 : QM^2 \\ &= AD\,.\,A'D : AM\,.\,A'M;\end{aligned}$$

but, in order to reduce the summation to the same as that in Prop. 2, Archimedes expresses $AM\,.\,A'M$ in a different form equivalent to the following.

Let AD $(=b)$ be divided into n equal parts of length h, and suppose that $AA'=a$, $CD=\frac{1}{2}c$.

Then $$AD\,.\,A'D = \tfrac{1}{4}a^2-\tfrac{1}{4}c^2,$$

and $$\begin{aligned}AM\,.\,A'M &= \tfrac{1}{4}a^2-(\tfrac{1}{2}c+rh)^2 \qquad (DM=rh)\\ &= AD\,.\,A'D-\{c\,.\,rh+(rh)^2\}\\ &= cb+b^2-\{c\,.\,rh+(rh)^2\}.\end{aligned}$$

Thus in this case we have

(frustum BF) : (inscribed figure)

$$= n(cb+b^2) : [n(cb+b^2)-\Sigma_1{}^n\{c\,.\,rh+(rh)^2\}]$$

and

(frustum BF) : (circumscribed figure)

$$= n(cb+b^2) : [n(cb+b^2)-\Sigma_1{}^{n-1}\{c\,.\,rh+(rh)^2\}].$$

And, since $b=nh$, we have, by means of Prop. 2,

$$\begin{aligned}n(cb+b^2) &: [n(cb+b^2)-\Sigma_1{}^n\{c\,.\,rh+(rh)^2\}]\\ &> (c+b) : \{c+b-(\tfrac{1}{2}c+\tfrac{1}{3}b)\}\\ &> n(cb+b^2) : [n(cb+b^2)-\Sigma_1{}^{n-1}\{c\,.\,rh+(rh)^2\}].\end{aligned}$$

The conclusion, confirmed as usual by the method of exhaustion, is that

$$(\text{frustum } BF) : (\text{segment of spheroid}) = (c+b) : \{c+b-(\tfrac{1}{2}c+\tfrac{1}{3}b)\}$$
$$= (c+b) : (\tfrac{1}{2}c+\tfrac{2}{3}b),$$

whence (volume of segment) : (volume of cone ABB')

$$= (\tfrac{3}{2}c+2b) : (c+b)$$
$$= (3CA-AD) : (2CA-AD), \text{ since } CA = \tfrac{1}{2}c+b.$$

As a particular case (Props. 27, 28), half the spheroid is double of the corresponding cone.

Props. 31, 32, concluding the treatise, deduce the similar formula for the volume of the greater segment, namely, in our figure,

(greater segmt.) : (cone or segmt. of cone with same base and axis)

$$= (CA+AD) : AD.$$

On Spirals.

The treatise *On Spirals* begins with a preface addressed to Dositheus in which Archimedes mentions the death of Conon as a grievous loss to mathematics, and then summarizes the main results of the treatises *On the Sphere and Cylinder* and *On Conoids and Spheroids*, observing that the last two propositions of Book II of the former treatise took the place of two which, as originally enunciated to Dositheus, were wrong; lastly, he states the main results of the treatise *On Spirals*, premising the definition of a spiral which is as follows:

'If a straight line one extremity of which remains fixed be made to revolve at a uniform rate in a plane until it returns to the position from which it started, and if, at the same time as the straight line is revolving, a point move at a uniform rate along the straight line, starting from the fixed extremity, the point will describe a spiral in the plane.'

As usual, we have a series of propositions preliminary to the main subject, first two propositions about uniform motion,

then two simple geometrical propositions, followed by propositions (5–9) which are all of one type. Prop. 5 states that, given a circle with centre O, a tangent to it at A, and c, the

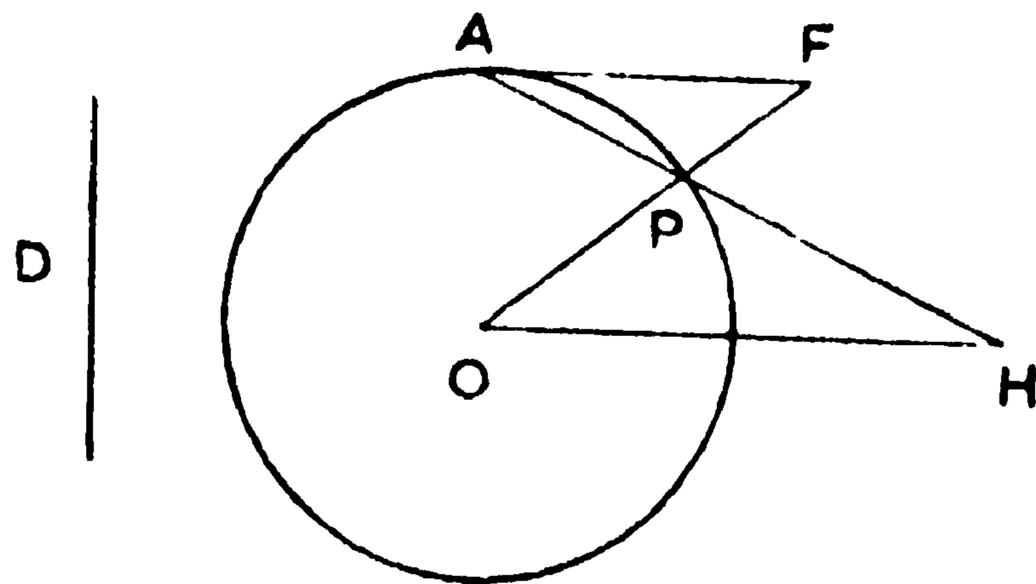


FIG. 1.

circumference of any circle whatever, it is possible to draw a straight line OPF meeting the circle in P and the tangent in F such that

$$FP : OP < (\text{arc } AP) : c.$$

Archimedes takes D a straight line greater than c, draws OH parallel to the tangent at A and then says 'let PH be placed equal to D *verging* (νεύουσα) towards A'. This is the usual phraseology of the type of problem known as νεῦσις where a straight line of given length has to be placed between two lines or curves in such a position that, if produced, it passes through a given point (this is the meaning of *verging*). Each of the propositions 5–9 depends on a νεῦσις of this kind,

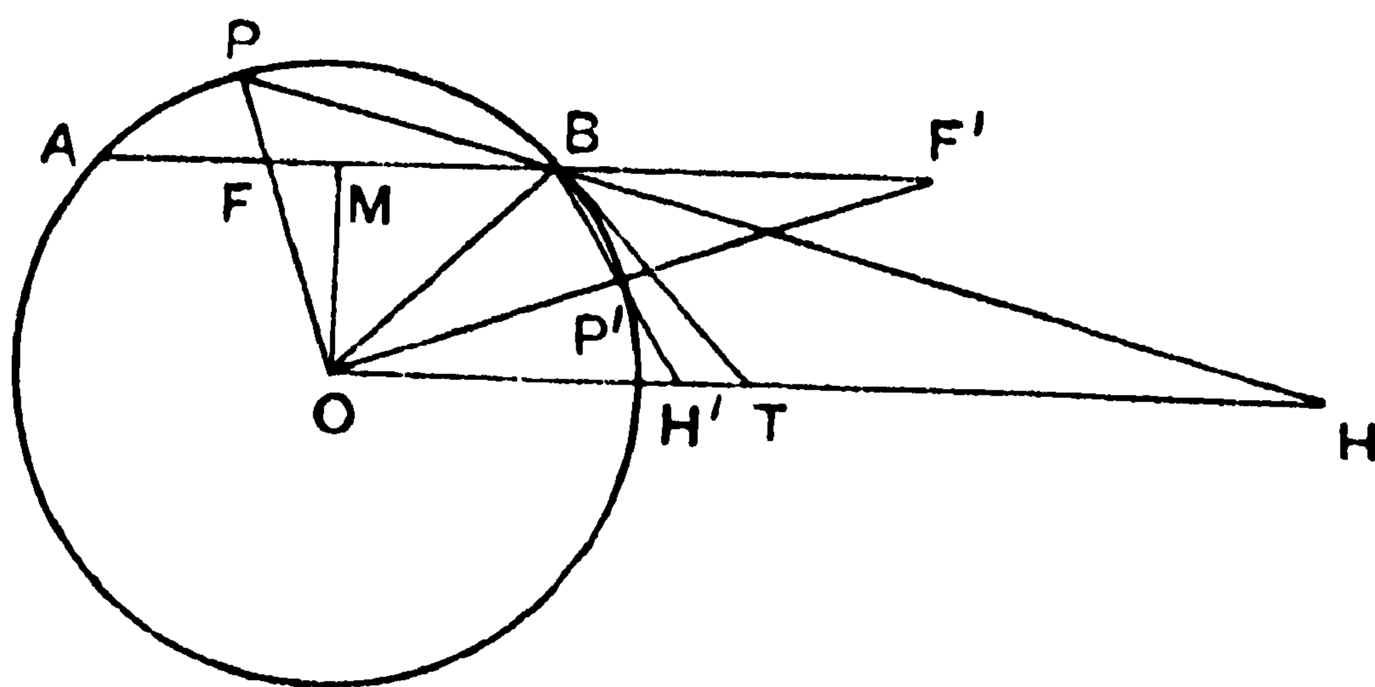


FIG. 2.

which Archimedes assumes as 'possible' without showing how it is effected. Except in the case of Prop. 5, the theoretical solution cannot be effected by means of the straight line and circle; it depends in general on the solution of an equation of the fourth degree, which can be solved by means of the

points of intersection of a certain rectangular hyperbola and a certain parabola. It is quite possible, however, that such problems were in practice often solved by a mechanical method, namely by placing a ruler, by trial, in the position of the required line: for it is only necessary to place the ruler so that it passes through the given point and then turn it round that point as a pivot till the intercept becomes of the given length. In Props. 6–9 we have a circle with centre O, a chord AB less than the diameter in it, OM the perpendicular from O on AB, BT the tangent at B, OT the straight line through O parallel to AB; $D:E$ is any ratio less or greater, as the case may be, than the ratio $BM:MO$. Props. 6, 7 (Fig. 2) show that it is possible to draw a straight line OFP

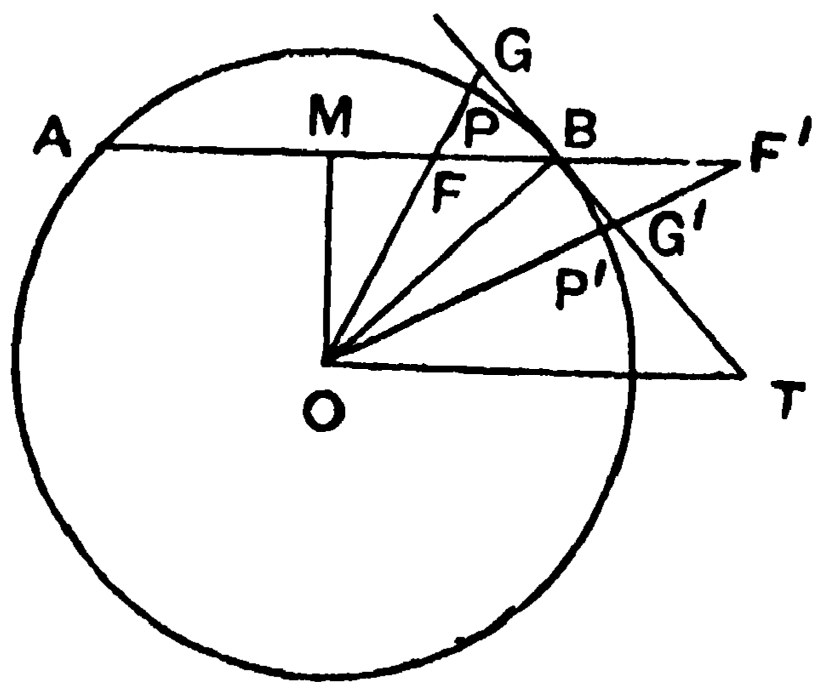


Fig. 3.

meeting AB in F and the circle in P such that $FP:PB=D:E$ (OP meeting AB in the case where $D:E < BM:MO$, and meeting AB produced when $D:E > BM:MO$). In Props. 8, 9 (Fig. 3) it is proved that it is possible to draw a straight line OFP meeting AB in F, the circle in P and the tangent at B in G, such that $FP:BG=D:E$ (OP meeting AB itself in the case where $D:E < BM:MO$, and meeting AB produced in the case where $D:E > BM:MO$).

We will illustrate by the constructions in Props. 7, 8, as it is these propositions which are actually cited later. Prop. 7. If $D:E$ is any ratio $> BM:MO$, it is required (Fig. 2) to draw $OP'F'$ meeting the circle in P' and AB produced in F' so that

$$F'P':P'B = D:E.$$

Draw OT parallel to AB, and let the tangent to the circle at B meet OT in T.

Then $D:E > BM:MO$, by hypothesis,

$> OB:BT$, by similar triangles.

Take a straight line $P'H'$ (less than BT) such that $D:E = OB:P'H'$, and place $P'H'$ between the circle and OT 'verging towards B' (construction assumed).

Then
$$F'P':P'B = OP':P'H'$$
$$= OB:P'H'$$
$$= D:E.$$

Prop. 8. If $D:E$ is any given ratio $< BM:MO$, it is required to draw $OFPG$ meeting AB in F, the circle in P, and the tangent at B to the circle in G so that

$$FP:BG = D:E.$$

If OT is parallel to AB and meets the tangent at B in T,

$$BM:MO = OB:BT, \text{ by similar triangles,}$$

whence
$$D:E < OB:BT.$$

Produce TB to C, making BC of such length that

$$D:E = OB:BC,$$

so that $BC > BT$.

Describe a circle through the three points O, T, C and let OB produced meet this circle in K.

'Then, since $BC > BT$, and OK is perpendicular to CT, it is possible to place QG [between the circle TKC and BC] equal to BK and verging towards O' (construction assumed).

Let QGO meet the original circle in P and AB in F. Then $OFPG$ is the straight line required.

For $CG \,.\, GT = OG \,.\, GQ = OG \,.\, BK.$

But $OF : OG = BT : GT$, by parallels,

whence $OF \,.\, GT = OG \,.\, BT.$

Therefore $CG \,.\, GT : OF \,.\, GT = OG \,.\, BK : OG \,.\, BT,$

whence
$$CG : OF = BK : BT$$
$$= BC : OB$$
$$= BC : OP.$$

Therefore $OP : OF = BC : CG,$

and hence $PF : OP = BG : BC,$

or $PF : BG = OB : BC = D : E.$

Pappus objects to Archimedes's use of the νεῦσις assumed in Prop. 8, 9 in these words:

'it seems to be a grave error into which geometers fall whenever any one discovers the solution of a plane problem by means of conics or linear (higher) curves, or generally solves it by means of a foreign kind, as is the case e.g. (1) with the problem in the fifth Book of the Conics of Apollonius relating to the parabola, and (2) when Archimedes assumes in his work on the spiral a νεῦσις of a "solid" character with reference to a circle; for it is possible without calling in the aid of anything solid to find the proof of the theorem given by Archimedes, that is, to prove that the circumference of the circle arrived at in the first revolution is equal to the straight line drawn at right angles to the initial line to meet the tangent to the spiral (i.e. the subtangent).'

There is, however, this excuse for Archimedes, that he only assumes that the problem *can* be solved and does not assume the actual solution. Pappus[1] himself gives a solution of the particular νεῦσις by means of conics. Apollonius wrote two Books of νεύσεις, and it is quite possible that by Archimedes's time there may already have been a collection of such problems to which tacit reference was permissible.

Prop. 10 repeats the result of the Lemma to Prop. 2 of *On*

[1] Pappus, iv, pp. 298–302.

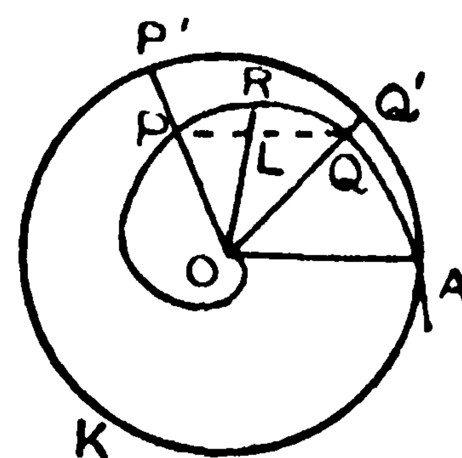
Conoids and Spheroids involving the summation of the series $1^2+2^2+3^2+\ldots+n^2$. Prop 11 proves another proposition in summation, namely that

$$(n-1)(na)^2 : \{a^2+(2a)^2+(3a)^2+\ldots+(\overline{n-1}\,a)^2\}$$
$$> (na)^2 : \{na\,.\,a+\tfrac{1}{3}(na-a)^2\}$$
$$> (n-1)(na)^2 : \{(2a)^2+(3a)^2+\ldots+(na)^2\}.$$

The same proposition is also true if the terms of the series are a^2, $(a+b)^2$, $(a+2b)^2 \ldots (a+\overline{n-1}\,b)^2$, and it is assumed in the more general form in Props. 25, 26.

Archimedes now introduces his Definitions, of the *spiral* itself, the *origin*, the *initial line*, the *first distance* (= the radius vector at the end of one revolution), the *second distance* (= the equal length added to the radius vector during the second complete revolution), and so on; the *first area* (the area bounded by the spiral described in the first revolution and the 'first distance'), the *second area* (that bounded by the spiral described in the second revolution and the 'second distance'), and so on; the *first circle* (the circle with the 'first distance' as radius), the *second circle* (the circle with radius equal to the sum of the 'first' and 'second distances', or twice the first distance), and so on.

Props. 12, 14, 15 give the fundamental property of the spiral connecting the length of the radius vector with the angle through which the initial line has revolved from its original position, and corresponding to the equation in polar coordinates $r = a\theta$. As Archimedes does not speak of angles greater than π, or 2π, he has, in the case of points on any turn after the first, to use multiples of the circumference of a circle as well as arcs of it. He uses the 'first circle' for this purpose. Thus, if P, Q are two points on the first turn,



$$OP : OQ = (\text{arc } AKP') : (\text{arc } AKQ');$$

if P, Q are points on the nth turn of the spiral, and c is the circumference of the first circle,

$$OP : OQ = \{(n-1)c+\text{arc } AKP'\} : \{(n-1)c+\text{arc } AKQ'\}.$$

Prop. 13 proves that, if a straight line touches the spiral, it

touches it at one point only. For, if possible, let the tangent at P touch the spiral at another point Q. Then, if we bisect the angle POQ by OL meeting PQ in L and the spiral in R, $OP + OQ = 2OR$ by the property of the spiral. But by the property of the triangle (assumed, but easily proved) $OP + OQ > 2OL$, so that $OL < OR$, and some point of PQ lies within the spiral. Hence PQ cuts the spiral, which is contrary to the hypothesis.

Props. 16, 17 prove that the angle made by the tangent at a point with the radius vector to that point is obtuse on the 'forward' side, and acute on the 'backward' side, of the radius vector.

Props. 18–20 give the fundamental proposition about the tangent, that is to say, they give the length of the *subtangent* at any point P (the distance between O and the point of intersection of the tangent with the perpendicular from O to OP). Archimedes always deals first with the first turn and then with any subsequent turn, and with each complete turn before parts or points of any particular turn. Thus he deals with tangents in this order, (1) the tangent at A the end of the first turn, (2) the tangent at the end of the second and any subsequent turn, (3) the tangent at any intermediate point of the first or any subsequent turn. We will take as illustrative the case of the tangent at any intermediate point P of the first turn (Prop. 20).

If OA be the initial line, P any point on the first turn, PT the tangent at P and OT perpendicular to OP, then it is to be proved that, if ASP be the circle through P with centre O, meeting PT in S, then

$$(\text{subtangent } OT) = (\text{arc } ASP).$$

I. If possible, let OT be greater than the arc ASP.

Measure off OU such that $OU >$ arc ASP but $< OT$.

Then the ratio $PO : OU$ is greater than the ratio $PO : OT$, i.e. greater than the ratio of $\frac{1}{2}PS$ to the perpendicular from O on PS.

Therefore (Prop. 7) we can draw a straight line OQF meeting TP produced in F, and the circle in Q, such that

$$FQ : PQ = PO : OU.$$

Let OF meet the spiral in Q'.

Then we have, *alternando*, since $PO = QO$,

$$FQ : QO = PQ : OU$$

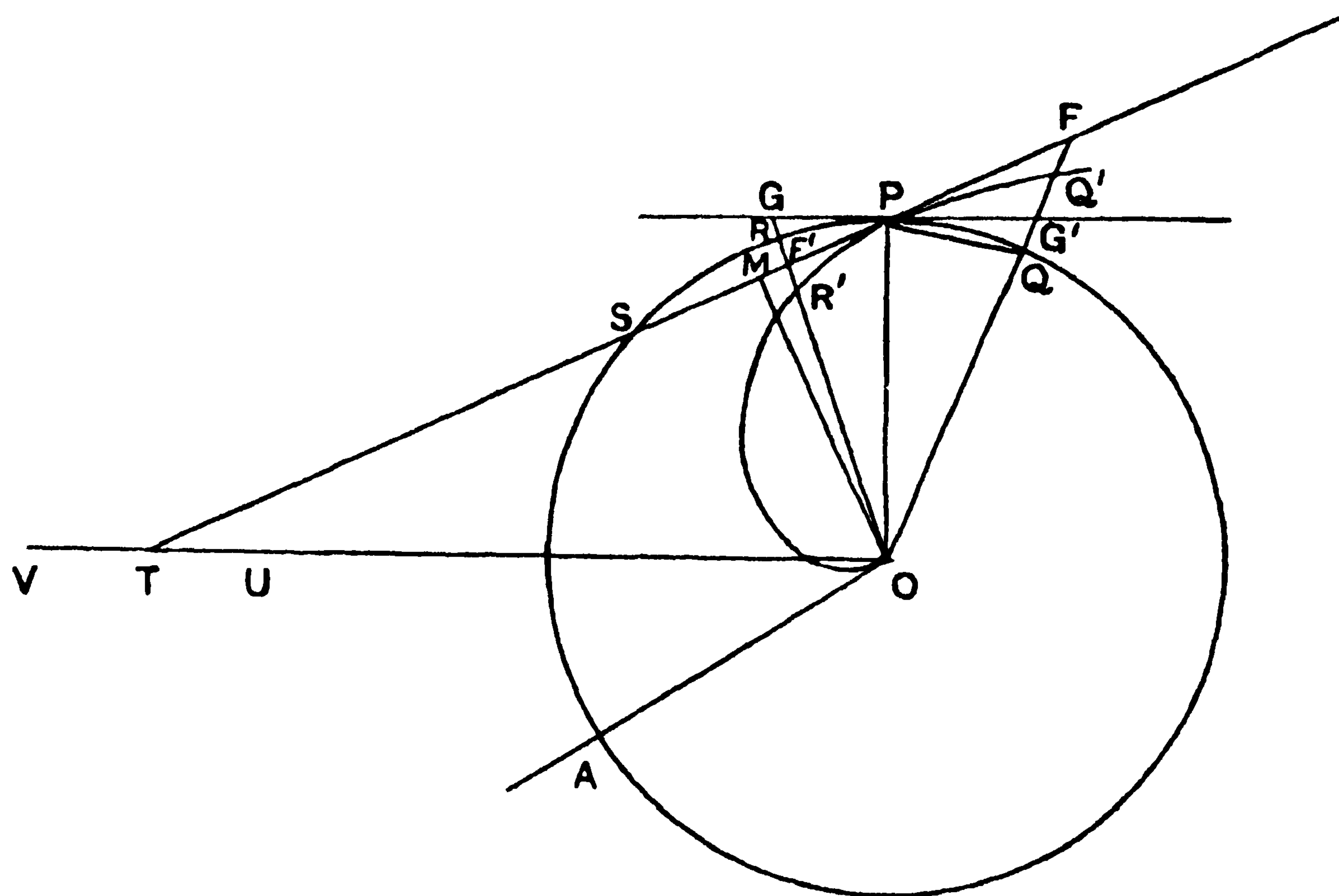
$$< (\text{arc } PQ) : (\text{arc } ASP), \text{ by hypothesis and } a\ fortiori.$$

Componendo, $FO : QO < (\text{arc } ASQ) : (\text{arc } ASP)$

$$< OQ' : OP.$$

But $QO = OP$; therefore $FO < OQ'$; which is impossible.

Therefore OT is not greater than the arc ASP.



II. Next suppose, if possible, that $OT <$ arc ASP.

Measure OV along OT such that OV is greater than OT but less than the arc ASP.

Then the ratio $PO : OV$ is less than the ratio $PO : OT$, i.e. than the ratio of $\frac{1}{2}PS$ to the perpendicular from O on PS; therefore it is possible (Prop. 8) to draw a straight line $OF'RG$ meeting PS, the circle PSA, and the tangent to the circle at P in F', R, G respectively, and such that

$$F'R : GP = PO : OV.$$

Let $OF'G$ meet the spiral in R'.

Then, since $PO = RO$, we have, *alternando*,

$$F'R : RO = GP : OV$$

$$> (\text{arc } PR) : (\text{arc } ASP), \text{ *a fortiori*,}$$

whence $$F'O : RO < (\text{arc } ASR) : (\text{arc } ASP)$$

$$< OR' : OP,$$

so that $F'O < OR'$; which is impossible.

Therefore OT is not less than the arc ASP. And it was proved not greater than the same arc. Therefore

$$OT = (\text{arc } ASP).$$

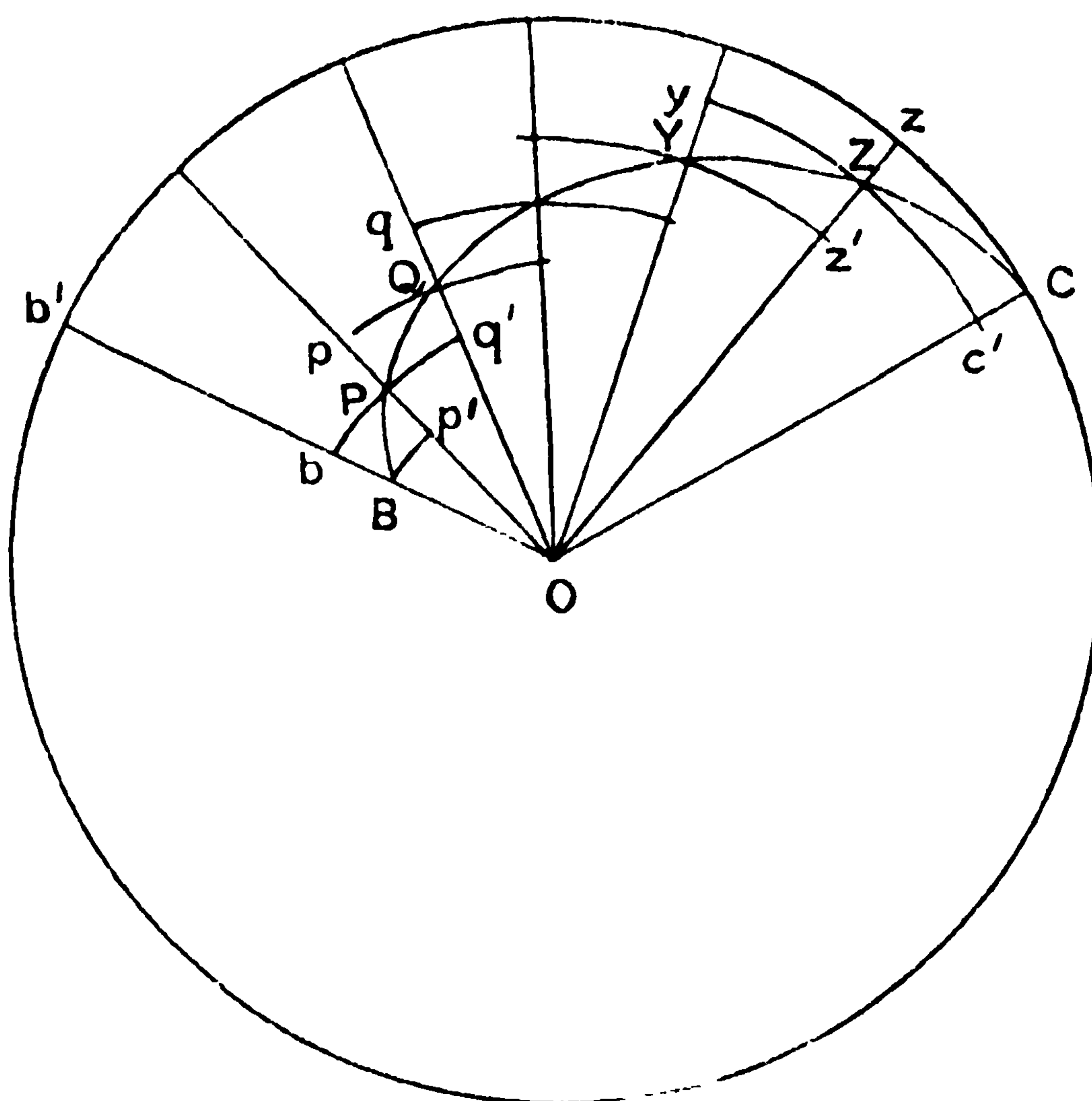
As particular cases (separately proved by Archimedes), if P be the extremity of the first turn and c_1 the circumference of the first circle, the subtangent $= c_1$; if P be the extremity of the second turn and c_2 the circumference of the 'second circle', the subtangent $= 2c_2$; and generally, if c_n be the circumference of the nth circle (the circle with the radius vector to the extremity of the nth turn as radius), the subtangent to the tangent at the extremity of the nth turn $= nc_n$.

If P is a point on the nth turn, not the extremity, and the circle with O as centre and OP as radius cuts the initial line in K, while p is the circumference of the circle, the subtangent to the tangent at $P = (n-1)p + \text{arc } KP$ (measured 'forward').[1]

The remainder of the book (Props. 21–8) is devoted to finding the areas of portions of the spiral and its several turns cut off by the initial line or any two radii vectores. We will illustrate by the general case (Prop. 26). Take OB, OC, two bounding radii vectores, including an arc BC of the spiral. With centre O and radius OC describe a circle. Divide the angle BOC into any number of equal parts by radii of this circle. The spiral meets these radii in points $P, Q \ldots Y, Z$ such that the radii vectores $OB, OP, OQ \ldots OZ, OC$

[1] On the whole course of Archimedes's proof of the property of the subtangent, see note in the Appendix.

are in arithmetical progression. Draw arcs of circles with radii OB, OP, OQ ... as shown; this produces a figure circumscribed to the spiral and consisting of the sum of small sectors of circles, and an inscribed figure of the same kind. As the first sector in the circumscribed figure is equal to the second sector in the inscribed, it is easily seen that the areas of the circumscribed and inscribed figures differ by the difference between the sectors OzC and OBp'; therefore, by increasing the number of divisions of the angle BOC, we can make the



difference between the areas of the circumscribed and inscribed figures as small as we please; we have, therefore, the elements necessary for the application of the method of exhaustion.

If there are n radii OB, OP ... OC, there are $(n-1)$ parts of the angle BOC. Since the angles of all the small sectors are equal, the sectors are as the square on their radii.

Thus (whole sector $Ob'C$) : (circumscribed figure)

$$= (n-1)OC^2 : (OP^2 + OQ^2 + \ldots + OC^2),$$

and (whole sector $Ob'C$) : (inscribed figure)

$$= (n-1)OC^2 : (OB^2 + OP^2 + OQ^2 + \ldots + OZ^2).$$

And $OB, OP, OQ, \dots OZ, OC$ is an arithmetical progression of n terms; therefore (cf. Prop. 11 and Cor.),

$$(n-1)OC^2 : (OP^2 + OQ^2 + \dots + OC^2)$$
$$< OC^2 : \{OC \cdot OB + \tfrac{1}{3}(OC - OB)^2\}$$
$$< (n-1)OC^2 : (OB^2 + OP^2 + \dots + OZ^2).$$

Compressing the circumscribed and inscribed figures together in the usual way, Archimedes proves by exhaustion that

$$(\text{sector } Ob'C) : (\text{area of spiral } OBC)$$
$$= OC^2 : \{OC \cdot OB + \tfrac{1}{3}(OC - OB)^2\}.$$

If $OB = b$, $OC = c$, and $(c-b) = (n-1)h$, Archimedes's result is the equivalent of saying that, when h diminishes and n increases indefinitely, while $c-b$ remains constant,

$$\text{limit of } h\{b^2 + (b+h)^2 + (b+2h)^2 + \dots + (b+\overline{n-2}h)^2\}$$
$$= (c-b)\{cb + \tfrac{1}{3}(c-b)^2\}$$
$$= \tfrac{1}{3}(c^3 - b^3);$$

that is, with our notation,

$$\int_b^c x^2 dx = \tfrac{1}{3}(c^3 - b^3).$$

In particular, the area included by the first turn and the initial line is bounded by the radii vectores 0 and $2\pi a$; the area, therefore, is to the circle with radius $2\pi a$ as $\frac{1}{3}(2\pi a)^2$ to $(2\pi a)^2$, that is to say, it is $\frac{1}{3}$ of the circle or $\frac{1}{3}\pi(2\pi a)^2$. This is separately proved in Prop. 24 by means of Prop. 10 and Corr. 1, 2.

The area of the ring added while the radius vector describes the second turn is the area bounded by the radii vectores $2\pi a$ and $4\pi a$, and is to the circle with radius $4\pi a$ in the ratio of $\{r_2 r_1 + \frac{1}{3}(r_2 - r_1)^2\}$ to r_2^2, where $r_1 = 2\pi a$ and $r_2 = 4\pi a$; the ratio is $7 : 12$ (Prop. 25).

If R_1 be the area of the first turn of the spiral bounded by the initial line, R_2 the area of the ring added by the second complete turn, R_3 that of the ring added by the third turn, and so on, then (Prop. 27)

$$R_3 = 2R_2, \quad R_4 = 3R_2, \quad R_5 = 4R_2, \dots R_n = (n-1)R_2.$$

Also $R_2 = 6R_1$.

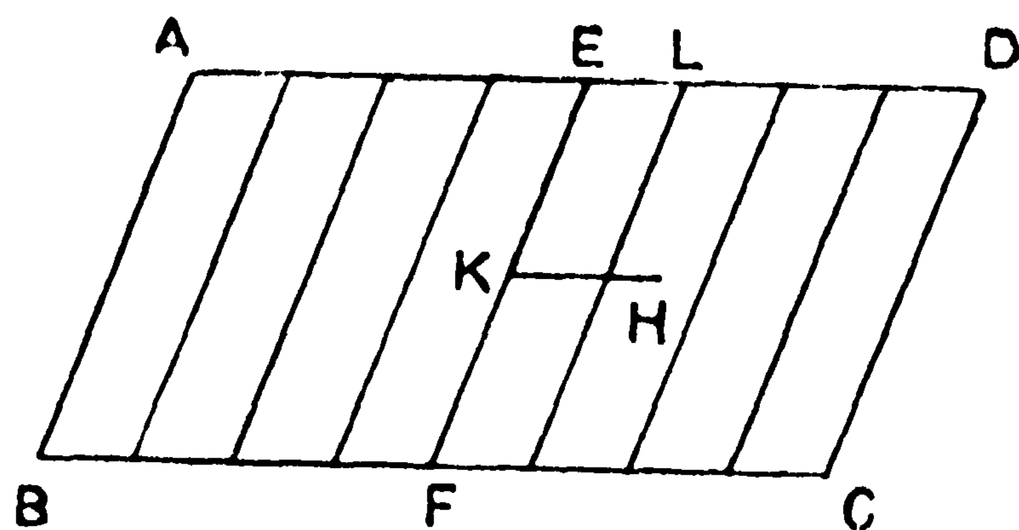
Lastly, if E be the portion of the sector $b'OC$ bounded by $b'B$, the arc $b'zC$ of the circle and the arc BC of the spiral, and F the portion cut off between the arc BC of the spiral, the radius OC and the arc intercepted between OB and OC of the circle with centre O and radius OB, it is proved that

$$E : F = \{OB + \tfrac{2}{3}(OC - OB)\} : \{OB + \tfrac{1}{3}(OC - OB)\} \text{ (Prop. 28).}$$

On Plane Equilibriums, I, II.

In this treatise we have the fundamental principles of mechanics established by the methods of geometry in its strictest sense. There were doubtless earlier treatises on mechanics, but it may be assumed that none of them had been worked out with such geometrical rigour. Archimedes begins with seven Postulates including the following principles. Equal weights at equal distances balance; if unequal weights operate at equal distances, the larger weighs down the smaller. If when equal weights are in equilibrium something be added to, or subtracted from, one of them, equilibrium is not maintained but the weight which is increased or is not diminished prevails. When equal and similar plane figures coincide if applied to one another, their centres of gravity similarly coincide; and in figures which are unequal but similar the centres of gravity will be 'similarly situated'. In any figure the contour of which is concave in one and the same direction the centre of gravity must be within the figure. Simple propositions (1–5) follow, deduced by *reductio ad absurdum*; these lead to the fundamental theorem, proved first for commensurable and then by *reductio ad absurdum* for incommensurable magnitudes, that *Two magnitudes, whether commensurable or incommensurable, balance at distances reciprocally proportional to the magnitudes* (Props. 6, 7). Prop. 8 shows how to find the centre of gravity of a part of a magnitude when the centres of gravity of the other part and of the whole magnitude are given. Archimedes then addresses himself to the main problems of Book I, namely to find the centres of gravity of (1) a parallelogram (Props. 9, 10), (2) a triangle (Props. 13, 14), and (3) a parallel-trapezium (Prop. 15), and here we have an illustration of the extraordinary rigour which he requires in his geometrical

proofs. We do not find him here assuming, as in *The Method*, that, if all the lines that can be drawn in a figure parallel to (and including) one side have their middle points in a straight line, the centre of gravity must lie somewhere on that straight line; he is not content to regard the figure as *made up* of an infinity of such parallel lines; pure geometry realizes that the parallelogram is made up of elementary parallelograms, indefinitely narrow if you please, but still parallelograms, and the triangle of elementary *trapezia*, not straight lines, so that to assume directly that the centre of gravity lies on the straight line bisecting the parallelograms would really be a *petitio principii*. Accordingly the result, no doubt discovered in the informal way, is clinched by a proof by *reductio ad absurdum* in each case. In the case of the parallelogram *ABCD* (Prop. 9), if the centre of gravity is not on the straight line *EF* bisecting two opposite sides, let it be at *H*. Draw *HK* parallel to *AD*. Then it is possible by bisecting *AE*, *ED*, then bisecting the halves, and so on, ultimately to reach a length less than *KH*. Let this be done, and through the



points of division of *AD* draw parallels to *AB* or *DC* making a number of equal and similar parallelograms as in the figure. The centre of gravity of each of these parallelograms is similarly situated with regard to it. Hence we have a number of equal magnitudes with their centres of gravity at equal distances along a straight line. Therefore the centre of gravity of the whole is on the line joining the centres of gravity of the two middle parallelograms (Prop. 5, Cor. 2). But this is impossible, because *H* is outside those parallelograms. Therefore the centre of gravity cannot but lie on *EF*.

Similarly the centre of gravity lies on the straight line bisecting the other opposite sides *AB*, *CD*; therefore it lies at the intersection of this line with *EF*, i.e. at the point of intersection of the diagonals.

The proof in the case of the triangle is similar (Prop. 13). Let AD be the median through A. The centre of gravity must lie on AD.

For, if not, let it be at H, and draw HI parallel to BC. Then, if we bisect DC, then bisect the halves, and so on, we shall arrive at a length DE less than IH. Divide BC into lengths equal to DE, draw parallels to DA through the points of division, and complete the small parallelograms as shown in the figure.

The centres of gravity of the whole parallelograms SN, TP, FQ lie on AD (Prop. 9); therefore the centre of gravity of the

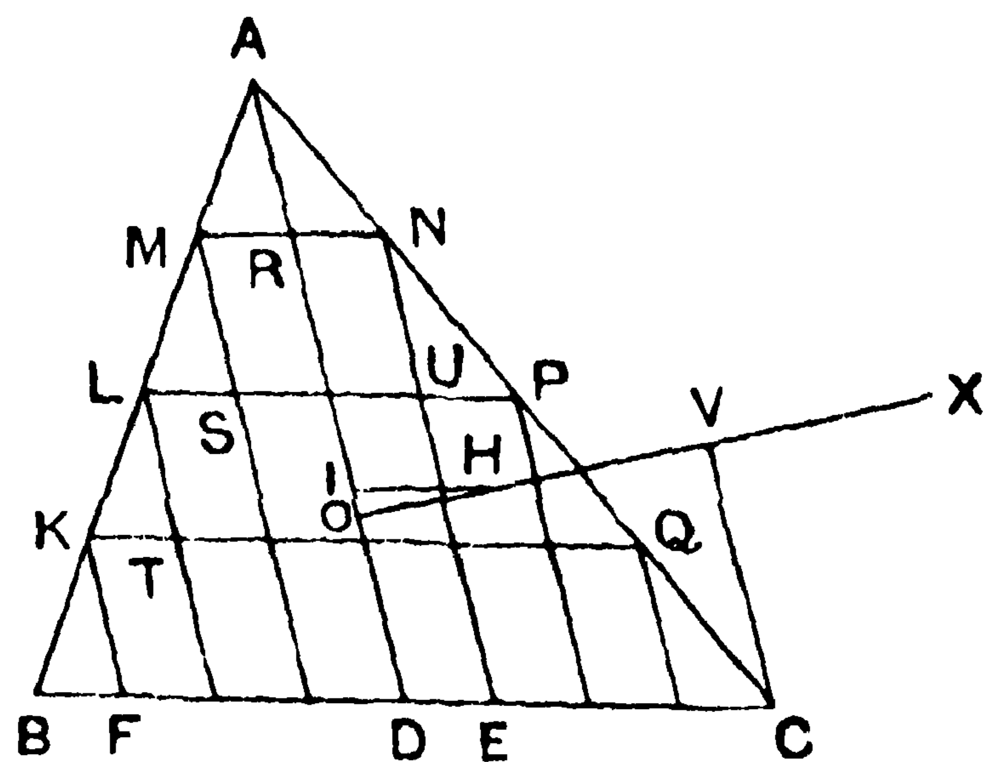


figure formed by them all lies on AD; let it be O. Join OH, and produce it to meet in V the parallel through C to AD.

Now it is easy to see that, if n be the number of parts into which DC, AC are divided respectively,

$$\begin{aligned}(\text{sum of small } \Delta\text{s } AMR,\ MLS \ldots ARN,\ NUP \ldots) : (\Delta ABC) &= n \,.\, AN^2 : AC^2 \\ &= 1 : n\,;\end{aligned}$$

whence

$$(\text{sum of small } \Delta\text{s}) : (\text{sum of parallelograms}) = 1 : (n-1).$$

Therefore the centre of gravity of the figure made up of all the small triangles is at a point X on OH produced such that

$$XH = (n-1)\,OH.$$

But $VH : HO < CE : ED$ or $(n-1) : 1$; therefore $XH > VH$.

It follows that the centre of gravity of all the small triangles taken together lies at X notwithstanding that all the triangles lie on one side of the parallel to AD drawn through X: which is impossible.

Hence the centre of gravity of the whole triangle cannot but lie on AD.

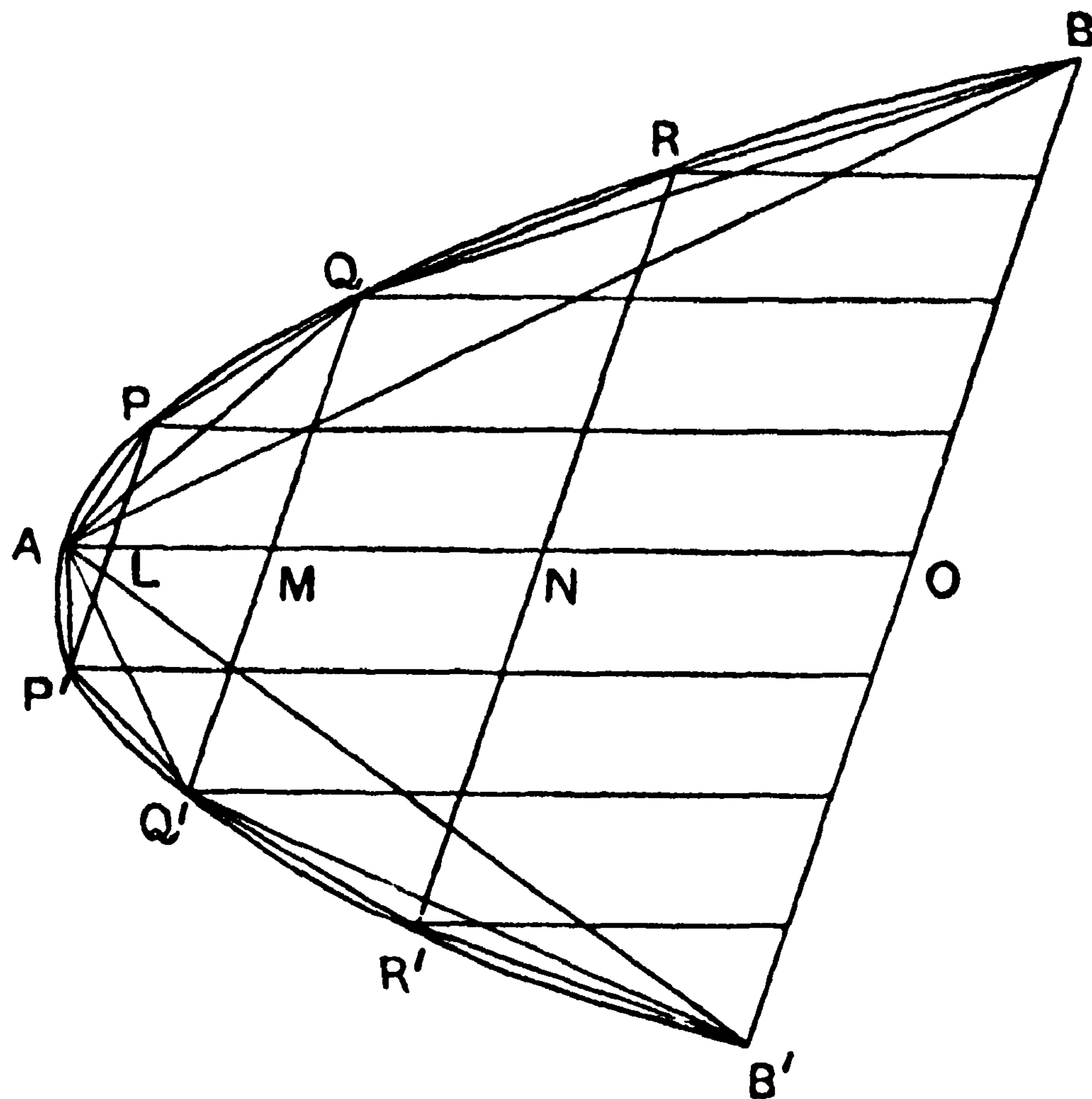
It lies, similarly, on either of the other two medians; so that it is at the intersection of any two medians (Prop. 14).

Archimedes gives alternative proofs of a direct character, both for the parallelogram and the triangle, depending on the postulate that the centres of gravity of similar figures are 'similarly situated' in regard to them (Prop. 10 for the parallelogram, Props. 11, 12 and part 2 of Prop. 13 for the triangle).

The geometry of Prop. 15 deducing the centre of gravity of a trapezium is also interesting. It is proved that, if AD, BC are the parallel sides (AD being the smaller), and EF is the straight line joining their middle points, the centre of gravity is at a point G on EF such that

$$GE : GF = (2\,BC + AD) : (2\,AD + BC).$$

Book II of the treatise is entirely devoted to finding the centres of gravity of a parabolic segment (Props. 1–8) and of a portion of it cut off by a parallel to the base (Props. 9, 10). Prop. 1 (really a particular case of I. 6, 7) proves that, if P, P'



be the areas of two parabolic segments and D, E their centres of gravity, the centre of gravity of both taken together is at a point C on DE such that

$$P : P' = CE : CD.$$

This is merely preliminary. Then begins the real argument, the course of which is characteristic and deserves to be set out. Archimedes uses a series of figures inscribed to the segment, as he says, 'in the recognized manner' (γνωρίμως). The rule is as follows. Inscribe in the segment the triangle ABB' with the same base and height; the vertex A is then the point of contact of the tangent parallel to BB'. Do the same with the remaining segments cut off by AB, AB', then with the segments remaining, and so on. If $BRQPAP'Q'R'B'$ is such a figure, the diameters through Q, Q', P, P', R, R' bisect the straight lines AB, AB', AQ, AQ', QB, $Q'B'$ respectively, and BB' is divided by the diameters into parts which are all equal. It is easy to prove also that PP', QQ', RR' are all parallel to BB', and that $AL:LM:MN:NO = 1:3:5:7$, the same relation holding if the number of sides of the polygon is increased; i.e. the segments of AO are always in the ratio of the successive odd numbers (Lemmas to Prop. 2). The centre of gravity of the inscribed figure lies on AO (Prop. 2). If there be two parabolic segments, and two figures inscribed in them 'in the recognized manner' with an equal number of sides, the centres of gravity divide the respective axes in the same proportion, for the ratio depends on the same ratio of odd numbers $1:3:5:7$ ... (Prop. 3). The centre of gravity of the parabolic segment itself lies on the diameter AO (this is proved in Prop. 4 by *reductio ad absurdum* in exactly the same way as for the triangle in I. 13). It is next proved (Prop. 5) that the centre of gravity of the segment is nearer to the vertex A than the centre of gravity of the inscribed figure is; but that it is possible to inscribe in the segment in the recognized manner a figure such that the distance between the centres of gravity of the segment and of the inscribed figure is less than any assigned length, for we have only to increase the number of sides sufficiently (Prop. 6). Incidentally, it is observed in Prop. 4 that, if in any segment the triangle with the same base and equal height is inscribed, the triangle is greater than half the segment, whence it follows that, each time we increase the number of sides in the inscribed figure, we take away more than half of the segments remaining over; and in Prop. 5 that corresponding segments on opposite sides of the axis, e.g. QRB, $Q'R'B'$ have their axes equal and therefore are equal in

area. Lastly (Prop. 7), if there be two parabolic segments, their centres of gravity divide their diameters in the same ratio (Archimedes enunciates this of similar segments only, but it is true of any two segments and is required of any two segments in Prop. 8). Prop. 8 now finds the centre of gravity of any segment by using the last proposition. It is the geometrical equivalent of the solution of a simple equation in the ratio (m, say) of AG to AO, where G is the centre of gravity of the segment.

Since the segment $= \frac{4}{3}(\Delta ABB')$, the sum of the two segments AQB, $AQ'B' = \frac{1}{3}(\Delta ABB')$.

Further, if QD, $Q'D'$ are the diameters of these segments, QD, $Q'D'$ are equal, and, since the centres of gravity H, H' of the segments divide QD, $Q'D'$ proportionally, HH' is parallel to QQ', and the centre of gravity of the two segments together is at K, the point where HH' meets AO.

Now $AO = 4AV$ (Lemma 3 to Prop. 2), and $QD = \frac{1}{2}AO - AV = AV$. But H divides QD in the same ratio as G divides AO (Prop. 7); therefore

$$VK = QH = m.QD = m.AV.$$

Taking moments about A of the segment, the triangle ABB' and the sum of the small segments, we have (dividing out by AV and $\Delta ABB'$)

$$\tfrac{1}{3}(1+m)+\tfrac{2}{3}.4 = \tfrac{4}{3}.4m,$$

or

$$15m = 9,$$

and $m = \frac{3}{5}$.

That is, $AG = \frac{3}{5}AO$, or $AG : GO = 3 : 2$.

The final proposition (10) finds the centre of gravity of the portion of a parabola cut off between two parallel chords PP', BB'. If PP' is the shorter of the chords and the diameter bisecting PP', BB' meets them in N, O respectively, Archimedes proves that, if NO be divided into five equal parts of which LM is the middle one (L being nearer to N than M is),

the centre of gravity G of the portion of the parabola between PP' and BB' divides LM in such a way that

$$LG : GM = BO^2 . (2PN + BO) : PN^2 . (2BO + PN).$$

The geometrical proof is somewhat difficult, and uses a very remarkable Lemma which forms Prop. 9. If a, b, c, d, x, y are straight lines satisfying the conditions

$$\left.\begin{aligned} &\frac{a}{b} = \frac{b}{c} = \frac{c}{d}\ (a > b > c > d),\\ &\frac{d}{a-d} = \frac{x}{\frac{3}{5}(a-c)},\\ \text{and}\quad &\frac{2a+4b+6c+3d}{5a+10b+10c+5d} = \frac{y}{a-c},\\ \text{then must}\quad &x+y = \tfrac{2}{5}a. \end{aligned}\right\}$$

The proof is entirely geometrical, but amounts of course to the elimination of three quantities b, c, d from the above four equations.

The Sand-reckoner (*Psammites* or *Arenarius*).

I have already described in a previous chapter the remarkable system, explained in this treatise and in a lost work, *Ἀρχαί*, *Principles*, addressed to Zeuxippus, for expressing very large numbers which were beyond the range of the ordinary Greek arithmetical notation. Archimedes showed that his system would enable any number to be expressed up to that which in our notation would require 80,000 million million ciphers and then proceeded to prove that this system more than sufficed to express the number of grains of sand which it would take to fill the universe, on a reasonable view (as it seemed to him) of the size to be attributed to the universe. Interesting as the book is for the course of the argument by which Archimedes establishes this, it is, in addition, a document of the first importance historically. It is here that we learn that Aristarchus put forward the Copernican theory of the universe, with the sun in the centre and the planets including the earth revolving round it, and that Aristarchus further discovered the angular diameter of the sun to be $\frac{1}{720}$th of the circle of the zodiac or half a degree. Since Archimedes, in order to calculate a safe figure (not too small) for the size

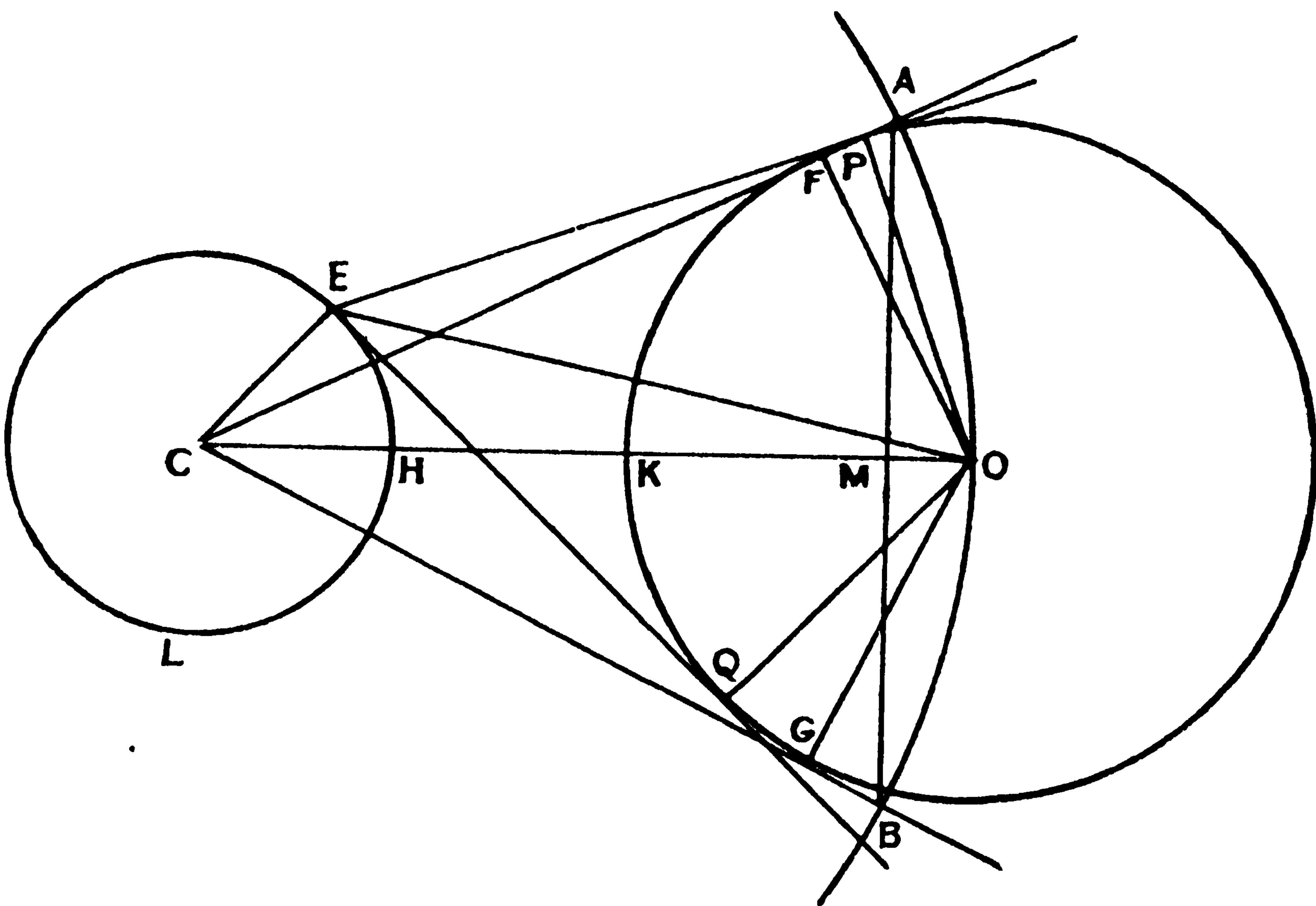
of the universe, has to make certain assumptions as to the sizes and distances of the sun and moon and their relation to the size of the universe, he takes the opportunity of quoting earlier views. Some have tried, he says, to prove that the perimeter of the earth is about 300,000 stades; in order to be quite safe he will take it to be about ten times this, or 3,000,000 stades, and not greater. The diameter of the earth, like most earlier astronomers, he takes to be greater than that of the moon but less than that of the sun. Eudoxus, he says, declared the diameter of the sun to be nine times that of the moon, Phidias, his own father, twelve times, while Aristarchus tried to prove that it is greater than 18 but less than 20 times the diameter of the moon; he will again be on the safe side and take it to be 30 times, but not more. The position is rather more difficult as regards the ratio of the distance of the sun to the size of the universe. Here he seizes upon a dictum of Aristarchus that the sphere of the fixed stars is so great that the circle in which he supposes the earth to revolve (round the sun) 'bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface'. If this is taken in a strictly mathematical sense, it means that the sphere of the fixed stars is infinite in size, which would not suit Archimedes's purpose; to get another meaning out of it he presses the point that Aristarchus's words cannot be taken quite literally because the centre, being without magnitude, cannot be in any ratio to any other magnitude; hence he suggests that a reasonable interpretation of the statement would be to suppose that, if we conceive a sphere with radius equal to the distance between the centre of the sun and the centre of the earth, then

(diam. of earth) : (diam. of said sphere)

= (diam. of said sphere) : (diam. of sphere of fixed stars).

This is, of course, an arbitrary interpretation; Aristarchus presumably meant no such thing, but merely that the size of the earth is negligible in comparison with that of the sphere of the fixed stars. However, the solution of Archimedes's problem demands some assumption of the kind, and, in making this assumption, he was no doubt aware that he was taking a liberty with Aristarchus for the sake of giving his hypothesis an air of authority.

Archimedes has, lastly, to compare the diameter of the sun with the circumference of the circle described by its centre. Aristarchus had made the apparent diameter of the sun $\frac{1}{720}$th of the said circumference; Archimedes will prove that the said circumference cannot contain as many as 1,000 sun's diameters, or that the diameter of the sun is greater than the side of a regular chiliagon inscribed in the circle. First he made an experiment of his own to determine the apparent diameter of the sun. With a small cylinder or disc in a plane at right angles to a long straight stick and moveable along it, he observed the sun at the moment when it cleared the horizon in rising, moving the disc till it just covered and just failed to cover the sun as he looked along the straight stick. He thus found the angular diameter to lie between $\frac{1}{164}R$ and $\frac{1}{200}R$, where R is a right angle. But as, under his assumptions, the size of the earth is not negligible in comparison with the sun's circle, he had to allow for parallax and find limits for the angle subtended by the sun at the centre of the earth. This he does by a geometrical argument very much in the manner of Aristarchus.



Let the circles with centres O, C represent sections of the sun and earth respectively, E the position of the observer observing

the sun when it has just cleared the horizon. Draw from E two tangents EP, EQ to the circle with centre O, and from C let CF, CG be drawn touching the same circle. With centre C and radius CO describe a circle: this will represent the path of the centre of the sun round the earth. Let this circle meet the tangents from C in A, B, and join AB meeting CO in M.

Archimedes's observation has shown that

$$\tfrac{1}{164}R > \angle PEQ > \tfrac{1}{200}R;$$

and he proceeds to prove that AB is less than the side of a regular polygon of 656 sides inscribed in the circle AOB, but greater than the side of an inscribed regular polygon of 1,000 sides, in other words, that

$$\tfrac{1}{164}R > \angle FCG > \tfrac{1}{250}R.$$

The first relation is obvious, for, since $CO > EO$,

$$\angle PEQ > \angle FCG.$$

Next, the perimeter of any polygon inscribed in the circle AOB is less than $\frac{44}{7}CO$ (i.e. $\frac{22}{7}$ times the diameter);

Therefore $\quad AB < \frac{1}{656}\cdot\frac{44}{7}CO$ or $\frac{11}{1148}CO$,

and, *a fortiori*, $\quad AB < \frac{1}{100}CO$.

Now, the triangles CAM, COF being equal in all respects, $AM = OF$, so that $AB = 2OF =$ (diameter of sun) $> CH + OK$, since the diameter of the sun is greater than that of the earth;

therefore $\quad CH + OK < \frac{1}{100}CO$, and $HK > \frac{99}{100}CO$.

And $\quad CO > CF$, while $HK < EQ$, so that $EQ > \frac{99}{100}CF$.

We can now compare the angles OCF, OEQ;

$$\text{for} \qquad \frac{\angle OCF}{\angle OEQ}\left[> \frac{\tan OCF}{\tan OEQ}\right] > \frac{EQ}{CF} > \tfrac{99}{100}, \textit{ a fortiori}.$$

Doubling the angles, we have

$$\begin{aligned}\angle FCG &> \tfrac{99}{100}\cdot\angle PEQ \\ &> \tfrac{99}{20000}R, \text{ since } \angle PEQ > \tfrac{1}{200}R, \\ &> \tfrac{1}{203}R.\end{aligned}$$

Hence AB is greater than the side of a regular polygon of 812 sides, and *a fortiori* greater than the side of a regular polygon of 1,000 sides, inscribed in the circle AOB.

The perimeter of the chiliagon, as of any regular polygon with more sides than six, inscribed in the circle AOB is greater than 3 times the diameter of the sun's orbit, but is less than 1,000 times the diameter of the sun, and *a fortiori* less than 30,000 times the diameter of the earth;

therefore (diameter of sun's orbit) < 10,000 (diam. of earth)
< 10,000,000,000 stades.

But (diam. of earth) : (diam. of sun's orbit)
= (diam. of sun's orbit) : (diam. of universe);

therefore the universe, or the sphere of the fixed stars, is less than $10{,}000^3$ times the sphere in which the sun's orbit is a great circle.

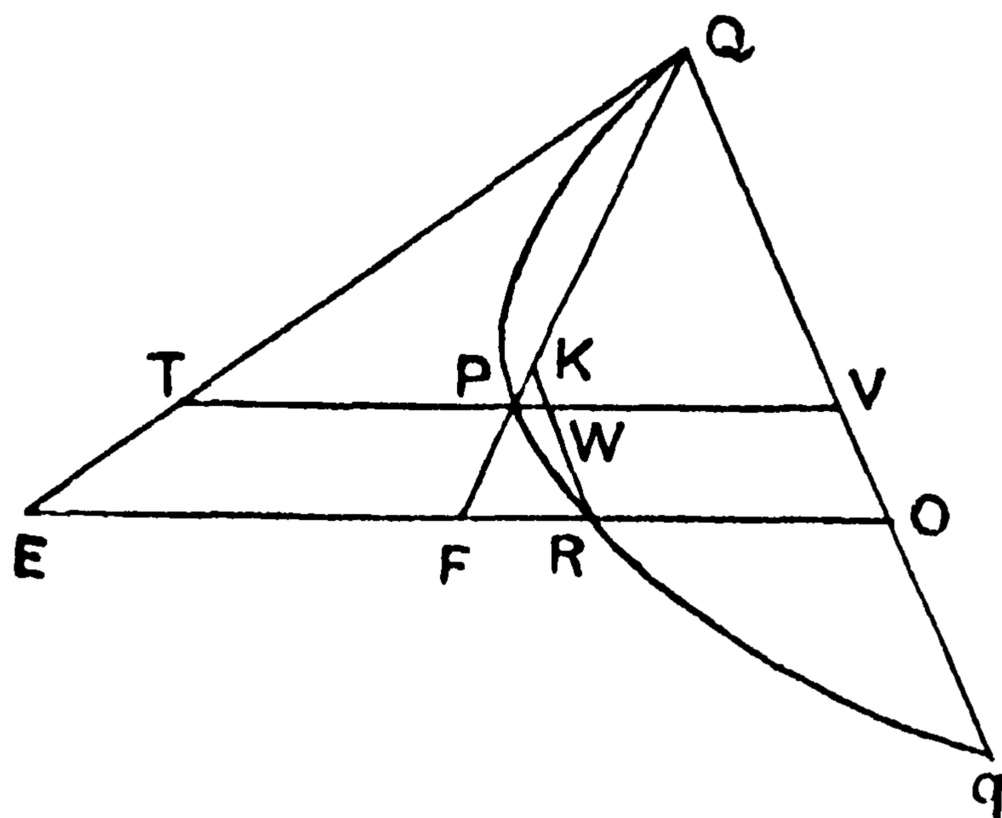
Archimedes takes a quantity of sand not greater than a poppy-seed and assumes that it contains not more than 10,000 grains; the diameter of a poppy-seed he takes to be not less than $\frac{1}{40}$th of a finger-breadth; thus a sphere of diameter 1 finger-breadth is not greater than 64,000 poppy-seeds and therefore contains not more than 640,000,000 grains of sand ('6 units of *second order* + 40,000,000 units of *first order*') and *a fortiori* not more than 1,000,000,000 ('10 units of *second order* of numbers'). Gradually increasing the diameter of the sphere by multiplying it each time by 100 (making the sphere 1,000,000 times larger each time) and substituting for 10,000 finger-breadths a stadium (< 10,000 finger-breadths), he finds the number of grains of sand in a sphere of diameter 10,000,000,000 stadia to be less than '1,000 units of *seventh order* of numbers' or 10^{51}, and the number in a sphere $10{,}000^3$ times this size to be less than '10,000,000 units of the *eighth order* of numbers' or 10^{63}.

The Quadrature of the Parabola.

In the preface, addressed to Dositheus after the death of Conon, Archimedes claims originality for the solution of the problem of finding the area of a segment of a parabola cut off by any chord, which he says he first discovered by means of mechanics and then confirmed by means of geometry, using the lemma that, if there are two unequal areas (or magnitudes

generally), then however small the excess of the greater over the lesser, it can by being continually added to itself be made to exceed the greater; in other words, he confirmed the solution by the method of exhaustion. One solution by means of mechanics is, as we have seen, given in *The Method*; the present treatise contains a solution by means of mechanics confirmed by the method of exhaustion (Props. 1–17), and then gives an entirely independent solution by means of pure geometry, also confirmed by exhaustion (Props. 18–24).

I. The mechanical solution depends upon two properties of the parabola proved in Props. 4, 5. If Qq be the base, and P



the vertex, of a parabolic segment, P is the point of contact of the tangent parallel to Qq, the diameter PV through P bisects Qq in V, and, if VP produced meets the tangent at Q in T, then $TP = PV$. These properties, along with the fundamental property that QV^2 varies as PV, Archimedes uses to prove that, if EO be any parallel to TV meeting QT, QP (produced, if necessary), the curve, and Qq in E, F, R, O respectively, then

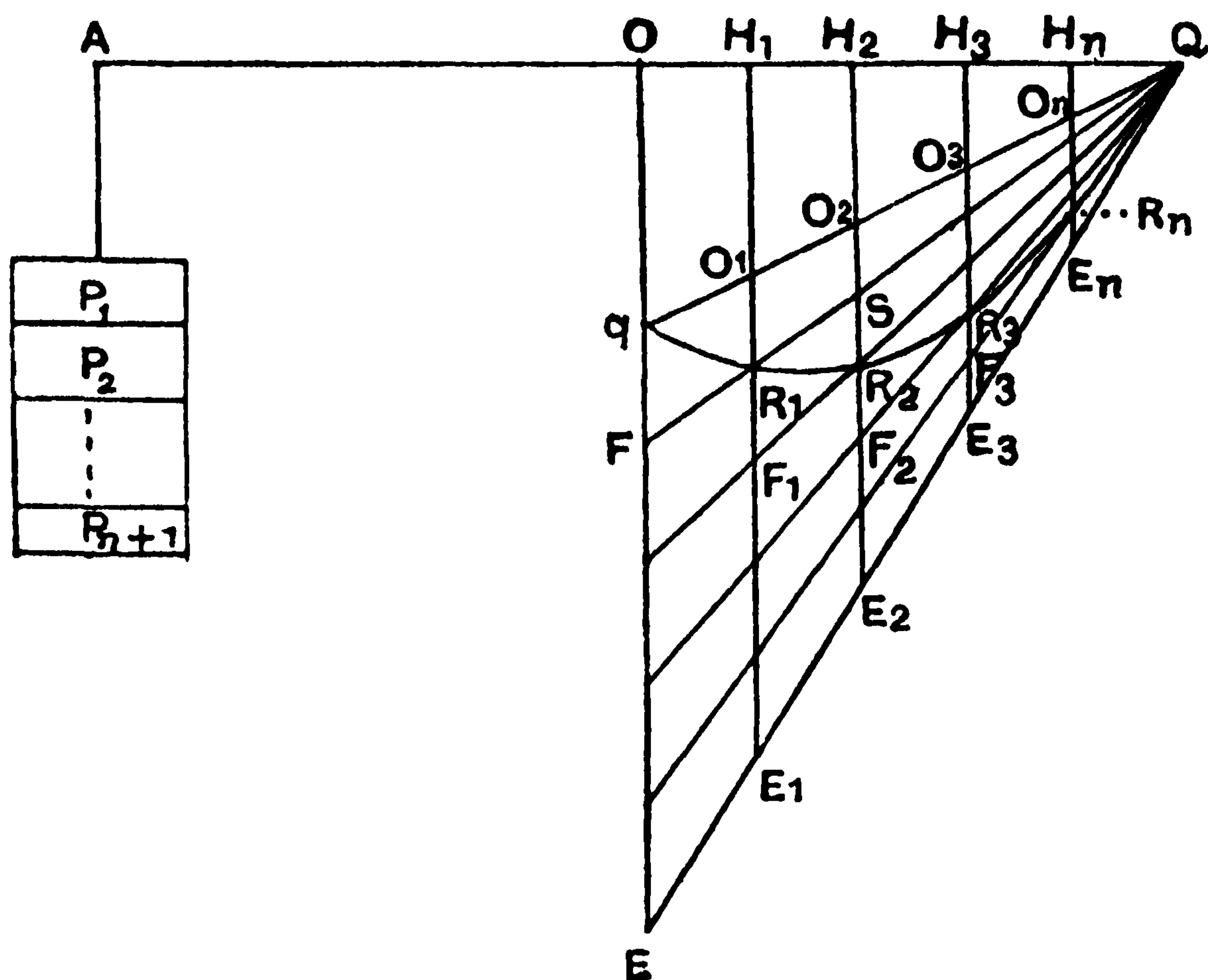
$$QV : VO = OF : FR,$$

and $$QO : Oq = ER : RO. \qquad \text{(Props. 4, 5.)}$$

Now suppose a parabolic segment QR_1q so placed in relation to a horizontal straight line QA through Q that the diameter bisecting Qq is at right angles to QA, i.e. vertical, and let the tangent at Q meet the diameter qO through q in E. Produce QO to A, making OA equal to OQ.

Divide Qq into any number of equal parts at $O_1, O_2 \dots O_n$, and through these points draw parallels to OE, i.e. vertical lines meeting OQ in $H_1, H_2, \dots$, EQ in $E_1, E_2, \dots$, and the

curve in R_1, R_2, ... Join QR_1, and produce it to meet OE in F, QR_2 meeting O_1E_1 in F_1, and so on.



Now Archimedes has proved in a series of propositions (6-13) that, if a trapezium such as $O_1E_1E_2O_2$ is suspended from H_1H_2, and an area P suspended at A balances $O_1E_1E_2O_2$ so suspended, it will take a greater area than P suspended at A to balance the same trapezium suspended from H_2 and a less area than P to balance the same trapezium suspended from H_1. A similar proposition holds with regard to a triangle such as E_nH_nQ suspended where it is and suspended at Q and H_n respectively.

Suppose (Props. 14, 15) the triangle QqE suspended where it is from OQ, and suppose that the trapezium EO_1, suspended where it is, is balanced by an area P_1 suspended at A, the trapezium E_1O_2, suspended where it is, is balanced by P_2 suspended at A, and so on, and finally the triangle E_nO_nQ, suspended where it is, is balanced by P_{n+1} suspended at A; then $P_1+P_2+\ldots+P_{n+1}$ at A balances the whole triangle, so that

$$P_1+P_2+\ldots+P_{n+1}=\tfrac{1}{3}\triangle EqQ,$$

since the whole triangle may be regarded as suspended from the point on OQ vertically above its centre of gravity.

Now
$$\begin{aligned} AO:OH_1 &= QO:OH_1 \\ &= Qq:qO_1 \\ &= E_1O_1:O_1R_1, \text{ by Prop. 5,} \\ &= (\text{trapezium } EO_1):(\text{trapezium } FO_1), \end{aligned}$$

that is, it takes the trapezium FO_1 suspended at A to balance the trapezium EO_1 suspended at H_1. And P_1 balances EO_1 where it is.

Therefore $(FO_1) > P_1$.

Similarly $(F_1O_2) > P_2$, and so on.

Again $AO:OH_1 = E_1O_1:O_1R_1$

$= (\text{trapezium } E_1O_2):(\text{trapezium } R_1O_2)$,

that is, (R_1O_2) at A will balance (E_1O_2) suspended at H_1, while P_2 at A balances (E_1O_2) suspended where it is

whence $P_2 > R_1O_2$.

Therefore $(F_1O_2) > P_2 > (R_1O_2)$,

$(F_2O_3) > P_3 > R_2O_3$, and so on;

and finally, $\Delta E_nO_nQ > P_{n+1} > \Delta R_nO_nQ$.

By addition,

$$(R_1O_2) + (R_2O_3) + \ldots + (\Delta R_nO_nQ) < P_2 + P_3 + \ldots + P_{n+1};$$

therefore, *a fortiori*,

$$(R_1O_2) + (R_2O_3) + \ldots + \Delta R_nO_nQ < P_1 + P_2 + \ldots + P_{n+1}$$
$$< (FO_1) + (F_1O_2) + \ldots + \Delta E_nO_nQ.$$

That is to say, we have an inscribed figure consisting of trapezia and a triangle which is less, and a circumscribed figure composed in the same way which is greater, than

$$P_1 + P_2 + \ldots + P_{n+1}, \text{ i.e. } \tfrac{1}{3}\Delta EqQ.$$

It is therefore inferred, and proved by the method of exhaustion, that the segment itself is *equal* to $\frac{1}{3}\Delta EqQ$ (Prop. 16).

In order to enable the method to be applied, it has only to be proved that, by increasing the number of parts in Qq sufficiently, the difference between the circumscribed and inscribed figures can be made as small as we please. This can be seen thus. We have first to show that all the parts, as qF, into which qE is divided are equal.

We have $E_1O_1:O_1R_1 = QO:OH_1 = (n+1):1$,

or $O_1R_1 = \dfrac{1}{n+1} \cdot E_1O_1$, whence also $O_2S = \dfrac{1}{n+1} \cdot O_2E_2$.

And $\qquad E_2O_2 : O_2R_2 = QO : OH_2 = (n+1) : 2,$

or
$$O_2R_2 = \frac{2}{n+1} \cdot O_2E_2.$$

It follows that $O_2S = SR_2$, and so on.

Consequently O_1R_1, O_2R_2, O_3R_3 ... are divided into 1, 2, 3 ... equal parts respectively by the lines from Q meeting qE.

It follows that the difference between the circumscribed and inscribed figures is equal to the triangle FqQ, which can be made as small as we please by increasing the number of divisions in Qq, i.e. in qE.

Since the area of the segment is equal to $\frac{1}{3}\Delta EqQ$, and it is easily proved (Prop. 17) that $\Delta EqQ = 4$ (triangle with same base and equal height with segment), it follows that the area of the segment $= \frac{4}{3}$ times the latter triangle.

It is easy to see that this solution is essentially the same as that given in *The Method* (see pp. 29–30, above), only in a more orthodox form (geometrically speaking). For there Archimedes took the sum of all the *straight lines*, as O_1R_1, O_2R_2 ..., as making up the segment notwithstanding that there are an infinite number of them and straight lines have no breadth. Here he takes inscribed and circumscribed trapezia proportional to the straight lines and having finite breadth, and then compresses the figures together into the segment itself by increasing indefinitely the number of trapezia in each figure, i.e. diminishing their breadth indefinitely.

The procedure is equivalent to an integration, thus:

If X denote the area of the triangle FqQ, we have, if n be the number of parts in Qq,

(circumscribed figure)

$$\begin{aligned} &= \text{sum of } \Delta\text{s } QqF,\ QR_1F_1,\ QR_2F_2, \ldots \\ &= \text{sum of } \Delta\text{s } QqF,\ QO_1R_1,\ QO_2S, \ldots \\ &= X\left\{1 + \frac{(n-1)^2}{n^2} + \frac{(n-2)^2}{n^2} + \ldots + \frac{1}{n^2}\right\} \\ &= \frac{1}{n^2X^2} \cdot X(X^2 + 2^2X^2 + 3^2X^2 + \ldots + n^2X^2). \end{aligned}$$

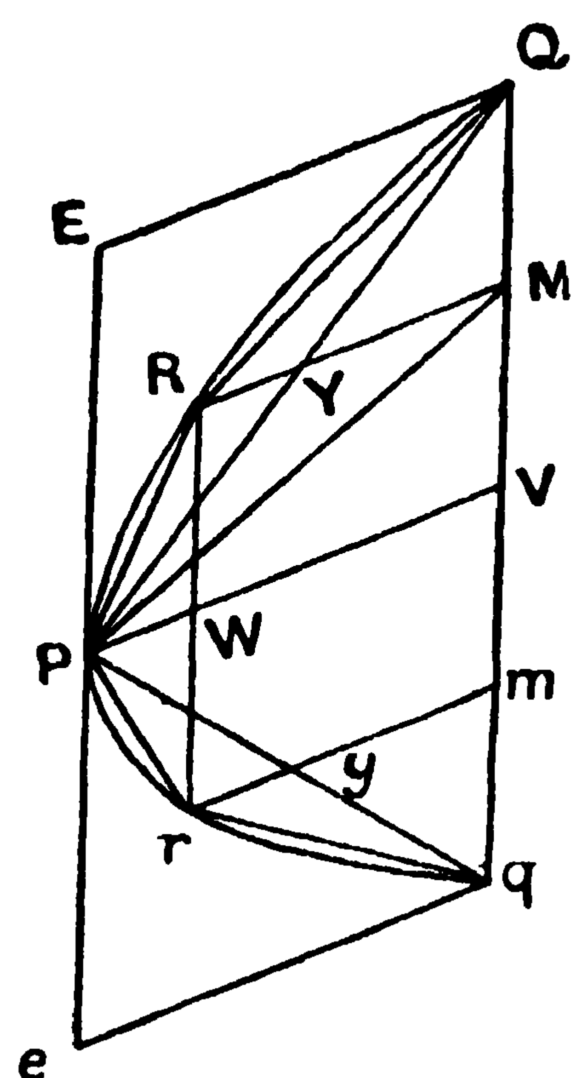
Similarly, we find that

$$\text{(inscribed figure)} = \frac{1}{n^2X^2} \cdot X\{X^2 + 2^2X^2 + \ldots + (n-1)^2X^2\}.$$

Taking the limit, we have, if A denote the area of the triangle EqQ, so that $A = nX$,

$$\text{area of segment} = \frac{1}{A^2}\int_0^A X^2 dX$$
$$= \tfrac{1}{3}A.$$

II. The purely geometrical method simply *exhausts* the parabolic segment by inscribing successive figures 'in the recognized manner' (see p. 79, above). For this purpose it is necessary to find, in terms of the triangle with the same base and height, the area added to the inscribed figure by doubling the number of sides other than the base of the segment.



Let QPq be the triangle inscribed 'in the recognized manner', P being the point of contact of the tangent parallel to Qq, and PV the diameter bisecting Qq. If QV, Vq be bisected in M, m, and RM, rm be drawn parallel to PV meeting the curve in R, r, the latter points are vertices of the next figure inscribed 'in the recognized manner', for RY, ry are diameters bisecting PQ, Pq respectively.

Now $QV^2 = 4\,RW^2$, so that $PV = 4PW$, or $RM = 3PW$.

But $\quad YM = \frac{1}{2}PV = 2PW$, so that $YM = 2\,RY$.

Therefore $\quad \Delta PRQ = \frac{1}{2}\Delta PQM = \frac{1}{4}\Delta PQV$.

Similarly

$\Delta Prq = \frac{1}{4}\Delta PVq$; whence $(\Delta PRQ + \Delta Prq) = \frac{1}{4}PQq$. (Prop. 21.)

In like manner it can be proved that the next addition to the inscribed figure adds $\frac{1}{4}$ of the sum of Δs PRQ, Prq, and so on.

Therefore the area of the inscribed figure

$$= \{1 + \tfrac{1}{4} + (\tfrac{1}{4})^2 + \ldots\}\,.\,\Delta PQq. \qquad \text{(Prop. 22.)}$$

Further, each addition to the inscribed figure is greater than half the segments of the parabola left over before the addition is made. For, if we draw the tangent at P and complete the parallelogram $EQqe$ with side EQ parallel to PV,

the triangle PQq is half of the parallelogram and therefore more than half the segment. And so on (Prop. 20).

We now have to sum n terms of the above geometrical series. Archimedes enunciates the problem in the form, Given a series of areas $A, B, C, D \ldots Z$, of which A is the greatest, and each is equal to four times the next in order, then (Prop. 23)

$$A + B + C + \ldots + Z + \tfrac{1}{3}Z = \tfrac{4}{3}A.$$

The algebraical equivalent of this is of course

$$1 + \tfrac{1}{4} + (\tfrac{1}{4})^2 + \ldots + (\tfrac{1}{4})^{n-1} = \tfrac{4}{3} - \tfrac{1}{3}(\tfrac{1}{4})^{n-1} = \frac{1-(\frac{1}{4})^n}{1-\frac{1}{4}}.$$

To find the area of the segment, Archimedes, instead of taking the limit, as we should, uses the method of *reductio ad absurdum*.

Suppose $K = \tfrac{4}{3}.\Delta PQq.$

(1) If possible, let the area of the segment be greater than K.

We then inscribe a figure 'in the recognized manner' such that the segment exceeds it by an area less than the excess of the segment over K. Therefore the inscribed figure must be greater than K, which is impossible since

$$A + B + C + \ldots + Z < \tfrac{4}{3}A,$$

where $A = \Delta PQq$ (Prop. 23).

(2) If possible, let the area of the segment be less than K.

If then $\Delta PQq = A$, $B = \tfrac{1}{4}A$, $C = \tfrac{1}{4}B$, and so on, until we arrive at an area X less than the excess of K over the area of the segment, we have

$$A + B + C + \ldots + X + \tfrac{1}{3}X = \tfrac{4}{3}A = K.$$

Thus K exceeds $A + B + C + \ldots + X$ by an area less than X, and exceeds the segment by an area greater than X.

It follows that $A + B + C + \ldots + X >$ (the segment); which is impossible (Prop. 22).

Therefore the area of the segment, being neither greater nor less than K, is equal to K or $\tfrac{4}{3}\Delta PQq$.

On Floating Bodies, I, II.

In Book I of this treatise Archimedes lays down the fundamental principles of the science of hydrostatics. These are

deduced from Postulates which are only two in number. The first which begins Book I is this:

'let it be assumed that a fluid is of such a nature that, of the parts of it which lie evenly and are continuous, that which is pressed the less is driven along by that which is pressed the more; and each of its parts is pressed by the fluid which is perpendicularly above it except when the fluid is shut up in anything and pressed by something else';

the second, placed after Prop. 7, says

'let it be assumed that, of bodies which are borne upwards in a fluid, each is borne upwards along the perpendicular drawn through its centre of gravity'.

Prop. 1 is a preliminary proposition about a sphere, and then Archimedes plunges *in medias res* with the theorem (Prop. 2) that '*the surface of any fluid at rest is a sphere the centre of which is the same as that of the earth*', and in the whole of Book I the surface of the fluid is always shown in the diagrams as spherical. The method of proof is similar to what we should expect in a modern elementary textbook, the main propositions established being the following. A solid which, size for size, is of equal weight with a fluid will, if let down into the fluid, sink till it is just covered but not lower (Prop. 3); a solid lighter than a fluid will, if let down into it, be only partly immersed, in fact just so far that the weight of the solid is equal to the weight of the fluid displaced (Props. 4, 5), and, if it is forcibly immersed, it will be driven upwards by a force equal to the difference between its weight and the weight of the fluid displaced (Prop. 6).

The important proposition follows (Prop. 7) that a solid heavier than a fluid will, if placed in it, sink to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced.

The problem of the Crown.

This proposition gives a method of solving the famous problem the discovery of which in his bath sent Archimedes home naked crying εὕρηκα, εὕρηκα, namely the problem of

determining the proportions of gold and silver in a certain crown.

Let W be the weight of the crown, w_1 and w_2 the weights of the gold and silver in it respectively, so that $W = w_1 + w_2$.

(1) Take a weight W of pure gold and weigh it in the fluid. The apparent loss of weight is then equal to the weight of the fluid displaced; this is ascertained by weighing. Let it be F_1.

It follows that the weight of the fluid displaced by a weight w_1 of gold is $\frac{w_1}{W} \cdot F_1$.

(2) Take a weight W of silver, and perform the same operation. Let the weight of the fluid displaced be F_2. Then the weight of the fluid displaced by a weight w_2 of silver is $\frac{w_2}{W} \cdot F_2$.

(3) Lastly weigh the crown itself in the fluid, and let F be loss of weight or the weight of the fluid displaced.

We have then $$\frac{w_1}{W} \cdot F_1 + \frac{w_2}{W} \cdot F_2 = F,$$

that is, $$w_1 F_1 + w_2 F_2 = (w_1 + w_2) F,$$

whence $$\frac{w_1}{w_2} = \frac{F_2 - F}{F - F_1}.$$

According to the author of the poem *de ponderibus et mensuris* (written probably about A.D. 500) Archimedes actually used a method of this kind. We first take, says our authority, two equal weights of gold and silver respectively and weigh them against each other when both are immersed in water; this gives the relation between their weights in water, and therefore between their losses of weight in water. Next we take the mixture of gold and silver and an equal weight of silver, and weigh them against each other in water in the same way.

Nevertheless I do not think it probable that this was the way in which the solution of the problem was *discovered*. As we are told that Archimedes discovered it in his bath, and that he noticed that, if the bath was full when he entered it, so much water overflowed as was displaced by his body, he is more likely to have discovered the solution by the alternative

method attributed to him by Vitruvius,[1] namely by measuring successively the *volumes* of fluid displaced by three equal weights, (1) the crown, (2) an equal weight of gold, (3) an equal weight of silver respectively. Suppose, as before, that the weight of the crown is W and that it contains weights w_1 and w_2 of gold and silver respectively. Then

(1) the crown displaces a certain volume of the fluid, V, say;

(2) the weight W of gold displaces a volume V_1, say, of the fluid;

therefore a weight w_1 of gold displaces a volume $\frac{w_1}{W} \cdot V_1$ of the fluid;

(3) the weight W of silver displaces V_2, say, of the fluid; therefore a weight w_2 of silver displaces $\frac{w_2}{W} \cdot V_2$.

It follows that $V = \frac{w_1}{W} \cdot V_1 + \frac{w_2}{W} \cdot V_2$,

whence we derive (since $W = w_1 + w_2$)

$$\frac{w_1}{w_2} = \frac{V_2 - V}{V - V_1},$$

the latter ratio being obviously equal to that obtained by the other method.

The last propositions (8 and 9) of Book I deal with the case of any segment of a sphere lighter than a fluid and immersed in it in such a way that either (1) the curved surface is downwards and the base is entirely outside the fluid, or (2) the curved surface is upwards and the base is entirely submerged, and it is proved that in either case the segment is in stable equilibrium when the axis is vertical. This is expressed here and in the corresponding propositions of Book II by saying that, 'if the figure be forced into such a position that the base of the segment touches the fluid (at one point), the figure will not remain inclined but will return to the upright position'.

Book II, which investigates fully the conditions of stability of a right segment of a paraboloid of revolution floating in a fluid for different values of the specific gravity and different ratios between the axis or height of the segment and the

[1] *De architectura*, ix. 3.

principal parameter of the generating parabola, is a veritable *tour de force* which must be read in full to be appreciated. Prop. 1 is preliminary, to the effect that, if a solid lighter than a fluid be at rest in it, the weight of the solid will be to that of the same volume of the fluid as the immersed portion of the solid is to the whole. The results of the propositions about the segment of a paraboloid may be thus summarized. Let h be the axis or height of the segment, p the principal parameter of the generating parabola, s the ratio of the specific gravity of the solid to that of the fluid (s always < 1). The segment is supposed to be always placed so that its base is either entirely above, or entirely below, the surface of the fluid, and what Archimedes proves in each case is that, if the segment is so placed with its axis inclined to the vertical at any angle, it will not rest there but will return to the position of stability.

I. If h is not greater than $\frac{3}{4}p$, the position of stability is with the axis vertical, whether the curved surface is downwards or upwards (Props. 2, 3).

II. If h is greater than $\frac{3}{4}p$, then, in order that the position of stability may be with the axis vertical, s must be not less than $(h-\frac{3}{4}p)^2/h^2$ if the curved surface is downwards, and not greater than $\{h^2-(h-\frac{3}{4}p)^2\}/h^2$ if the curved surface is upwards (Props. 4, 5).

III. If $h>\frac{3}{4}p$, but $h/\frac{1}{2}p<15/4$, the segment, if placed with one point of the base touching the surface, will never remain there whether the curved surface be downwards or upwards (Props. 6, 7). (The segment will move in the direction of bringing the axis nearer to the vertical position.)

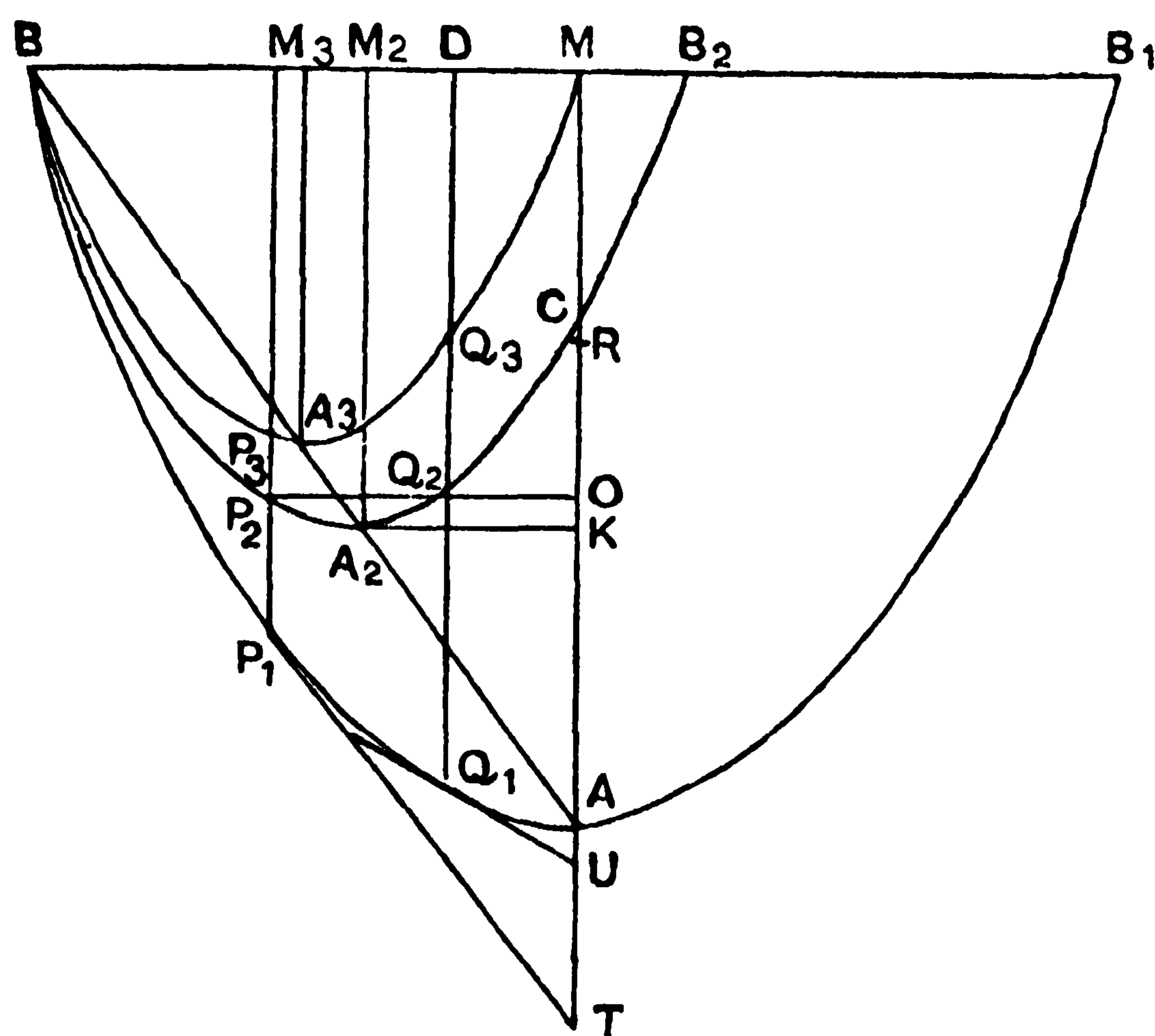
IV. If $h>\frac{3}{4}p$, but $h/\frac{1}{2}p<15/4$, and if s is less than $(h-\frac{3}{4}p)^2/h^2$ in the case where the curved surface is downwards, but greater than $\{h^2-(h-\frac{3}{4}p)^2\}/h^2$ in the case where the curved surface is upwards, then the position of stability is one in which the axis is not vertical but inclined to the surface of the fluid at a certain angle (Props. 8, 9). (The angle is drawn in an auxiliary figure. The construction for it in Prop. 8 is equivalent to the solution of the following equation in θ,

$$\tfrac{1}{4}p\cot^2\theta = \tfrac{2}{3}(h-k)-\tfrac{1}{2}p,$$

where k is the axis of the segment of the paraboloid cut off by the surface of the fluid.)

V. Prop. 10 investigates the positions of stability in the cases where $h/\frac{1}{2}p > 15/4$, the base is entirely above the surface, and s has values lying between five pairs of ratios respectively. Only in the case where s is not less than $(h-\frac{3}{4}p)^2/h^2$ is the position of stability that in which the axis is vertical.

BAB_1 is a section of the paraboloid through the axis AM. C is a point on AM such that $AC = 2CM$, K is a point on CA such that $AM:CK = 15:4$. CO is measured along CA such that $CO = \frac{1}{2}p$, and R is a point on AM such that $MR = \frac{3}{2}CO$. A_2 is the point in which the perpendicular to AM from K meets AB, and A_3 is the middle point of AB. BA_2B_2, BA_3M are parabolic segments on A_2M_2, A_3M_3 (parallel to AM) as axes



and similar to the original segment. (The parabola BA_2B_2 is proved to pass through C by using the above relation $AM:CK = 15:4$ and applying Prop. 4 of the *Quadrature of the Parabola.*) The perpendicular to AM from O meets the parabola BA_2B_2 in two points P_2, Q_2, and straight lines through these points parallel to AM meet the other parabolas in P_1, Q_1 and P_3, Q_3 respectively. P_1T and Q_1U are tangents to the original parabola meeting the axis MA produced in T, U. Then

(i) if s is not less than $AR^2:AM^2$ or $(h-\frac{3}{4}p)^2:h^2$, there is stable equilibrium when AM is vertical;

(ii) if $s < AR^2 : AM^2$ but $> Q_1Q_3^2 : AM^2$, the solid will not rest with its base touching the surface of the fluid in one point only, but in a position with the base entirely out of the fluid and the axis making with the surface an angle greater than U;

(iiia) if $s = Q_1Q_3^2 : AM^2$, there is stable equilibrium with one point of the base touching the surface and AM inclined to it at an angle equal to U;

(iiib) if $s = P_1P_3^2 : AM^2$, there is stable equilibrium with one point of the base touching the surface and with AM inclined to it at an angle equal to T;

(iv) if $s > P_1P_3^2 : AM^2$ but $< Q_1Q_3^2 : AM^2$, there will be stable equilibrium in a position in which the base is more submerged;

(v) if $s < P_1P_3^2 : AM^2$, there will be stable equilibrium with the base entirely out of the fluid and with the axis AM inclined to the surface at an angle less than T.

It remains to mention the traditions regarding other investigations by Archimedes which have reached us in Greek or through the Arabic.

(α) The Cattle-Problem.

This is a difficult problem in indeterminate analysis. It is required to find the number of bulls and cows of each of four colours, or to find 8 unknown quantities. The first part of the problem connects the unknowns by seven simple equations; and the second part adds two more conditions to which the unknowns must be subject. If W, w be the numbers of white bulls and cows respectively and (X, x), (Y, y), (Z, z) represent the numbers of the other three colours, we have first the following equations:

(I) $$W = (\tfrac{1}{2} + \tfrac{1}{3})\,X + Y, \qquad (\alpha)$$
$$X = (\tfrac{1}{4} + \tfrac{1}{5})\,Z + Y, \qquad (\beta)$$
$$Z = (\tfrac{1}{6} + \tfrac{1}{7})\,W + Y, \qquad (\gamma)$$

(II) $$w = (\tfrac{1}{3} + \tfrac{1}{4})\,(X + x), \qquad (\delta)$$
$$x = (\tfrac{1}{4} + \tfrac{1}{5})\,(Z + z), \qquad (\epsilon)$$
$$z = (\tfrac{1}{5} + \tfrac{1}{6})\,(Y + y), \qquad (\zeta)$$
$$y = (\tfrac{1}{6} + \tfrac{1}{7})\,(W + w). \qquad (\eta)$$

Secondly, it is required that

$$W+X = \text{a square}, \quad (\theta)$$

$$Y+Z = \text{a triangular number}. \quad (\iota)$$

There is an ambiguity in the text which makes it just possible that $W+X$ need only be the product of two whole numbers instead of a square as in (θ). Jul. Fr. Wurm solved the problem in the simpler form to which this change reduces it. The complete problem is discussed and partly solved by Amthor.[1]

The general solution of the first seven equations is

$$W = 2.3.7.53.4657n = 10366482n,$$

$$X = 2.3^2.89.4657n = 7460514n,$$

$$Y = 3^4.11.4657n = 4149387n,$$

$$Z = 2^2.5.79.4657n = 7358060n,$$

$$w = 2^3.3.5.7.23.373n = 7206360n,$$

$$x = 2.3^2.17.15991n = 4893246n,$$

$$y = 3^2.13.46489n = 5439213n,$$

$$z = 2^2.3.5.7.11.761n = 3515820n.$$

It is not difficult to find such a value of n that $W+X =$ a square number; it is $n = 3.11.29.4657\xi^2 = 4456749\xi^2$, where ξ is any integer. We then have to make $Y+Z$ a triangular number, i.e. a number of the form $\frac{1}{2}q(q+1)$. This reduces itself to the solution of the 'Pellian' equation

$$t^2 - 4729494u^2 = 1,$$

which leads to prodigious figures; one of the eight unknown quantities alone would have more than 206,500 digits!

(β) *On semi-regular polyhedra.*

In addition, Archimedes investigated polyhedra of a certain type. This we learn from Pappus.[2] The polyhedra in question are semi-regular, being contained by equilateral and equi-

[1] *Zeitschrift für Math. u. Physik* (Hist.-litt. Abt.) xxv. (1880), pp. 156 sqq.

[2] Pappus, v, pp. 352–8.

angular, but not similar, polygons; those discovered by Archimedes were 13 in number. If we for convenience designate a polyhedron contained by m regular polygons of α sides, n regular polygons of β sides, &c., by $(m_\alpha, n_\beta \ldots)$, the thirteen Archimedean polyhedra, which we will denote by $P_1, P_2 \ldots P_{13}$, are as follows:

Figure with 8 faces: $P_1 \equiv (4_3, 4_6)$.

Figures with 14 faces: $P_2 \equiv (8_3, 6_4)$, $P_3 \equiv (6_4, 8_6)$,
$P_4 \equiv (8_3, 6_8)$.

Figures with 26 faces: $P_5 \equiv (8_3, 18_4)$, $P_6 \equiv (12_4, 8_6, 6_8)$.

Figures with 32 faces: $P_7 \equiv (20_3, 12_5)$, $P_8 \equiv (12_5, 20_6)$,
$P_9 \equiv (20_3, 12_{10})$.

Figure with 38 faces: $P_{10} \equiv (32_3, 6_4)$.

Figures with 62 faces: $P_{11} \equiv (20_3, 30_4, 12_5)$,
$P_{12} \equiv (30_4, 20_6, 12_{10})$.

Figure with 92 faces: $P_{13} \equiv (80_3, 12_5)$.

Kepler[1] showed how these figures can be obtained. A method of obtaining some of them is indicated in a fragment of a scholium to the Vatican MS. of Pappus. If a solid angle of one of the regular solids be cut off symmetrically by a plane, i.e. in such a way that the plane cuts off the same length from each of the edges meeting at the angle, the section is a regular polygon which is a triangle, square or pentagon according as the solid angle is formed of three, four, or five plane angles. If certain equal portions be so cut off from all the solid angles respectively, they will leave regular polygons inscribed in the faces of the solid; this happens (A) when the cutting planes bisect the sides of the faces and so leave in each face a polygon of the same kind, and (B) when the cutting planes cut off a smaller portion from each angle in such a way that a regular polygon is left in each face which has double the number of sides (as when we make, say, an octagon out of a square by cutting off the necessary portions,

[1] Kepler, *Harmonice mundi* in *Opera* (1864), v, pp. 123–6.

symmetrically, from the corners). We have seen that, according to Heron, two of the semi-regular solids had already been discovered by Plato, and this would doubtless be his method. The methods (A) and (B) applied to the five regular solids give the following out of the 13 semi-regular solids. We obtain (1) from the tetrahedron, P_1 by cutting off angles so as to leave hexagons in the faces; (2) from the cube, P_2 by leaving squares, and P_4 by leaving octagons, in the faces; (3) from the octahedron, P_2 by leaving triangles, and P_3 by leaving hexagons, in the faces; (4) from the icosahedron, P_7 by leaving triangles, and P_8 by leaving hexagons, in the faces; (5) from the dodecahedron, P_7 by leaving pentagons, and P_9 by leaving decagons in the faces.

Of the remaining six, four are obtained by cutting off all the edges symmetrically and equally by planes parallel to the edges, and then cutting off angles. Take first the cube. (1) Cut off from each four parallel edges portions which leave an octagon as the section of the figure perpendicular to the edges; then cut off equilateral triangles from the corners (see Fig. 1); this gives P_5 containing 8 equilateral triangles and 18 squares. (P_5 is also obtained by bisecting all the edges of P_2 and cutting off corners.) (2) Cut off from the edges of the cube a smaller portion so as to leave in each face a square such that the octagon described in it has its side equal to the breadth of the section in which each edge is cut; then cut off hexagons from each angle (see Fig. 2); this

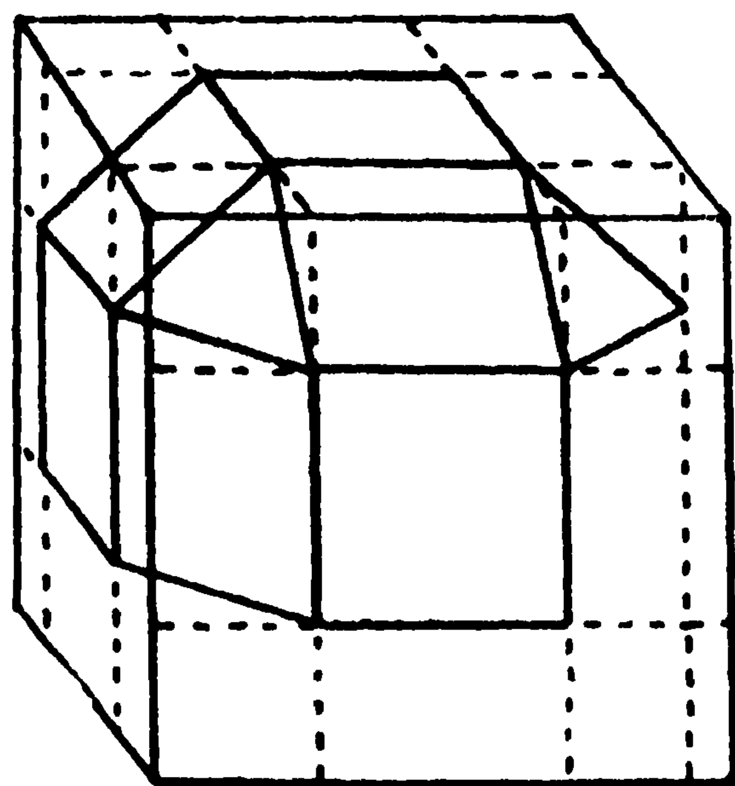


Fig. 1.

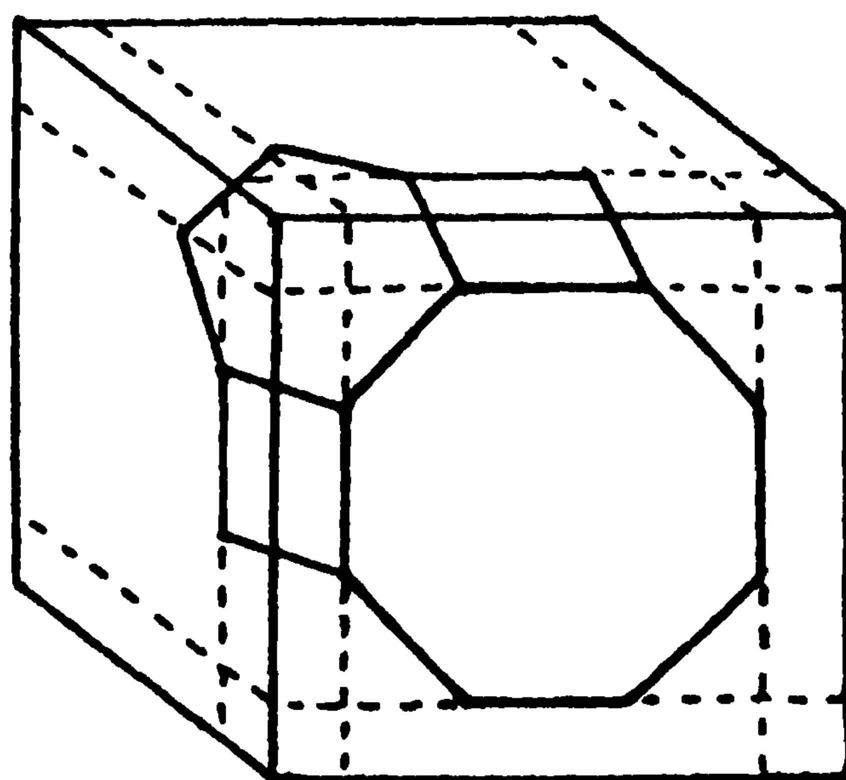


Fig. 2.

gives 6 octagons in the faces, 12 squares under the edges and 8 hexagons at the corners; that is, we have P_6. An exactly

similar procedure with the icosahedron and dodecahedron produces P_{11} and P_{12} (see Figs. 3, 4 for the case of the icosahedron).

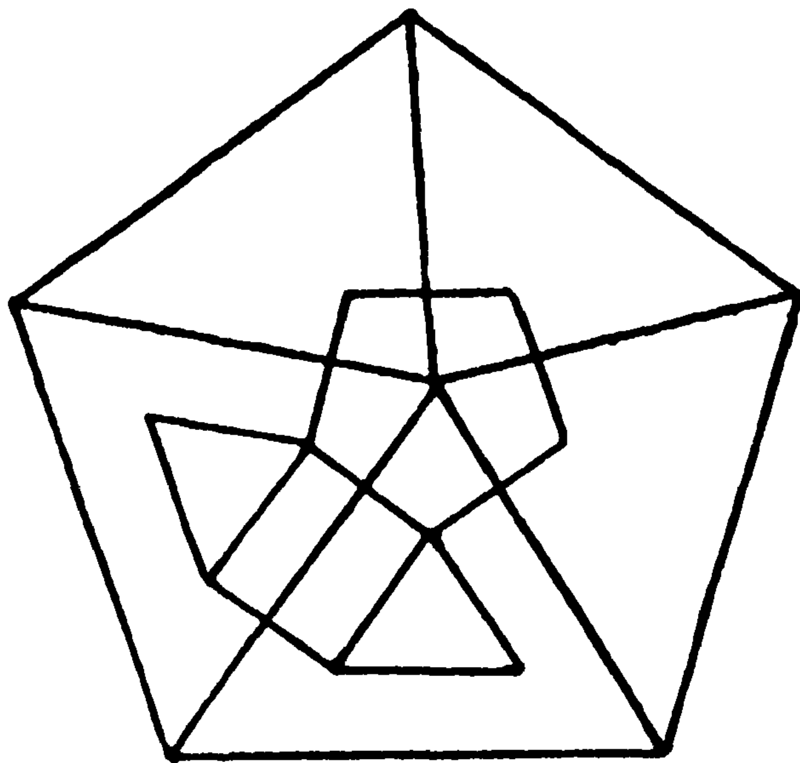


Fig. 3.

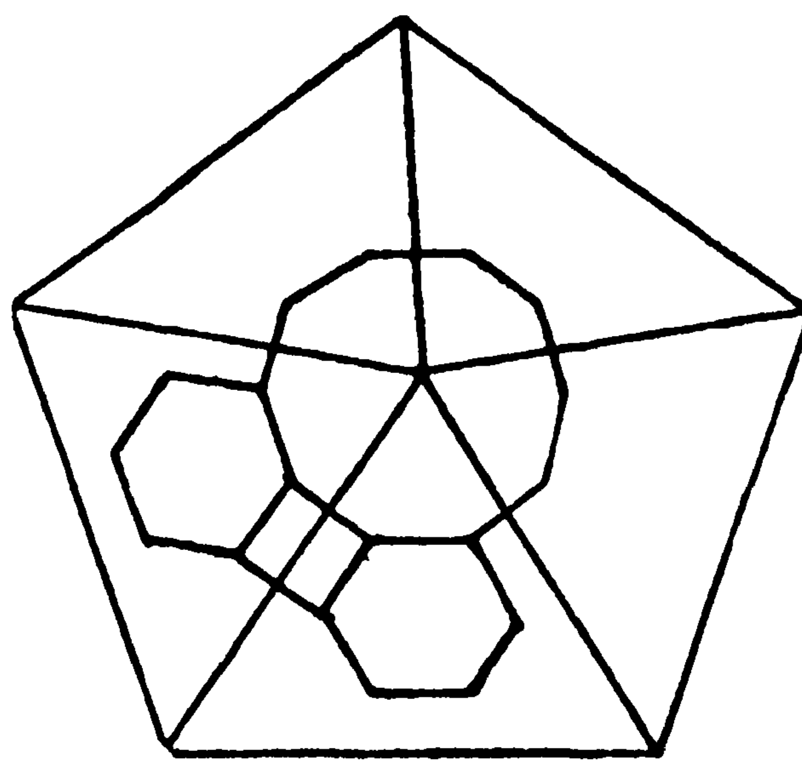


Fig. 4.

The two remaining solids P_{10}, P_{13} cannot be so simply produced. They are represented in Figs. 5, 6, which I have

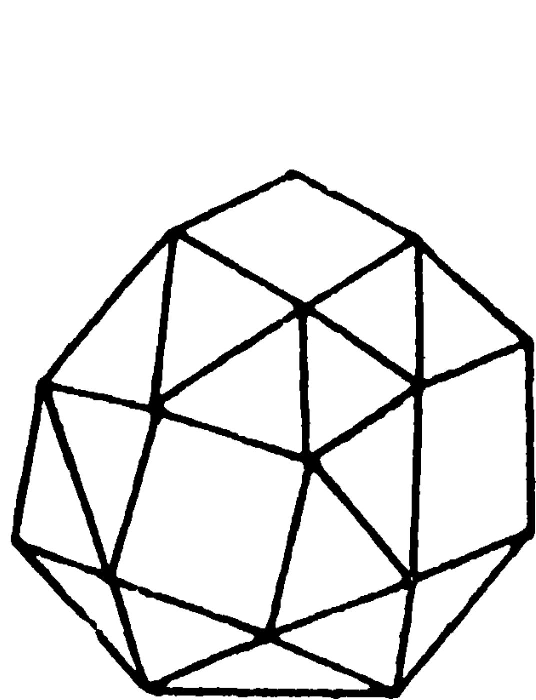


Fig. 5.

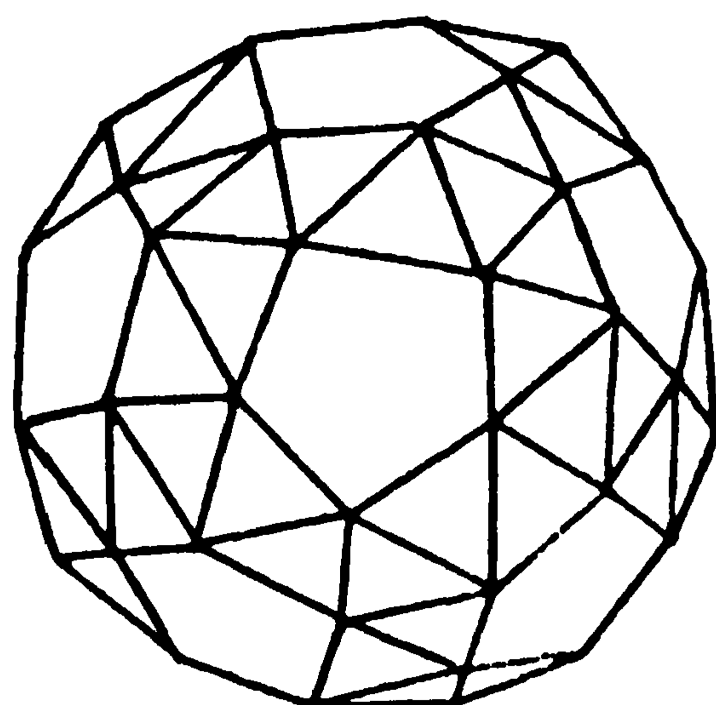


Fig. 6.

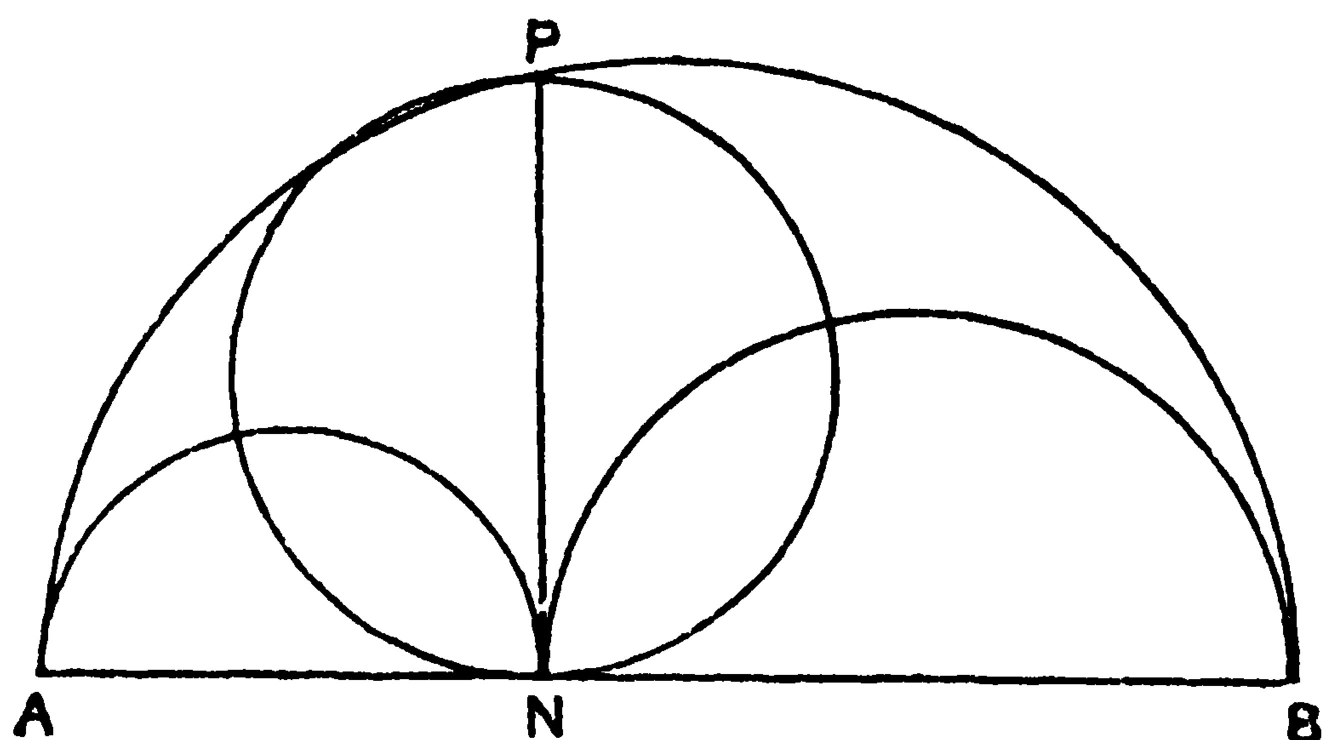
taken from Kepler. P_{10} is the *snub cube* in which each solid angle is formed by the angles of four equilateral triangles and one square; P_{13} is the *snub dodecahedron*, each solid angle of which is formed by the angles of four equilateral triangles and *one regular pentagon.*

We are indebted to Arabian tradition for

(γ) *The Liber Assumptorum.*

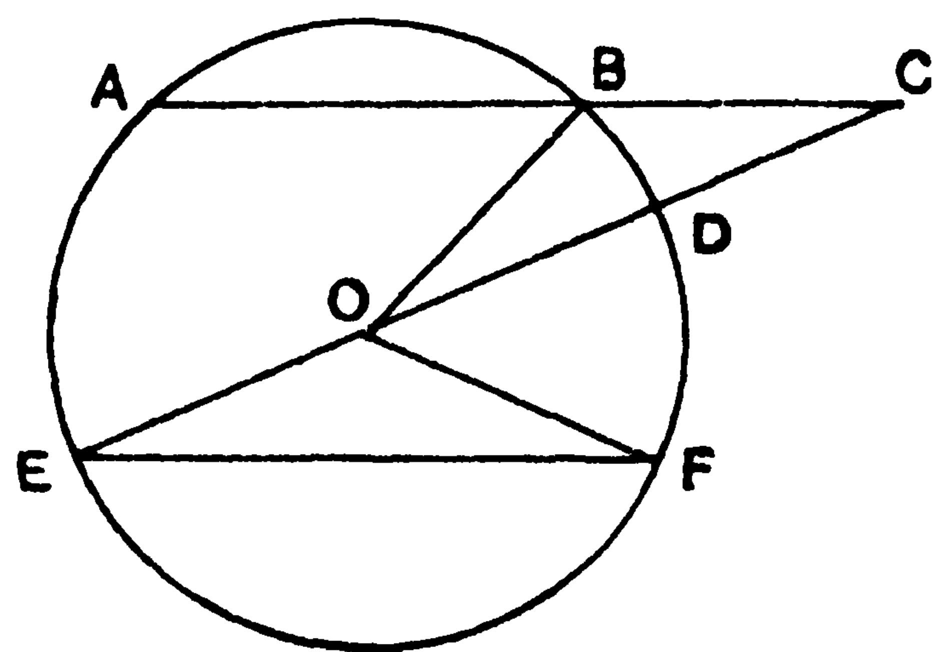
Of the theorems contained in this collection many are so elegant as to afford a presumption that they may really be due to Archimedes. In three of them the figure appears which was called *ἄρβηλος*, a shoemaker's knife, consisting of three semicircles with a common diameter as shown in the annexed figure. If N be the point at which the diameters

of the two smaller semicircles adjoin, and NP be drawn at right angles to AB meeting the external semicircle in P, the area of the ἄρβηλος (included between the three semicircular arcs) is equal to the circle on PN as diameter (Prop. 4). In Prop. 5 it is shown that, if a circle be described in the space between the arcs AP, AN and the straight line PN touching



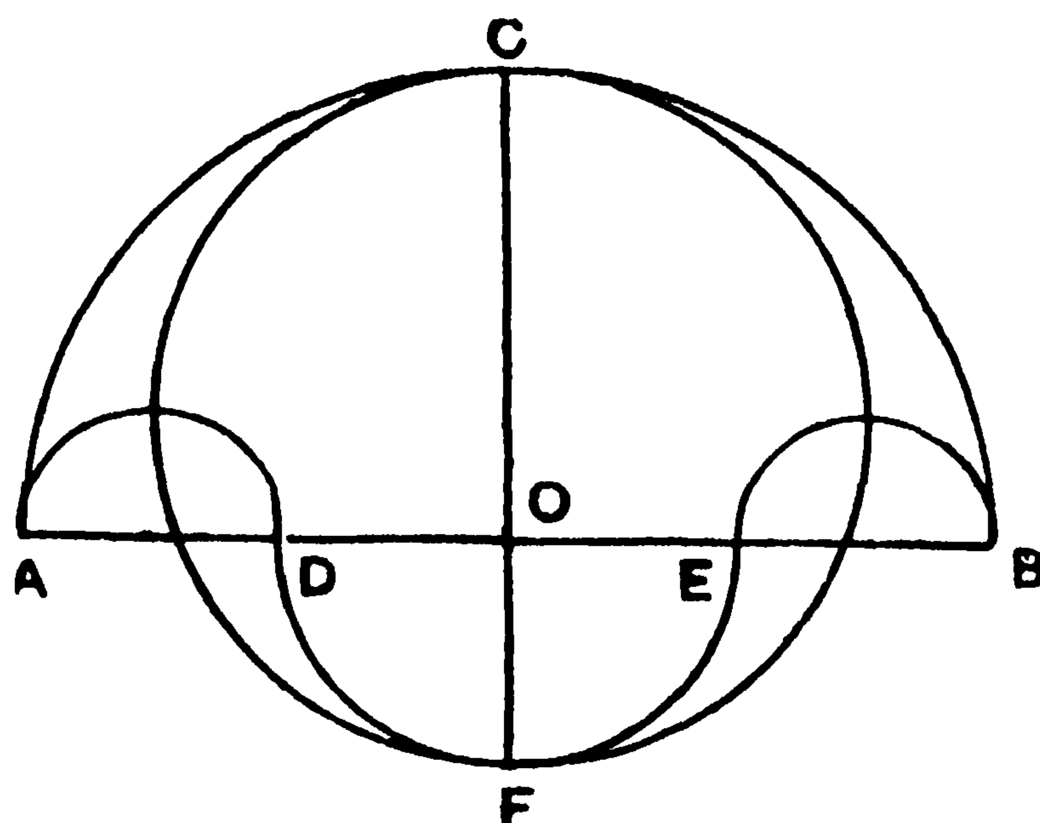
all three, and if a circle be similarly described in the space between the arcs PB, NB and the straight line PN touching all three, the two circles are equal. If one circle be described in the ἄρβηλος touching all three semicircles, Prop. 6 shows that, if the ratio of AN to NB be given, we can find the relation between the diameter of the circle inscribed to the ἄρβηλος and the straight line AB; the proof is for the particular case $AN = \frac{3}{2}BN$, and shows that the diameter of the inscribed circle $= \frac{6}{19}AB$.

Prop. 8 is of interest in connexion with the problem of



trisecting any angle. If AB be any chord of a circle with centre O, and BC on AB produced be made equal to the radius, draw CO meeting the circle in D, E; then will the arc BD be one-third of the arc AE (or BF, if EF be the chord through E parallel to AB). The problem is by this theorem reduced to a νεῦσις (cf. vol. i, p. 241).

Lastly, we may mention the elegant theorem about the area of the *Salinon* (presumably 'salt-cellar') in Prop. 14. *ACB* is a semicircle on *AB* as diameter, *AD*, *EB* are equal lengths measured from *A* and *B* on *AB*. Semicircles are drawn with *AD*, *EB* as diameters on the side towards *C*, and



a semicircle with *DE* as diameter is drawn on the other side of *AB*. *CF* is the perpendicular to *AB* through *O*, the centre of the semicircles *ACB*, *DFE*. Then is the area bounded by all the semicircles (the *Salinon*) equal to the circle on *CF* as diameter.

The Arabians, through whom the Book of Lemmas has reached us, attributed to Archimedes other works (1) on the Circle, (2) on the Heptagon in a Circle, (3) on Circles touching one another, (4) on Parallel Lines, (5) on Triangles, (6) on the properties of right-angled triangles, (7) a book of Data, (8) De clepsydris: statements which we are not in a position to check. But the author of a book on the finding of chords in a circle,[1] Abū'l Raiḥān Muḥ. al-Bīrūnī, quotes some alternative proofs as coming from the first of these works.

(δ) *Formula for area of triangle.*

More important, however, is the mention in this same work of Archimedes as the discoverer of two propositions hitherto attributed to Heron, the first being the problem of finding the perpendiculars of a triangle when the sides are given, and the second the famous formula for the area of a triangle in terms of the sides,

$$\sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

[1] See ***Bibliotheca mathematica***, xi_3, pp. 11-78.

Long as the present chapter is, it is nevertheless the most appropriate place for ERATOSTHENES of Cyrene. It was to him that Archimedes dedicated *The Method*, and the *Cattle-Problem* purports, by its heading, to have been sent through him to the mathematicians of Alexandria. It is evident from the preface to *The Method* that Archimedes thought highly of his mathematical ability. He was, indeed, recognized by his contemporaries as a man of great distinction in all branches of knowledge, though in each subject he just fell short of the highest place. On the latter ground he was called Beta, and another nickname applied to him, *Pentathlos*, has the same implication, representing as it does an all-round athlete who was not the first runner or wrestler but took the second prize in these contests as well as in others. He was very little younger than Archimedes; the date of his birth was probably 284 B.C. or thereabouts. He was a pupil of the philosopher Ariston of Chios, the grammarian Lysanias of Cyrene, and the poet Callimachus; he is said also to have been a pupil of Zeno the Stoic, and he may have come under the influence of Arcesilaus at Athens, where he spent a considerable time. Invited, when about 40 years of age, by Ptolemy Euergetes to be tutor to his son (Philopator), he became librarian at Alexandria; his obligation to Ptolemy he recognized by the column which he erected with a graceful epigram inscribed on it. This is the epigram, with which we are already acquainted (vol. i, p. 260), relating to the solutions, discovered up to date, of the problem of the duplication of the cube, and commending his own method by means of an appliance called μεσόλαβον, itself represented in bronze on the column.

Eratosthenes wrote a book with the title Πλατωνικός, and, whether it was a sort of commentary on the *Timaeus* of Plato, or a dialogue in which the principal part was played by Plato, it evidently dealt with the fundamental notions of mathematics in connexion with Plato's philosophy. It was naturally one of the important sources of Theon of Smyrna's work on the mathematical matters which it was necessary for the student of Plato to know; and Theon cites the work twice by name. It seems to have begun with the famous problem of Delos, telling the story quoted by Theon how the god required, as a means of stopping a plague, that the altar

there, which was cubical in form, should be doubled in size. The book evidently contained a disquisition on *proportion* (ἀναλογία); a quotation by Theon on this subject shows that Eratosthenes incidentally dealt with the fundamental definitions of geometry and arithmetic. The principles of music were discussed in the same work.

We have already described Eratosthenes's solution of the problem of Delos, and his contribution to the theory of arithmetic by means of his *sieve* (κόσκινον) for finding successive prime numbers.

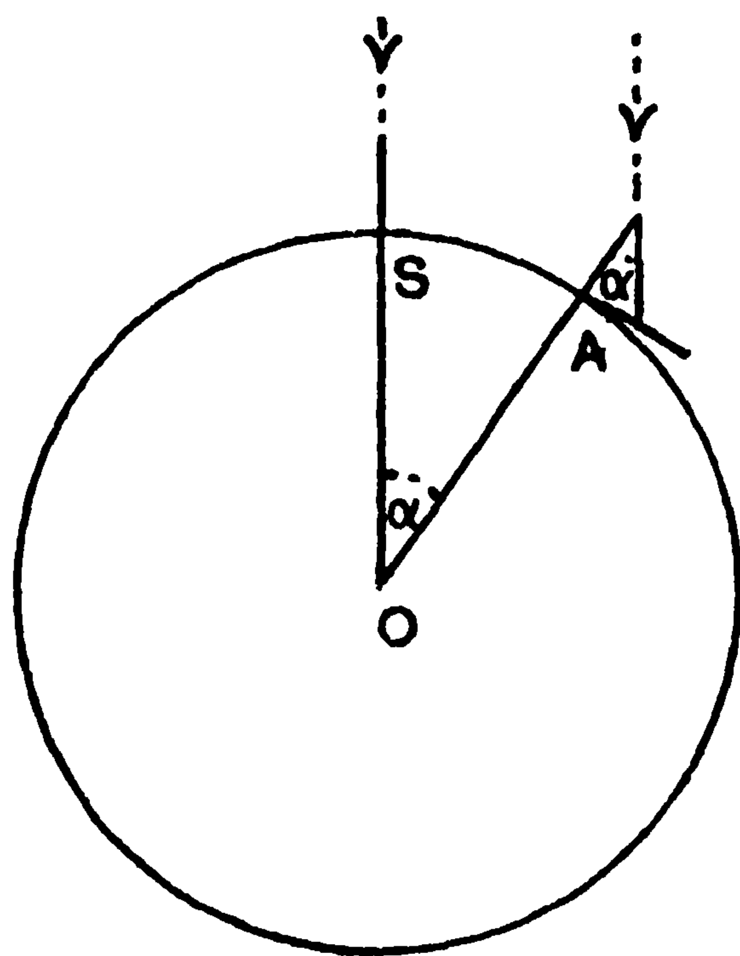
He wrote also an independent work *On means*. This was in two Books, and was important enough to be mentioned by Pappus along with works by Euclid, Aristaeus and Apollonius as forming part of the *Treasury of Analysis*[1]; this proves that it was a systematic geometrical treatise. Another passage of Pappus speaks of certain loci which Eratosthenes called 'loci with reference to means' (τόποι πρὸς μεσότητας)[2]; these were presumably discussed in the treatise in question. What kind of loci these were is quite uncertain; Pappus (if it is not an interpolator who speaks) merely says that these loci 'belong to the aforesaid classes of loci', but as the classes are numerous (including 'plane', 'solid', 'linear', 'loci on surfaces', &c.), we are none the wiser. Tannery conjectured that they were loci of points such that their distances from three fixed straight lines furnished a 'médiété', i.e. loci (straight lines and conics) which we should represent in trilinear coordinates by such equations as $2y = x+z$, $y^2 = xz$, $y(x+z) = 2xz$, $x(x-y) = z(y-z)$, $x(x-y) = y(y-z)$, the first three equations representing the arithmetic, geometric and harmonic means, while the last two represent the 'subcontraries' to the harmonic and geometric means respectively. Zeuthen has a different conjecture.[3] He points out that, if QQ' be the polar of a given point C with reference to a conic, and $CPOP'$ be drawn through C meeting QQ' in O and the conic in P, P', then CO is the harmonic mean to CP, CP'; the locus of O for all transversals CPP' is then the straight line QQ'. If A, G are points on PP' such that CA is the arithmetic, and CG the

[1] Pappus, vii, p. 636. 24. [2] *Ib.*, p. 662. 15 sq.
[3] Zeuthen, *Die Lehre von den Kegelschnitten im Altertum*, 1886, pp. 320, 321.

geometric mean between CP, CP', the loci of A, G respectively are conics. Zeuthen therefore suggests that these loci and the corresponding loci of the points on CPP' at a distance from C equal to the subcontraries of the geometric and harmonic means between CP and CP' are the 'loci with reference to means' of Eratosthenes; the latter two loci are 'linear', i.e. higher curves than conics. Needless to say, we have no confirmation of this conjecture.

Eratosthenes's measurement of the Earth.

But the most famous scientific achievement of Eratosthenes was his measurement of the earth. Archimedes mentions, as we have seen, that some had tried to prove that the circumference of the earth is about 300,000 stades. This was evidently the measurement based on observations made at Lysimachia (on the Hellespont) and Syene. It was observed that, while both these places were on one meridian, the head of Draco was in the zenith at Lysimachia, and Cancer in the zenith at Syene; the arc of the meridian separating the two in the heavens was taken to be 1/15th of the complete circle. The distance between the two towns was estimated at 20,000 stades, and accordingly the whole circumference of the earth was reckoned at 300,000 stades. Eratosthenes improved on this. He observed (1) that at Syene, at noon, at the summer solstice, the sun cast no shadow from an upright gnomon (this was confirmed by the observation that a well dug at the same place was entirely lighted up at the same time), while (2) at the same moment the gnomon fixed upright at Alexandria (taken to be on the same meridian with Syene) cast a shadow corresponding to an angle between the gnomon and the sun's rays of 1/50th of a complete circle or four right angles. The sun's rays are of course assumed to be parallel at the two places represented by S and A in the annexed figure. If α be the angle made at A by the sun's rays with the gnomon (OA produced), the angle SOA is also equal to



α, or 1/50th of four right angles. Now the distance from *S* to *A* was known by measurement to be 5,000 stades; it followed that the circumference of the earth was 250,000 stades. This is the figure given by Cleomedes, but Theon of Smyrna and Strabo both give it as 252,000 stades. The reason of the discrepancy is not known; it is possible that Eratosthenes corrected 250,000 to 252,000 for some reason, perhaps in order to get a figure divisible by 60 and, incidentally, a round number (700) of stades for one degree. If Pliny is right in saying that Eratosthenes made 40 stades equal to the Egyptian *σχοῖνος*, then, taking the *σχοῖνος* at 12,000 Royal cubits of 0·525 metres, we get 300 such cubits, or 157·5 metres, i.e. 516·73 feet, as the length of the stade. On this basis 252,000 stades works out to 24,662 miles, and the diameter of the earth to about 7,850 miles, only 50 miles shorter than the true polar diameter, a surprisingly close approximation, however much it owes to happy accidents in the calculation.

We learn from Heron's *Dioptra* that the measurement of the earth by Eratosthenes was given in a separate work *On the Measurement of the Earth*. According to Galen[1] this work dealt generally with astronomical or mathematical geography, treating of 'the size of the equator, the distance of the tropic and polar circles, the extent of the polar zone, the size and distance of the sun and moon, total and partial eclipses of these heavenly bodies, changes in the length of the day according to the different latitudes and seasons'. Several details are preserved elsewhere of results obtained by Eratosthenes, which were doubtless contained in this work. He is supposed to have estimated the distance between the tropic circles or twice the obliquity of the ecliptic at 11/83rds of a complete circle or 47° 42′ 39″; but from Ptolemy's language on this subject it is not clear that this estimate was not Ptolemy's own. What Ptolemy says is that he himself found the distance between the tropic circles to lie always between 47° 40′ and 47° 45′, 'from which we obtain *about* (σχεδόν) the same ratio as that of Eratosthenes, which Hipparchus also used. For the distance between the tropics becomes (or *is found to be*, γίνεται) very nearly 11 parts

Galen, *Instit. Logica*, 12 (p. 26 Kalbfleisch).

out of 83 contained in the whole meridian circle'.[1] The mean of Ptolemy's estimates, 47° 42′ 30″, is of course nearly 11/83rds of 360°. It is consistent with Ptolemy's language to suppose that Eratosthenes adhered to the value of the obliquity of the ecliptic discovered before Euclid's time, namely 24°, and Hipparchus does, in his extant *Commentary on the Phaenomena of Aratus and Eudoxus*, say that the summer tropic is 'very nearly 24° north of the equator'.

The *Doxographi* state that Eratosthenes estimated the distance of the moon from the earth at 780,000 stades and the distance of the sun from the earth at 804,000,000 stades (the versions of Stobaeus and Joannes Lydus admit 4,080,000 as an alternative for the latter figure, but this obviously cannot be right). Macrobius[2] says that Eratosthenes made the 'measure' of the sun to be 27 times that of the earth. It is not certain whether measure means 'solid content' or 'diameter' in this case; the other figures on record make the former more probable, in which case the diameter of the sun would be three times that of the earth. Macrobius also tells us that Eratosthenes's estimates of the distances of the sun and moon were obtained by means of lunar eclipses.

Another observation by Eratosthenes, namely that at Syene (which is under the summer tropic) and throughout a circle round it with a radius of 300 stades the upright gnomon throws no shadow at noon, was afterwards made use of by Posidonius in his calculation of the size of the sun. Assuming that the circle in which the sun apparently moves round the earth is 10,000 times the size of a circular section of the earth through its centre, and combining with this hypothesis the datum just mentioned, Posidonius arrived at 3,000,000 stades as the diameter of the sun.

Eratosthenes wrote a poem called *Hermes* containing a good deal of descriptive astronomy; only fragments of this have survived. The work *Catasterismi* (literally 'placings among the stars') which is extant can hardly be genuine in the form in which it has reached us; it goes back, however, to a genuine work by Eratosthenes which apparently bore the same name; alternatively it is alluded to as **Κατάλογοι** or by the general

[1] Ptolemy, *Syntaxis*, i. 12, pp. 67. 22–68. 6.
[2] Macrobius, *In Somn. Scip.* i. 20. 9.

word Ἀστρονομία (Suidas), which latter word is perhaps a mistake for Ἀστροθεσία corresponding to the title Ἀστροθεσίαι ζῳδίων found in the manuscripts. The work as we have it contains the story, mythological and descriptive, of the constellations, &c., under forty-four heads; there is little or nothing belonging to astronomy proper.

Eratosthenes is also famous as the first to attempt a scientific chronology beginning from the siege of Troy; this was the subject of his *Χρονογραφίαι*, with which must be connected the separate Ὀλυμπιονῖκαι in several books. Clement of Alexandria gives a short *résumé* of the main results of the former work, and both works were largely used by Apollodorus. Another lost work was on the Octaëteris (or eight-years' period), which is twice mentioned, by Geminus and Achilles; from the latter we learn that Eratosthenes regarded the work on the same subject attributed to Eudoxus as not genuine. His *Geographica* in three books is mainly known to us through Suidas's criticism of it. It began with a history of geography down to his own time; Eratosthenes then proceeded to mathematical geography, the spherical form of the earth, the negligibility in comparison with this of the unevennesses caused by mountains and valleys, the changes of features due to floods, earthquakes and the like. It would appear from Theon of Smyrna's allusions that Eratosthenes estimated the height of the highest mountain to be 10 stades or about 1/8000th part of the diameter of the earth.

XIV

CONIC SECTIONS. APOLLONIUS OF PERGA

A. HISTORY OF CONICS UP TO APOLLONIUS

Discovery of the conic sections by Menaechmus.

WE have seen that Menaechmus solved the problem of the two mean proportionals (and therefore the duplication of the cube) by means of conic sections, and that he is credited with the discovery of the three curves; for the epigram of Eratosthenes speaks of 'the *triads* of Menaechmus', whereas of course only two conics, the parabola and the rectangular hyperbola, actually appear in Menaechmus's solutions. The question arises, how did Menaechmus come to think of obtaining curves by cutting a cone? On this we have no information whatever. Democritus had indeed spoken of a section of a cone parallel and very near to the base, which of course would be a circle, since the cone would certainly be the right circular cone. But it is probable enough that the attention of the Greeks, whose observation nothing escaped, would be attracted to the shape of a section of a cone or a cylinder by a plane obliquely inclined to the axis when it occurred, as it often would, in real life; the case where the solid was cut right through, which would show an ellipse, would presumably be noticed first, and some attempt would be made to investigate the nature and geometrical measure of the elongation of the figure in relation to the circular sections of the same solid; these would in the first instance be most easily ascertained when the solid was a right cylinder; it would then be a natural question to investigate whether the curve arrived at by cutting the cone had the same property as that obtained by cutting the cylinder. As we have seen, the

observation that an ellipse can be obtained from a cylinder as well as a cone is actually made by Euclid in his *Phaenomena*: 'if', says Euclid, 'a cone or a cylinder be cut by a plane not parallel to the base, the resulting section is a section of an acute-angled cone which is similar to a θυρεός (shield).' After this would doubtless follow the question what sort of curves they are which are produced if we cut a cone by a plane which does not cut through the cone completely, but is either parallel or not parallel to a generator of the cone, whether these curves have the same property with the ellipse and with one another, and, if not, what exactly are their fundamental properties respectively.

As it is, however, we are only told how the first writers on conics obtained them in actual practice. We learn on the authority of Geminus[1] that the ancients defined a cone as the surface described by the revolution of a right-angled triangle about one of the sides containing the right angle, and that they knew no cones other than right cones. Of these they distinguished three kinds; according as the vertical angle of the cone was less than, equal to, or greater than a right angle, they called the cone acute-angled, right-angled, or obtuse-angled, and from each of these kinds of cone they produced one and only one of the three sections, the section being always made perpendicular to one of the generating lines of the cone; the curves were, on this basis, called 'section of an acute-angled cone' (= an ellipse), 'section of a right-angled cone' (= a parabola), and 'section of an obtuse-angled cone' (= a hyperbola) respectively. These names were still used by Euclid and Archimedes.

Menaechmus's probable procedure.

Menaechmus's constructions for his curves would presumably be the simplest and the most direct that would show the desired properties, and for the parabola nothing could be simpler than a section of a right-angled cone by a plane at right angles to one of its generators. Let *OBC* (Fig. 1) represent

[1] Eutocius, *Comm. on Conics* of Apollonius.

a section through the axis OL of a right-angled cone, and conceive a section through AG (perpendicular to OA) and at right angles to the plane of the paper.

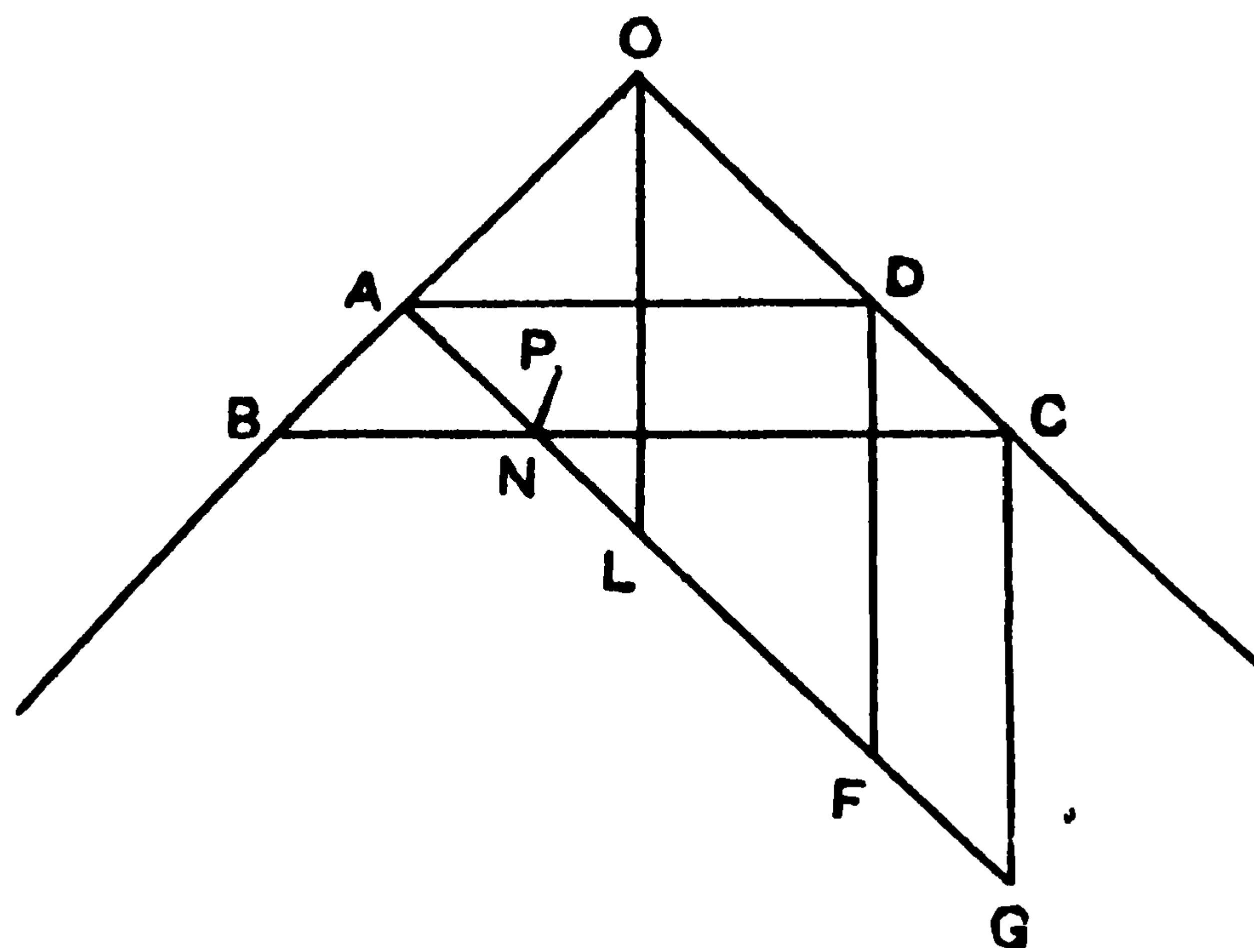


Fig. 1.

If P is any point on the curve, and PN perpendicular to AG, let BC be drawn through N perpendicular to the axis of the cone. Then P is on the circular section of the cone about BC as diameter.

Draw AD parallel to BC, and DF, CG parallel to OL meeting AL produced in F, G. Then AD, AF are both bisected by OL.

If now $$PN = y, \quad AN = x,$$

$$y^2 = PN^2 = BN \, . \, NC.$$

But B, A, C, G are concyclic, so that

$$BN \, . \, NC = AN \, . \, NG$$
$$= AN \, . \, AF$$
$$= AN \, . \, 2AL.$$

Therefore $$y^2 = AN \, . \, 2AL$$
$$= 2AL \, . \, x,$$

and $2AL$ is the 'parameter' of the principal ordinates y.

In the case of the hyperbola Menaechmus had to obtain the

particular hyperbola which we call rectangular or equilateral, and also to obtain its property with reference to its asymptotes, a considerable advance on what was necessary in the case of the parabola. Two methods of obtaining the particular hyperbola were possible, namely (1) to obtain the hyperbola arising from the section of any obtuse-angled cone by a plane at right angles to a generator, and then to show how a rectangular hyperbola can be obtained as a particular case by finding the vertical angle which the cone must have to give a rectangular hyperbola when cut in the particular way, or (2) to obtain the rectangular hyperbola direct by cutting another kind of cone by a section not necessarily perpendicular to a generator.

(1) Taking the first method, we draw (Fig. 2) a cone with its vertical angle BOC obtuse. Imagine a section perpendicular to the plane of the paper and passing through AG which is perpendicular to OB. Let GA produced meet CO produced in A', and complete the same construction as in the case of the parabola.

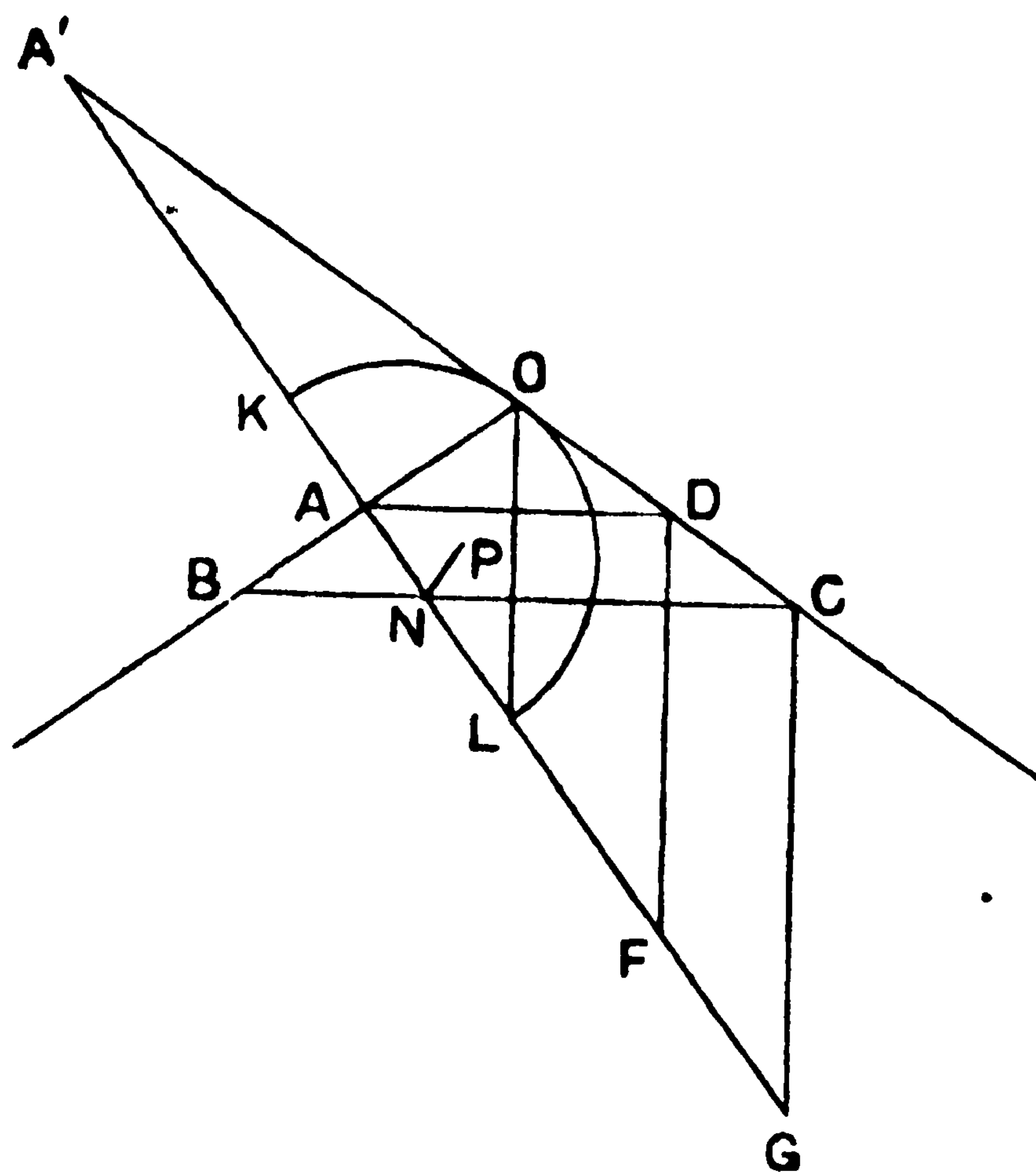


FIG. 2.

In this case we have

$$PN^2 = BN \,.\, NC = AN \,.\, NG.$$

For, let the right-angled cone HOK (Fig. 3) be cut by a plane through $A'AN$ parallel to the axis OM and cutting the sides of the axial triangle HOK in A', A, N respectively. Let P be the point on the curve for which PN is the principal ordinate. Draw OC parallel to HK. We have at once

Fig. 3.

$$PN^2 = HN \,.\, NK$$
$$= MK^2 - MN^2$$
$$= CN^2 - CA^2, \text{ since } MK = OM, \text{ and } MN = OC = CA.$$

This is the property of the rectangular hyperbola having $A'A$ as axis. To obtain a particular rectangular hyperbola with axis of given length we have only to choose the cutting plane so that the intercept $A'A$ may have the given length.

But Menaechmus had to prove the asymptote-property of his rectangular hyperbola. As he can hardly be supposed to have got as far as Apollonius in investigating the relations of the hyperbola to its asymptotes, it is probably safe to assume that he obtained the particular property in the simplest way, i.e. directly from the property of the curve in relation to its axes.

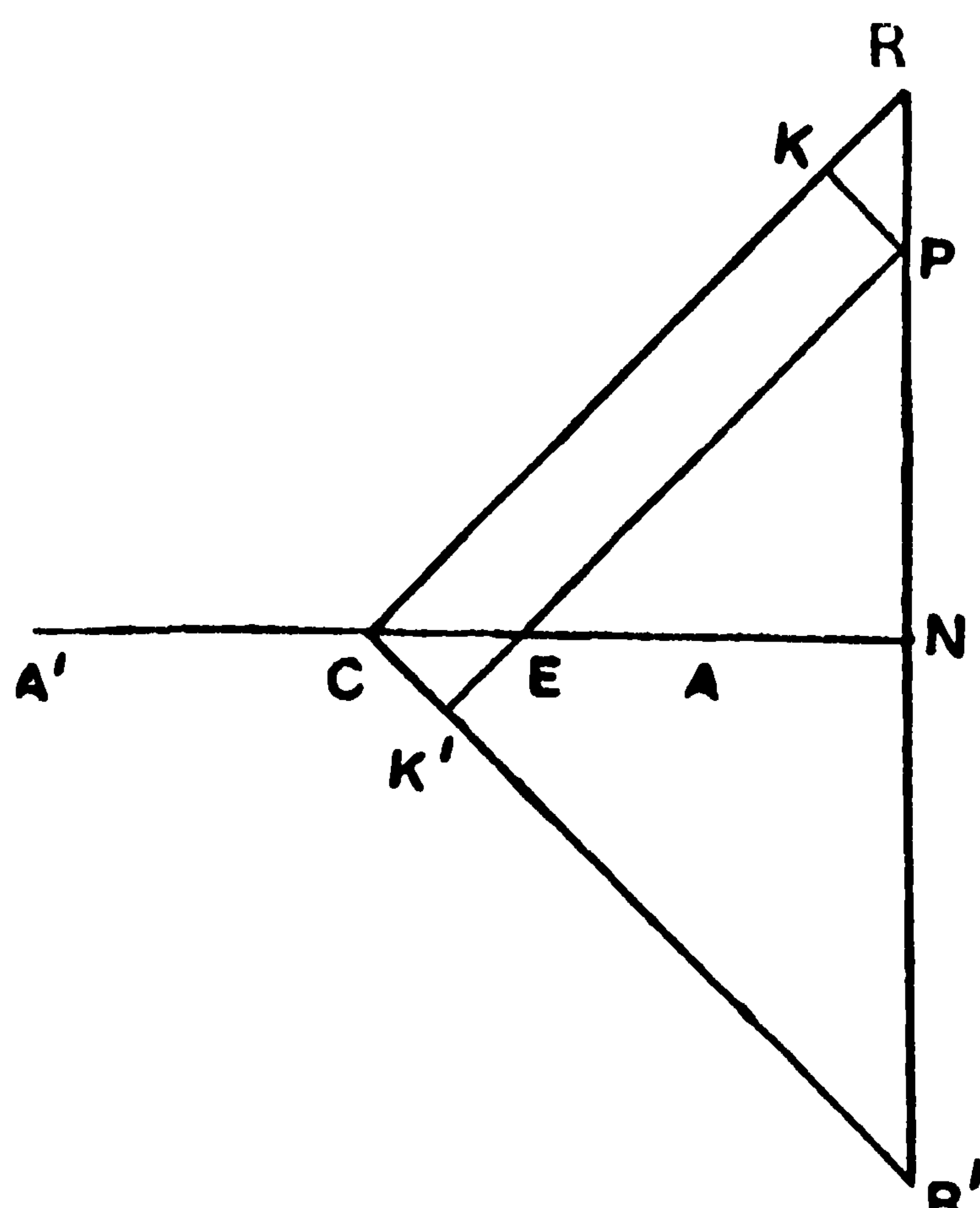


Fig. 4.

If (Fig. 4) CR, CR' be the asymptotes (which are therefore

at right angles) and $A'A$ the axis of a rectangular hyperbola, P any point on the curve, PN the principal ordinate, draw PK, PK' perpendicular to the asymptotes respectively. Let PN produced meet the asymptotes in R, R'.

Now, by the axial property,

$$\begin{aligned} CA^2 &= CN^2 - PN^2 \\ &= RN^2 - PN^2 \\ &= RP \,.\, PR' \\ &= 2PK \,.\, PK', \text{ since } \angle PRK \text{ is half a right angle;} \end{aligned}$$

therefore $$PK \,.\, PK' = \tfrac{1}{2}CA^2.$$

Works by Aristaeus and Euclid.

If Menaechmus was really the discoverer of the three conic sections at a date which we must put at about 360 or 350 B.C., the subject must have been developed very rapidly, for by the end of the century there were two considerable works on conics in existence, works which, as we learn from Pappus, were considered worthy of a place, alongside the *Conics* of Apollonius, in the *Treasury of Analysis*. Euclid flourished about 300 B.C., or perhaps 10 or 20 years earlier; but his *Conics* in four books was preceded by a work of Aristaeus which was still extant in the time of Pappus, who describes it as 'five books of *Solid Loci* connected (or continuous, συνεχῆ) with the conics'. Speaking of the relation of Euclid's *Conics* in four books to this work, Pappus says (if the passage is genuine) that Euclid gave credit to Aristaeus for his discoveries in conics and did not attempt to anticipate him or wish to construct anew the same system. In particular, Euclid, when dealing with what Apollonius calls the three- and four-line locus, 'wrote so much about the locus as was possible by means of the conics of Aristaeus, without claiming completeness for his demonstrations'.[1] We gather from these remarks that Euclid's *Conics* was a compilation and rearrangement of the geometry of the conics so far as known in his

[1] Pappus, vii, p. 678. 4.

time; whereas the work of Aristaeus was more specialized and more original.

'Solid loci' and 'solid problems'.

'Solid loci' are of course simply conics, but the use of the title 'Solid loci' instead of 'conics' seems to indicate that the work was in the main devoted to conics regarded as loci. As we have seen, 'solid loci' which are conics are distinguished from 'plane loci', on the one hand, which are straight lines and circles, and from 'linear loci' on the other, which are curves higher than conics. There is some doubt as to the real reason why the term 'solid loci' was applied to the conic sections. We are told that 'plane' loci are so called because they are generated in a plane (but so are some of the higher curves, such as the *quadratrix* and the spiral of Archimedes), and that 'solid loci' derived their name from the fact that they arise as sections of solid figures (but so do some higher curves, e.g. the spiric curves which are sections of the σπεῖρα or *tore*). But some light is thrown on the subject by the corresponding distinction which Pappus draws between 'plane', 'solid' and 'linear' *problems*.

'Those problems', he says, 'which can be solved by means of a straight line and a circumference of a circle may properly be called *plane*; for the lines by means of which such problems are solved have their origin in a plane. Those, however, which are solved by using for their discovery one or more of the sections of the cone have been called *solid*; for their construction requires the use of surfaces of solid figures, namely those of cones. There remains a third kind of problem, that which is called *linear*; for other lines (curves) besides those mentioned are assumed for the construction, the origin of which is more complicated and less natural, as they are generated from more irregular surfaces and intricate movements.'[1]

The true significance of the word 'plane' as applied to problems is evidently, not that straight lines and circles have their origin in a plane, but that the problems in question can be solved by the ordinary plane methods of transformation of

[1] Pappus, iv, p. 270. 5–17.

areas, manipulation of simple equations between areas and, in particular, the application of areas; in other words, plane problems were those which, if expressed algebraically, depend on equations of a degree not higher than the second. Problems, however, soon arose which did not yield to 'plane' methods. One of the first was that of the duplication of the cube, which was a problem of geometry in three dimensions or solid geometry. Consequently, when it was found that this problem could be solved by means of conics, and that no higher curves were necessary, it would be natural to speak of them as 'solid' loci, especially as they were in fact produced from sections of a solid figure, the cone. The propriety of the term would be only confirmed when it was found that, just as the duplication of the cube depended on the solution of a pure cubic equation, other problems such as the trisection of an angle, or the cutting of a sphere into two segments bearing a given ratio to one another, led to an equation between volumes in one form or another, i. e. a mixed cubic equation, and that this equation, which was also a solid problem, could likewise be solved by means of conics.

Aristaeus's *Solid Loci*.

The *Solid Loci* of Aristaeus, then, presumably dealt with loci which proved to be conic sections. In particular, he must have discussed, however imperfectly, the locus with respect to three or four lines the synthesis of which Apollonius says that he found inadequately worked out in Euclid's *Conics*. The theorems relating to this locus are enunciated by Pappus in this way:

'If three straight lines be given in position and from one and the same point straight lines be drawn to meet the three straight lines at given angles, and if the ratio of the rectangle contained by two of the straight lines so drawn to the square on the remaining one be given, then the point will lie on a solid locus given in position, that is, on one of the three conic sections. And if straight lines be so drawn to meet, at given angles, four straight lines given in position, and the ratio of the rectangle contained by two of the lines so drawn to the rectangle contained by the remaining two be given, then in

the same way the point will lie on a conic section given in position.'[1]

The reason why Apollonius referred in this connexion to Euclid and not to Aristaeus was probably that it was Euclid's work that was on the same lines as his own.

A very large proportion of the standard properties of conics admit of being stated in the form of locus-theorems; if a certain property holds with regard to a certain point, then that point lies on a conic section. But it may be assumed that Aristaeus's work was not merely a collection of the ordinary propositions transformed in this way; it would deal with new locus-theorems not implied in the fundamental definitions and properties of the conics, such as those just mentioned, the theorems of the three- and four-line locus. But one (to us) ordinary property, the focus-directrix property, was, as it seems to me, in all probability included.

Focus-directrix property known to Euclid.

It is remarkable that the directrix does not appear at all in Apollonius's great treatise on conics. The focal properties of the central conics are given by Apollonius, but the foci are obtained in a different way, without any reference to the directrix; the focus of the parabola does not appear at all. We may perhaps conclude that neither did Euclid's *Conics* contain the focus-directrix property; for, according to Pappus, Apollonius based his first four books on Euclid's four books, while filling them out and adding to them. Yet Pappus gives the proposition as a lemma to Euclid's *Surface-Loci*, from which we cannot but infer that it was assumed in that treatise without proof. If, then, Euclid did not take it from his own *Conics*, what more likely than that it was contained in Aristaeus's *Solid Loci*?

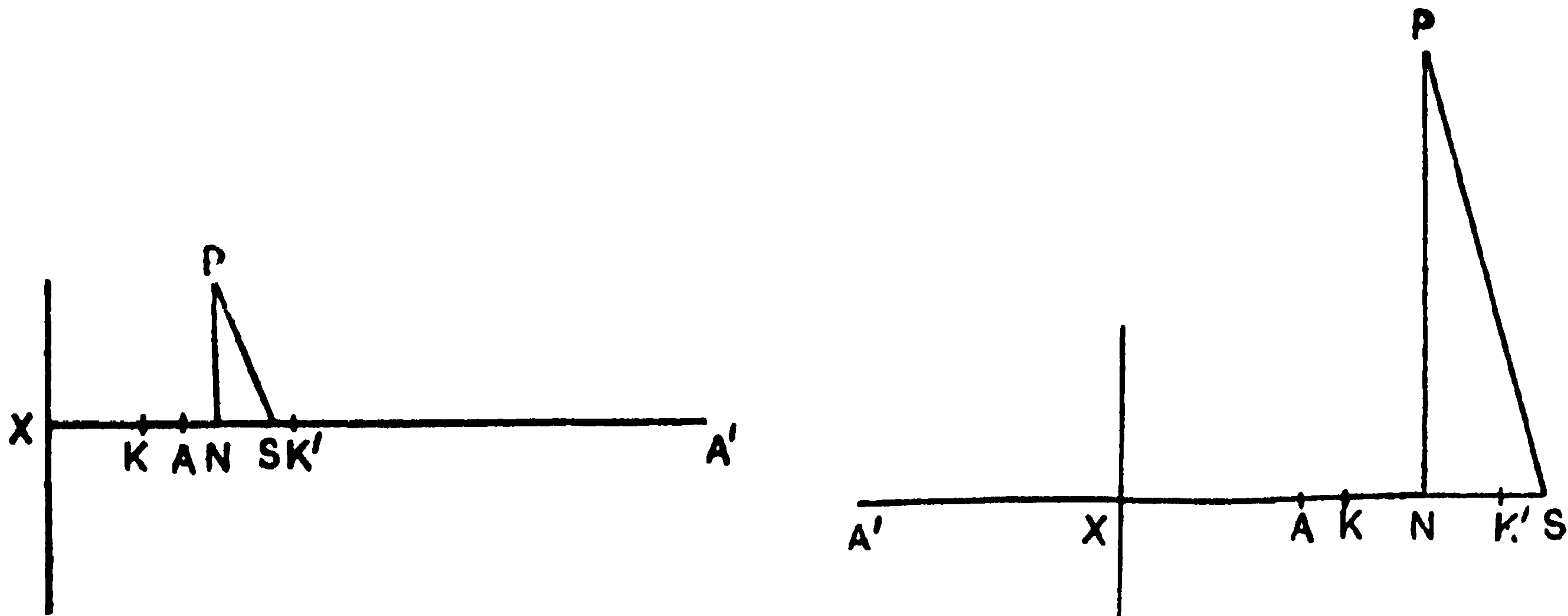
Pappus's enunciation of the theorem is to the effect that the locus of a point such that its distance from a given point is in a given ratio to its distance from a fixed straight line is a conic section, and is an ellipse, a parabola, or a hyperbola according as the given ratio is less than, equal to, or greater than unity.

[1] Pappus, vii, p. 678. 15–24.

Proof from Pappus.

The proof in the case where the given ratio is different from unity is shortly as follows.

Let S be the fixed point, SX the perpendicular from S on the fixed line. Let P be any point on the locus and PN



perpendicular to SX, so that SP is to NX in the given ratio (e);

thus $$e^2 = (PN^2+SN^2):NX^2.$$

Take K on SX such that

$$e^2 = SN^2:NK^2;$$

then, if K' be another point on SN, produced if necessary, such that $NK = NK'$,

$$e^2:1 = (PN^2+SN^2):NX^2 = SN^2:NK^2$$
$$= PN^2:(NX^2-NK^2)$$
$$= PN^2:XK\,.\,XK'.$$

The positions of N, K, K' change with the position of P. If A, A' be the points on which N falls when K, K' coincide with X respectively, we have

$$SA:AX = SN:NK = e:1 = SN:NK' = SA':A'X.$$

Therefore $\quad SX:SA = SK:SN = (1+e):e,$

whence $\quad (1+e):e = (SX-SK):(SA-SN)$

$$= XK:AN.$$

Similarly it can be shown that

$$(1 \sim e) : e = XK' : A'N.$$

By multiplication, $XK . XK' : AN . A'N = (1 \sim e^2) : e^2$;

and it follows from above, *ex aequali*, that

$$PN^2 : AN . A'N = (1 \sim e^2) : 1,$$

which is the property of a central conic.

When $e < 1$, A and A' lie on the same side of X, while N lies on AA', and the conic is an ellipse; when $e > 1$, A and A' lie on opposite sides of X, while N lies on $A'A$ produced, and the conic is a hyperbola.

The case where $e = 1$ and the curve is a parabola is easy and need not be reproduced here.

The treatise would doubtless contain other loci of types similar to that which, as Pappus says, was used for the trisection of an angle: I refer to the proposition already quoted (vol. i, p. 243) that, if A, B are the base angles of a triangle with vertex P, and $\angle B = 2\angle A$, the locus of P is a hyperbola with eccentricity 2.

Propositions included in Euclid's *Conics*.

That Euclid's *Conics* covered much of the same ground as the first three Books of Apollonius is clear from the language of Apollonius himself. Confirmation is forthcoming in the quotations by Archimedes of propositions (1) 'proved in the elements of conics', or (2) assumed without remark as already known. The former class include the fundamental ordinate properties of the conics in the following forms:

(1) for the ellipse,

$$PN^2 : AN . A'N = P'N'^2 : AN' . A'N' = BC^2 : AC^2;$$

(2) for the hyperbola,

$$PN^2 : AN . A'N = P'N'^2 : AN' . A'N';$$

(3) for the parabola, $PN^2 = p_a . AN$;

the principal tangent properties of the parabola;

the property that, if there are two tangents drawn from one point to any conic section whatever, and two intersecting

chords drawn parallel to the tangents respectively, the rectangles contained by the segments of the chords respectively are to one another as the squares of the parallel tangents;

the by no means easy proposition that, if in a parabola the diameter through P bisects the chord QQ' in V, and QD is drawn perpendicular to PV, then

$$QV^2 : QD^2 = p : p_a,$$

where p_a is the parameter of the principal ordinates and p is the parameter of the ordinates to the diameter PV.

Conic sections in Archimedes.

But we must equally regard Euclid's *Conics* as the source from which Archimedes took most of the other ordinary properties of conics which he assumes without proof. Before summarizing these it will be convenient to refer to Archimedes's terminology. We have seen that the axes of an ellipse are not called axes but *diameters*, greater and lesser; the axis of a parabola is likewise its *diameter* and the other diameters are 'lines parallel to the diameter', although in a segment of a parabola the diameter bisecting the base is the 'diameter' of the segment. The two 'diameters' (axes) of an ellipse are *conjugate*. In the case of the hyperbola the 'diameter' (axis) is the portion of it within the (single-branch) hyperbola; the centre is not called the 'centre', but the point in which the 'nearest lines to the section of an obtuse-angled cone' (the asymptotes) meet; the half of the axis (CA) is 'the line adjacent to the axis' (of the hyperboloid of revolution obtained by making the hyperbola revolve about its 'diameter'), and $A'A$ is double of this line. Similarly CP is the line 'adjacent to the axis' of a segment of the hyperboloid, and $P'P$ double of this line. It is clear that Archimedes did not yet treat the two branches of a hyperbola as forming one curve; this was reserved for Apollonius.

The main properties of conics assumed by Archimedes in addition to those above mentioned may be summarized thus.

Central Conics.

1. The property of the ordinates to any diameter PP',

$$QV^2 : PV . P'V = Q'V'^2 : PV' . P'V'.$$

In the case of the hyperbola Archimedes does not give any expression for the constant ratios $PN^2 : AN . A'N$ and $QV^2 : PV . P'V$ respectively, whence we conclude that he had no conception of diameters or radii of a hyperbola not meeting the curve.

2. The straight line drawn from the centre of an ellipse, or the point of intersection of the asymptotes of a hyperbola, through the point of contact of any tangent, bisects all chords parallel to the tangent.

3. In the ellipse the tangents at the extremities of either of two conjugate diameters are both parallel to the other diameter.

4. If in a hyperbola the tangent at P meets the transverse axis in T, and PN is the principal ordinate, $AN > AT$. (It is not easy to see how this could be proved except by means of the general property that, if PP' be any diameter of a hyperbola, QV the ordinate to it from Q, and QT the tangent at Q meeting $P'P$ in T, then $TP : TP' = PV : P'V$.)

5. If a cone, right or oblique, be cut by a plane meeting all the generators, the section is either a circle or an ellipse.

6. If a line between the asymptotes meets a hyperbola and is bisected at the point of concourse, it will touch the hyperbola.

7. If x, y are straight lines drawn, in fixed directions respectively, from a point on a hyperbola to meet the asymptotes, the rectangle xy is constant.

8. If PN be the principal ordinate of P, a point on an ellipse, and if NP be produced to meet the auxiliary circle in p, the ratio $pN : PN$ is constant.

9. The criteria of similarity of conics and segments of conics are assumed in practically the same form as Apollonius gives them.

The Parabola.

1. The fundamental properties appear in the alternative forms

$$PN^2 : P'N'^2 = AN : AN', \text{ or } PN^2 = p_a . AN,$$

$$QV^2 : Q'V'^2 = PV : PV', \text{ or } QV^2 = p . PV.$$

Archimedes applies the term *parameter* (ἁ παρ' ἂν δύνανται αἱ ἀπὸ τᾶς τομᾶς) to the parameter of the principal ordinates

only: p is simply the line to which the rectangle equal to QV^2 and of width equal to PV is applied.

2. Parallel chords are bisected by one straight line parallel to the axis, which passes through the point of contact of the tangent parallel to the chords.

3. If the tangent at Q meet the diameter PV in T, and QV be the ordinate to the diameter, $PV = PT$.

By the aid of this proposition a tangent to the parabola can be drawn (*a*) at a point on it, (*b*) parallel to a given chord.

4. Another proposition assumed is equivalent to the property of the subnormal, $NG = \frac{1}{2}p_a$.

5. If QQ' be a chord of a parabola perpendicular to the axis and meeting the axis in M, while QVq another chord parallel to the tangent at P meets the diameter through P in V, and RHK is the principal ordinate of any point R on the curve meeting PV in H and the axis in K, then $PV:PH >$ or $= MK:KA$; 'for this is proved' (*On Floating Bodies*, II. 6).

Where it was proved we do not know; the proof is not altogether easy.[1]

6. All parabolas are similar.

As we have seen, Archimedes had to specialize in the parabola for the purpose of his treatises on the *Quadrature of the Parabola, Conoids and Spheroids, Floating Bodies*, Book II, and *Plane Equilibriums*, Book II; consequently he had to prove for himself a number of special propositions, which have already been given in their proper places. A few others are assumed without proof, doubtless as being easy deductions from the propositions which he does prove. They refer mainly to similar parabolic segments so placed that their bases are in one straight line and have one common extremity.

1. If any three similar and similarly situated parabolic segments BQ_1, BQ_2, BQ_3 lying along the same straight line as bases ($BQ_1 < BQ_2 < BQ_3$), and if E be any point on the tangent at B to one of the segments, and EO a straight line through E parallel to the axis of one of the segments and meeting the segments in R_3, R_2, R_1 respectively and BQ_3 in O, then

$$R_3R_2 : R_2R_1 = (Q_2Q_3 : BQ_3) . (BQ_1 : Q_1Q_2).$$

[1] See *Apollonius of Perga*, ed. Heath, p. liv.

2. If two similar parabolic segments with bases BQ_1, BQ_2 be placed as in the last proposition, and if BR_1R_2 be any straight line through B meeting the segments in R_1, R_2 respectively,

$$BQ_1 : BQ_2 = BR_1 : BR_2.$$

These propositions are easily deduced from the theorem proved in the *Quadrature of the Parabola*, that, if through E, a point on the tangent at B, a straight line ERO be drawn parallel to the axis and meeting the curve in R and any chord BQ through B in O, then

$$ER : RO = BO : OQ.$$

3. On the strength of these propositions Archimedes assumes the solution of the problem of placing, between two parabolic segments similar to one another and placed as in the above propositions, a straight line of a given length and in a direction parallel to the diameters of either parabola.

Euclid and Archimedes no doubt adhered to the old method of regarding the three conics as arising from sections of three kinds of right circular cones (right-angled, obtuse-angled and acute-angled) by planes drawn in each case at right angles to a generator of the cone. Yet neither Euclid nor Archimedes was unaware that the 'section of an acute-angled cone', or ellipse, could be otherwise produced. Euclid actually says in his *Phaenomena* that 'if a cone or cylinder (presumably right) be cut by a plane not parallel to the base, the resulting section is a section of an acute-angled cone which is similar to a θυρεός (shield)'. Archimedes knew that the non-circular sections even of an oblique circular cone made by planes cutting all the generators are ellipses; for he shows us how, given an ellipse, to draw a cone (in general oblique) of which it is a section and which has its vertex outside the plane of the ellipse on any straight line through the centre of the ellipse in a plane at right angles to the ellipse and passing through one of its axes, whether the straight line is itself perpendicular or not perpendicular to the plane of the ellipse; drawing a cone in this case of course means finding the circular sections of the surface generated by a straight line always passing through the given vertex and all the several points of the given ellipse. The method of proof would equally serve

for the other two conics, the hyperbola and parabola, and we can scarcely avoid the inference that Archimedes was equally aware that the parabola and the hyperbola could be found otherwise than by the old method.

The first, however, to base the theory of conics on the production of all three in the most general way from any kind of circular cone, right or oblique, was Apollonius, to whose work we now come.

B. APOLLONIUS OF PERGA

Hardly anything is known of the life of Apollonius except that he was born at Perga, in Pamphylia, that he went when quite young to Alexandria, where he studied with the successors of Euclid and remained a long time, and that he flourished (γέγονε) in the reign of Ptolemy Euergetes (247–222 B.C.). Ptolemaeus Chennus mentions an astronomer of the same name, who was famous during the reign of Ptolemy Philopator (222–205 B.C.), and it is clear that our Apollonius is meant. As Apollonius dedicated the fourth and following Books of his *Conics* to King Attalus I (241–197 B.C.) we have a confirmation of his approximate date. He was probably born about 262 B.C., or 25 years after Archimedes. We hear of a visit to Pergamum, where he made the acquaintance of Eudemus of Pergamum, to whom he dedicated the first two Books of the *Conics* in the form in which they have come down to us; they were the first two instalments of a second edition of the work.

The text of the *Conics*.

The *Conics* of Apollonius was at once recognized as the authoritative treatise on the subject, and later writers regularly cited it when quoting propositions in conics. Pappus wrote a number of lemmas to it; Serenus wrote a commentary, as also, according to Suidas, did Hypatia. Eutocius (fl. A.D. 500) prepared an edition of the first four Books and wrote a commentary on them; it is evident that he had before him slightly differing versions of the completed work, and he may also have had the first unrevised edition which had got into premature circulation, as Apollonius himself complains in the Preface to Book I.

The edition of Eutocius suffered interpolations which were probably made in the ninth century when, under the auspices of Leon, mathematical studies were revived at Constantinople; for it was at that date that the uncial manuscripts were written, from which our best manuscripts, V (= Cod. Vat. gr. 206 of the twelfth to thirteenth century) for the *Conics*, and W (= Cod. Vat. gr. 204 of the tenth century) for Eutocius, were copied.

Only the first four Books survive in Greek; the eighth Book is altogether lost, but the three Books V–VII exist in Arabic. It was Aḥmad and al-Ḥasan, two sons of Muḥ. b. Mūsā b. Shākir, who first contemplated translating the *Conics* into Arabic. They were at first deterred by the bad state of their manuscripts; but afterwards Aḥmad obtained in Syria a copy of Eutocius's edition of Books I–IV and had them translated by Hilāl b. Abī Hilāl al-Ḥimṣī (died 883/4). Books V–VII were translated, also for Aḥmad, by Thābit b. Qurra (826 901) from another manuscript. Naṣīraddīn's recension of this translation of the seven Books, made in 1248, is represented by two copies in the Bodleian, one of the year 1301 (No. 943) and the other of 1626 containing Books V–VII only (No. 885).

A Latin translation of Books I–IV was published by Johannes Baptista Memus at Venice in 1537; but the first important edition was the translation by Commandinus (Bologna, 1566), which included the lemmas of Pappus and the commentary of Eutocius, and was the first attempt to make the book intelligible by means of explanatory notes. For the Greek text Commandinus used Cod. Marcianus 518 and perhaps also Vat. gr. 205, both of which were copies of V, but not V itself.

The first published version of Books V–VII was a Latin translation by Abraham Echellensis and Giacomo Alfonso Borelli (Florence, 1661) of a reproduction of the Books written in 983 by Abū 'l Fatḥ al-Iṣfahānī.

The *editio princeps* of the Greek text is the monumental work of Halley (Oxford, 1710). The original intention was that Gregory should edit the four Books extant in Greek, with Eutocius's commentary and a Latin translation, and that Halley should translate Books V–VII from the Arabic into

Latin. Gregory, however, died while the work was proceeding, and Halley then undertook responsibility for the whole. The Greek manuscripts used were two, one belonging to Savile and the other lent by D. Baynard; their whereabouts cannot apparently now be traced, but they were both copies of Paris. gr. 2356, which was copied in the sixteenth century from Paris. gr. 2357 of the sixteenth century, itself a copy of V. For the three Books in Arabic Halley used the Bodleian MS. 885, but also consulted (*a*) a compendium of the three Books by 'Abdelmelik al-Shīrāzī (twelfth century), also in the Bodleian (913), (*b*) Borelli's edition, and (*c*) Bodl. 943 above mentioned, by means of which he revised and corrected his translation when completed. Halley's edition is still, so far as I know, the only available source for Books V–VII, except for the beginning of Book V (up to Prop. 7) which was edited by L. Nix (Leipzig, 1889).

The Greek text of Books I–IV is now available, with the commentaries of Eutocius, the fragments of Apollonius, &c., in the definitive edition of Heiberg (Teubner, 1891–3).

Apollonius's own account of the *Conics*.

A general account of the contents of the great work which, according to Geminus, earned for him the title of the 'great geometer' cannot be better given than in the words of the writer himself. The prefaces to the several Books contain interesting historical details, and, like the prefaces of Archimedes, state quite plainly and simply in what way the treatise differs from those of his predecessors, and how much in it is claimed as original. The strictures of Pappus (or more probably his interpolator), who accuses him of being a braggart and unfair towards his predecessors, are evidently unfounded. The prefaces are quoted by v. Wilamowitz-Moellendorff as specimens of admirable Greek, showing how perfect the style of the great mathematicians could be when they were free from the trammels of mathematical terminology.

Book I. General Preface.

Apollonius to Eudemus, greeting.

If you are in good health and things are in other respects as you wish, it is well; with me too things are moderately

well. During the time I spent with you at Pergamum I observed your eagerness to become acquainted with my work in conics; I am therefore sending you the first book, which I have corrected, and I will forward the remaining books when I have finished them to my satisfaction. I dare say you have not forgotten my telling you that I undertook the investigation of this subject at the request of Naucrates the geometer, at the time when he came to Alexandria and stayed with me, and, when I had worked it out in eight books, I gave them to him at once, too hurriedly, because he was on the point of sailing; they had therefore not been thoroughly revised, indeed I had put down everything just as it occurred to me, postponing revision till the end. Accordingly I now publish, as opportunities serve from time to time, instalments of the work as they are corrected. In the meantime it has happened that some other persons also, among those whom I have met, have got the first and second books before they were corrected; do not be surprised therefore if you come across them in a different shape.

Now of the eight books the first four form an elementary introduction. The first contains the modes of producing the three sections and the opposite branches (of the hyperbola), and the fundamental properties subsisting in them, worked out more fully and generally than in the writings of others. The second book contains the properties of the diameters and the axes of the sections as well as the asymptotes, with other things generally and necessarily used for determining limits of possibility (*διορισμοί*); and what I mean by diameters and axes respectively you will learn from this book. The third book contains many remarkable theorems useful for the syntheses of solid loci and for *diorismi*; the most and prettiest of these theorems are new, and it was their discovery which made me aware that Euclid did not work out the synthesis of the locus with respect to three and four lines, but only a chance portion of it, and that not successfully; for it was not possible for the said synthesis to be completed without the aid of the additional theorems discovered by me. The fourth book shows in how many ways the sections of cones can meet one another and the circumference of a circle; it contains other things in addition, none of which have been discussed by earlier writers, namely the questions in how many points a section of a cone or a circumference of a circle can meet [a double-branch hyperbola, or two double-branch hyperbolas can meet one another].

The rest of the books are more by way of surplusage (*περιουσιαστικώτερα*): one of them deals somewhat fully with

minima and *maxima*, another with equal and similar sections of cones, another with theorems of the nature of determinations of limits, and the last with determinate conic problems. But of course, when all of them are published, it will be open to all who read them to form their own judgement about them, according to their own individual tastes. Farewell.

The preface to Book II merely says that Apollonius is sending the second Book to Eudemus by his son Apollonius, and begs Eudemus to communicate it to earnest students of the subject, and in particular to Philonides the geometer whom Apollonius had introduced to Eudemus at Ephesus. There is no preface to Book III as we have it, although the preface to Book IV records that it also was sent to Eudemus.

Preface to Book IV.

Apollonius to Attalus, greeting.

Some time ago I expounded and sent to Eudemus of Pergamum the first three books of my conics which I have compiled in eight books, but, as he has passed away, I have resolved to dedicate the remaining books to you because of your earnest desire to possess my works. I am sending you on this occasion the fourth book. It contains a discussion of the question, in how many points at most it is possible for sections of cones to meet one another and the circumference of a circle, on the assumption that they do not coincide throughout, and further in how many points at most a section of a cone or the circumference of a circle can meet the hyperbola with two branches, [or two double-branch hyperbolas can meet one another]; and, besides these questions, the book considers a number of others of a similar kind. Now the first question Conon expounded to Thrasydaeus, without, however, showing proper mastery of the proofs, and on this ground Nicoteles of Cyrene, not without reason, fell foul of him. The second matter has merely been mentioned by Nicoteles, in connexion with his controversy with Conon, as one capable of demonstration; but I have not found it demonstrated either by Nicoteles himself or by any one else. The third question and the others akin to it I have not found so much as noticed by any one. All the matters referred to, which I have not found anywhere, required for their solution many and various novel theorems, most of which I have, as a matter of fact, set out in the first three books, while the rest are contained in the present book. These theorems are of considerable use both for the syntheses of problems and for

diorismi. Nicoteles indeed, on account of his controversy with Conon, will not have it that any use can be made of the discoveries of Conon for the purpose of *diorismi*; he is, however, mistaken in this opinion, for, even if it is possible, without using them at all, to arrive at results in regard to limits of possibility, yet they at all events afford a readier means of observing some things, e.g. that several or so many solutions are possible, or again that no solution is possible; and such foreknowledge secures a satisfactory basis for investigations, while the theorems in question are again useful for the analyses of *diorismi*. And, even apart from such usefulness, they will be found worthy of acceptance for the sake of the demonstrations themselves, just as we accept many other things in mathematics for this reason and for no other.

The prefaces to Books V VII now to be given are reproduced for Book V from the translation of L. Nix and for Books VI, VII from that of Halley.

Preface to Book V.

Apollonius to Attalus, greeting.

In this fifth book I have laid down propositions relating to *maximum* and *minimum* straight lines. You must know that my predecessors and contemporaries have only superficially touched upon the investigation of the shortest lines, and have only proved what straight lines touch the sections and. conversely, what properties they have in virtue of which they are tangents. For my part, I have proved these properties in the first book (without however making any use, in the proofs, of the doctrine of the shortest lines), inasmuch as I wished to place them in close connexion with that part of the subject in which I treat of the production of the three conic sections, in order to show at the same time that in each of the three sections countless properties and necessary results appear, as they do with reference to the original (transverse) diameter. The propositions in which I discuss the shortest lines I have separated into classes, and I have dealt with each individual case by careful demonstration; I have also connected the investigation of them with the investigation of the greatest lines above mentioned, because I considered that those who cultivate this science need them for obtaining a knowledge of the analysis, and determination of limits of possibility, of problems as well as for their synthesis: in addition to which, the subject is one of those which seem worthy of study for their own sake. Farewell.

Preface to Book VI.

Apollonius to Attalus, greeting.

I send you the sixth book of the conics, which embraces propositions about conic sections and segments of conics equal and unequal, similar and dissimilar, besides some other matters left out by those who have preceded me. In particular, you will find in this book how, in a given right cone, a section can be cut which is equal to a given section, and how a right cone can be described similar to a given cone but such as to contain a given conic section. And these matters in truth I have treated somewhat more fully and clearly than those who wrote before my time on these subjects. Farewell.

Preface to Book VII.

Apollonius to Attalus, greeting.

I send to you with this letter the seventh book on conic sections. In it are contained a large number of new propositions concerning diameters of sections and the figures described upon them; and all these propositions have their uses in many kinds of problems, especially in the determination of the limits of their possibility. Several examples of these occur in the determinate conic problems solved and demonstrated by me in the eighth book, which is by way of an appendix, and which I will make a point of sending to you as soon as possible. Farewell.

Extent of claim to originality.

We gather from these prefaces a very good idea of the plan followed by Apollonius in the arrangement of the subject and of the extent to which he claims originality. The first four Books form, as he says, an elementary introduction, by which he means an exposition of the elements of conics, that is, the definitions and the fundamental propositions which are of the most general use and application; the term 'elements' is in fact used with reference to conics in exactly the same sense as Euclid uses it to describe his great work. The remaining Books beginning with Book V are devoted to more specialized investigation of particular parts of the subject. It is only for a very small portion of the *content* of the treatise that Apollonius claims originality; in the first three Books the claim is confined to certain propositions bearing on the 'locus with respect to three or four lines'; and in the fourth Book (on the number of points at which two conics

straight line through the centre of the circle perpendicular to its plane, a straight line passing through the point and produced indefinitely in both directions is made to move, while always passing through the fixed point, so as to pass successively through all the points of the circle; the straight line thus describes a double cone which is in general oblique or, as Apollonius calls it, *scalene.* Then, before proceeding to the geometry of a cone, Apollonius gives a number of definitions which, though of course only required for conics, are stated as applicable to any curve.

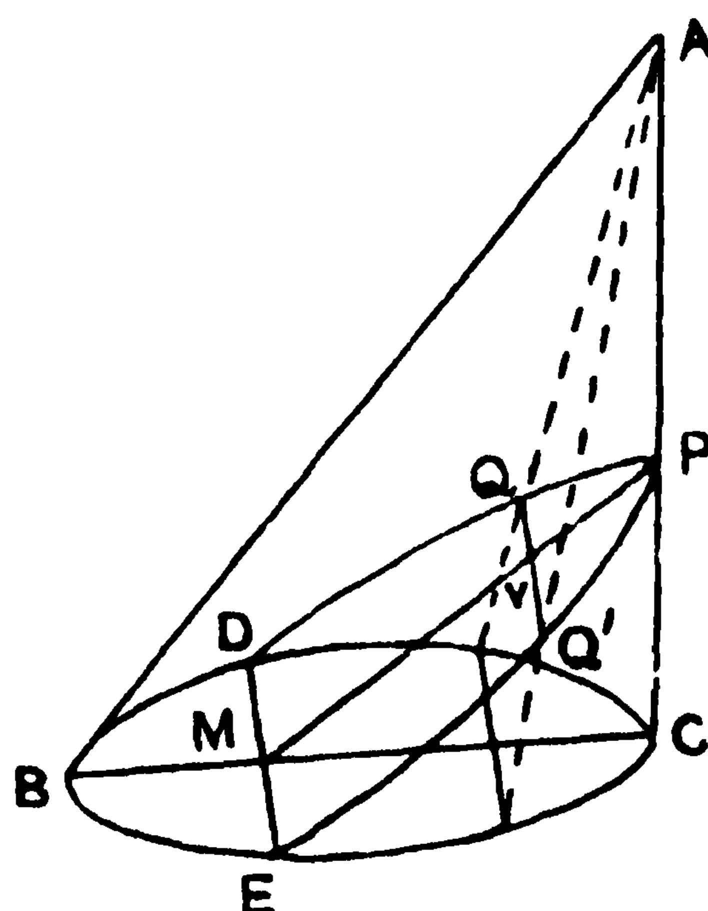
'In any curve,' says Apollonius, 'I give the name *diameter* to any straight line which, drawn from the curve, bisects all the straight lines drawn in the curve (chords) parallel to any straight line, and I call the extremity of the straight line (i.e. the diameter) which is at the curve a *vertex* of the curve and each of the parallel straight lines (chords) an ordinate (lit. drawn ordinate-wise, *τεταγμένως κατῆχθαι*) to the diameter.'

He then extends these terms to a pair of curves (the primary reference being to the double-branch hyperbola), giving the name *transverse diameter* to any straight line bisecting all the chords in both curves which are parallel to a given straight line (this gives two vertices where the diameter meets the curves respectively), and the name *erect diameter* (*ὀρθία*) to any straight line which bisects all straight lines drawn between one curve and the other which are parallel to any straight line; the *ordinates* to any diameter are again the parallel straight lines bisected by it. *Conjugate diameters* in any curve or pair of curves are straight lines each of which bisects chords parallel to the other. *Axes* are the particular diameters which cut at right angles the parallel chords which they bisect; and *conjugate axes* are related in the same way as conjugate diameters. Here we have practically our modern definitions, and there is a great advance on Archimedes's terminology.

The conics obtained in the most general way from an oblique cone.

Having described a cone (in general oblique), Apollonius defines the *axis* as the straight line drawn from the vertex to

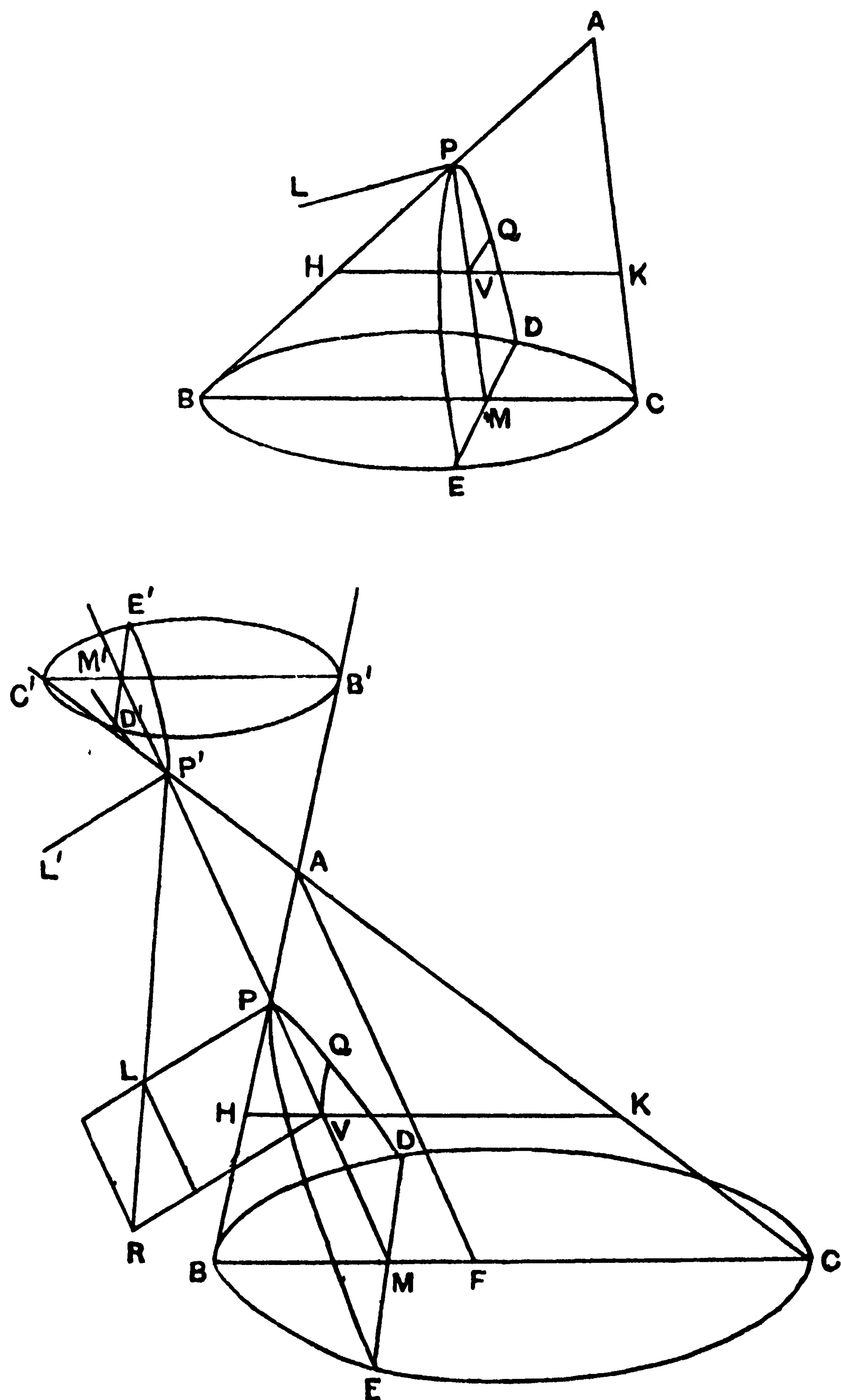
the centre of the circular base. After proving that all sections parallel to the base are also circles, and that there is another set of circular sections subcontrary to these, he proceeds to consider sections of the cone drawn in any manner. Taking any triangle through the axis (the base of the triangle being consequently a diameter of the circle which is the base of the cone), he is careful to make his section cut the base in a straight line perpendicular to the particular diameter which is the base of the axial triangle. (There is no loss of generality in this, for, if any section is taken, without reference to any axial triangle, we have only to select the particular axial triangle the base of which is that diameter of the circular base which is at right angles to the straight line in which the section of the cone cuts the base.) Let ABC be any axial triangle, and let any section whatever cut the base in a straight line DE at right angles to BC; if then PM be the intersection of the cutting plane and the axial triangle, and if QQ' be any chord in the section parallel to DE, Apollonius proves that QQ' is bisected by PM. In other words, PM is a *diameter* of the section. Apollonius is careful to explain that,



'if the cone is a right cone, the straight line in the base (DE) will be at right angles to the common section (PM) of the cutting plane and the triangle through the axis, but, if the cone is scalene, it will not in general be at right angles to PM, but will be at right angles to it only when the plane through the axis (i.e. the axial triangle) is at right angles to the base of the cone' (I. 7).

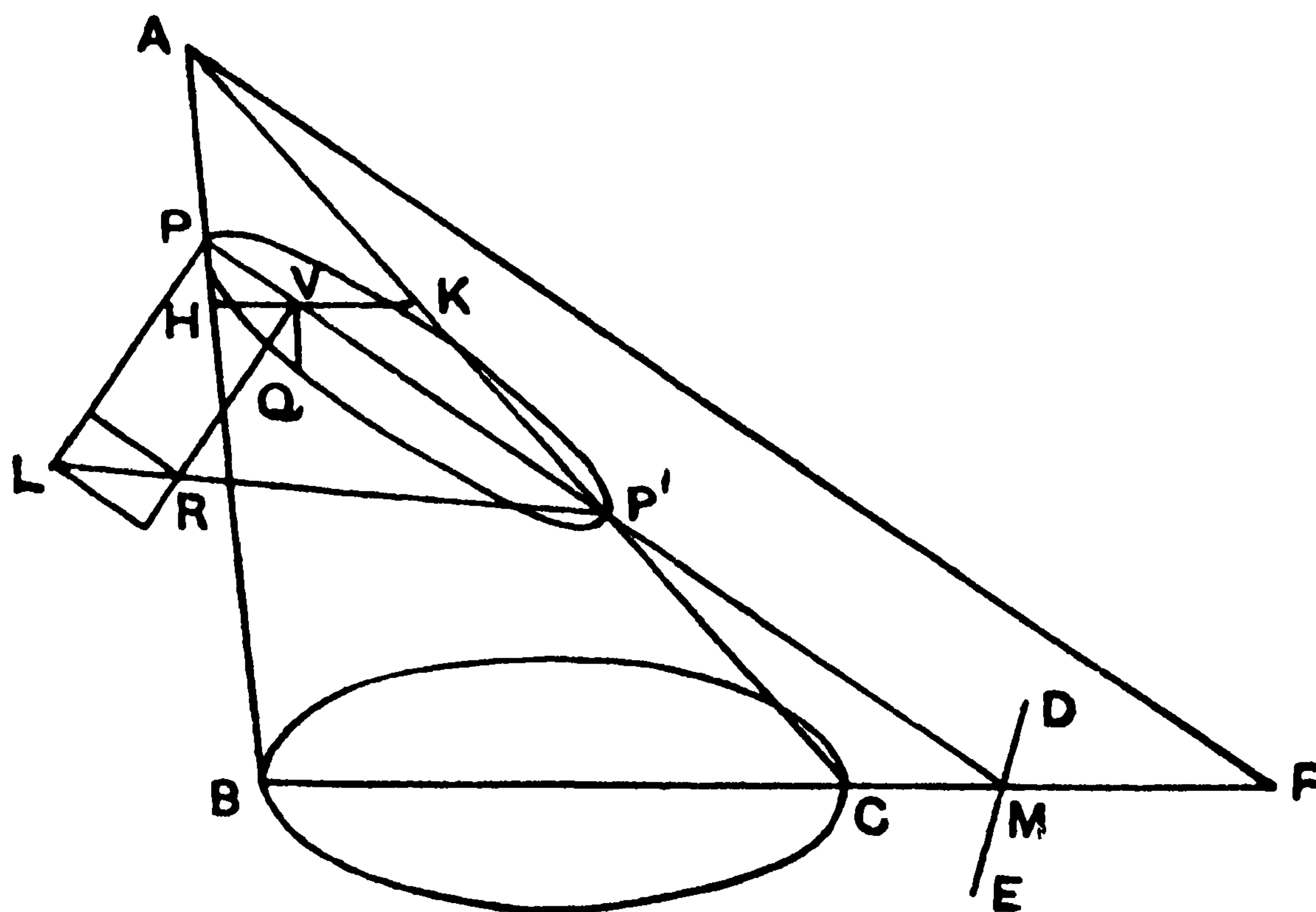
That is to say, Apollonius works out the properties of the conics in the most general way with reference to a diameter which is not one of the principal diameters or axes, but in general has its ordinates obliquely inclined to it. The axes do not appear in his exposition till much later, after it has been shown that each conic has the same property with reference to any diameter as it has with reference to the original diameter arising out of the construction; the axes then appear

as particular cases of the new diameter of reference. The three sections, the parabola, hyperbola, and ellipse are made in the manner shown in the figures. In each case they pass



through a straight line *DE* in the plane of the base which is at right angles to *BC*, the base of the axial triangle, or to *BC* produced. The diameter *PM* is in the case of the

parabola parallel to AC; in the case of the hyperbola it meets the other half of the double cone in P'; and in the case of the ellipse it meets the cone itself again in P'. We draw, in



the cases of the hyperbola and ellipse, AF parallel to PM to meet BC or BC produced in F.

Apollonius expresses the properties of the three curves by means of a certain straight line PL drawn at right angles to PM in the plane of the section.

In the case of the parabola, PL is taken such that

$$PL : PA = BC^2 : BA \,.\, AC;$$

and in the case of the hyperbola and ellipse such that

$$PL : PP' = BF \,.\, FC : AF^2.$$

In the latter two cases we join $P'L$, and then dr parallel to PL to meet $P'L$, produced if necessary, in R.

If HK be drawn through V parallel to BC and meeting AB, AC in H, K respectively, HK is the diameter of the circular section of the cone made by a plane parallel to the base.

Therefore $\qquad QV^2 = HV \,.\, VK.$

Then (1) for the parabola we have, by parallels and similar triangles,

$$HV : PV = BC : CA,$$

and $\qquad VK : PA = BC : BA.$

Therefore $QV^2 : PV . PA = HV . VK : PV . PA$

$= BC^2 : BA . AC$

$= PL : PA$, by hypothesis,

$= PL . PV : PV . PA$,

whence $QV^2 = PL . PV$.

(2) In the case of the hyperbola and ellipse,

$$HV : PV = BF : FA,$$

$$VK : P'V = FC : AF.$$

Therefore $QV^2 : PV . P'V = HV . VK : PV . P'V$

$= BF . FC : AF^2$

$= PL : PP'$, by hypothesis,

$= RV : P'V$

$= PV . VR : PV . P'V$,

whence $QV^2 = PV . VR$.

New names, 'parabola', 'ellipse', 'hyperbola'.

Accordingly, in the case of the parabola, the square of the ordinate (QV^2) is equal to the rectangle *applied* to PL and with width equal to the abscissa (PV);

in the case of the hyperbola the rectangle applied to PL which is equal to QV^2 and has its width equal to the abscissa PV *overlaps* or *exceeds* (ὑπερβάλλει) by the small rectangle LR which is similar and similarly situated to the rectangle contained by PL, PP';

in the case of the ellipse the corresponding rectangle *falls short* (ἐλλείπει) by a rectangle similar and similarly situated to the rectangle contained by PL, PP'.

Here then we have the properties of the three curves expressed in the precise language of the Pythagorean application of areas, and the curves are named accordingly: *parabola* (παραβολή) where the rectangle is exactly *applied*, *hyperbola* (ὑπερβολή) where it *exceeds*, and *ellipse* (ἔλλειψις) where it *falls short*.

PL is called the *latus rectum* (ὀρθία) or the *parameter of the ordinates* (παρ' ἣν δύνανται αἱ καταγόμεναι τεταγμένως) in each case. In the case of the central conics, the diameter PP' is the *transverse* (ἡ πλαγία) or *transverse diameter*; while, even more commonly, Apollonius speaks of the diameter and the corresponding parameter together, calling the latter the *latus rectum* or *erect side* (ὀρθία πλευρά) and the former the *transverse side* of the *figure* (εἶδος) *on*, or *applied to*, the diameter.

Fundamental properties equivalent to Cartesian equations.

If p is the parameter, and d the corresponding diameter, the properties of the curves are the equivalent of the Cartesian equations, referred to the diameter and the tangent at its extremity as axes (in general oblique),

$y^2 = px$ (the parabola),

$y^2 = px \pm \frac{p}{d}x^2$ (the hyperbola and ellipse respectively).

Thus Apollonius expresses the fundamental property of the central conics, like that of the parabola, as an equation between areas, whereas in Archimedes it appears as a proportion

$$y^2 : (a^2 \pm x^2) = b^2 : a^2,$$

which, however, is equivalent to the Cartesian equation referred to axes with the centre as origin. The latter property with reference to the original diameter is separately proved in I. 21, to the effect that QV^2 varies as $PV.P'V$, as is really evident from the fact that $QV^2 : PV.P'V = PL : PP'$, seeing that $PL : PP'$ is constant for any fixed diameter PP'.

Apollonius has a separate proposition (I. 14) to prove that the opposite branches of a hyperbola have the same diameter and equal *latera recta* corresponding thereto. As he was the first to treat the double-branch hyperbola fully, he generally discusses the *hyperbola* (i.e. the single branch) along with the ellipse, and *the opposites*, as he calls the double-branch hyperbola, separately. The properties of the single-branch hyperbola are, where possible, included in one enunciation with those of the ellipse and circle, the enunciation beginning,

'If in a hyperbola, an ellipse, or the circumference of a circle'; sometimes, however, the double-branch hyperbola and the ellipse come in one proposition, e.g. in I. 30: 'If in an ellipse or the opposites (i.e. the double hyperbola) a straight line be drawn through the centre meeting the curve on both sides of the centre, it will be bisected at the centre.' The property of conjugate diameters in an ellipse is proved in relation to the original diameter of reference and its conjugate in I. 15, where it is shown that, if DD' is the diameter conjugate to PP' (i.e. the diameter drawn ordinate-wise to PP'), just as PP' bisects all chords parallel to DD', so DD' bisects all chords parallel to PP'; also, if DL' be drawn at right angles to DD' and such that $DL'.DD' = PP'^2$ (or DL' is a third proportional to DD', PP'), then the ellipse has the same property in relation to DD' as diameter and DL' as parameter that it has in relation to PP' as diameter and PL as the corresponding parameter. Incidentally it appears that $PL.PP' = DD'^2$, or PL is a third proportional to PP', DD', as indeed is obvious from the property of the curve $QV^2 : PV.PV' = PL : PP' = DD'^2 : PP'^2$. The next proposition, I. 16, introduces the *secondary diameter* of the double-branch hyperbola (i.e. the diameter conjugate to the transverse diameter of reference), which does not meet the curve; this diameter is defined as that straight line drawn through the centre parallel to the ordinates of the transverse diameter which is bisected at the centre and is of length equal to the mean proportional between the 'sides of the figure', i.e. the transverse diameter PP' and the corresponding parameter PL. The *centre* is defined as the middle point of the diameter of reference, and it is proved that all other diameters are bisected at it (I. 30).

Props. 17–19, 22–9, 31–40 are propositions leading up to and containing the tangent properties. On lines exactly like those of Eucl. III. 16 for the circle, Apollonius proves that, if a straight line is drawn through the vertex (i.e. the extremity of the diameter of reference) parallel to the ordinates to the diameter, it will fall outside the conic, and no other straight line can fall between the said straight line and the conic; therefore the said straight line touches the conic (I. 17, 32). Props. I. 33, 35 contain the property of the tangent at any point on the parabola, and Props. I. 34, 36 the property of

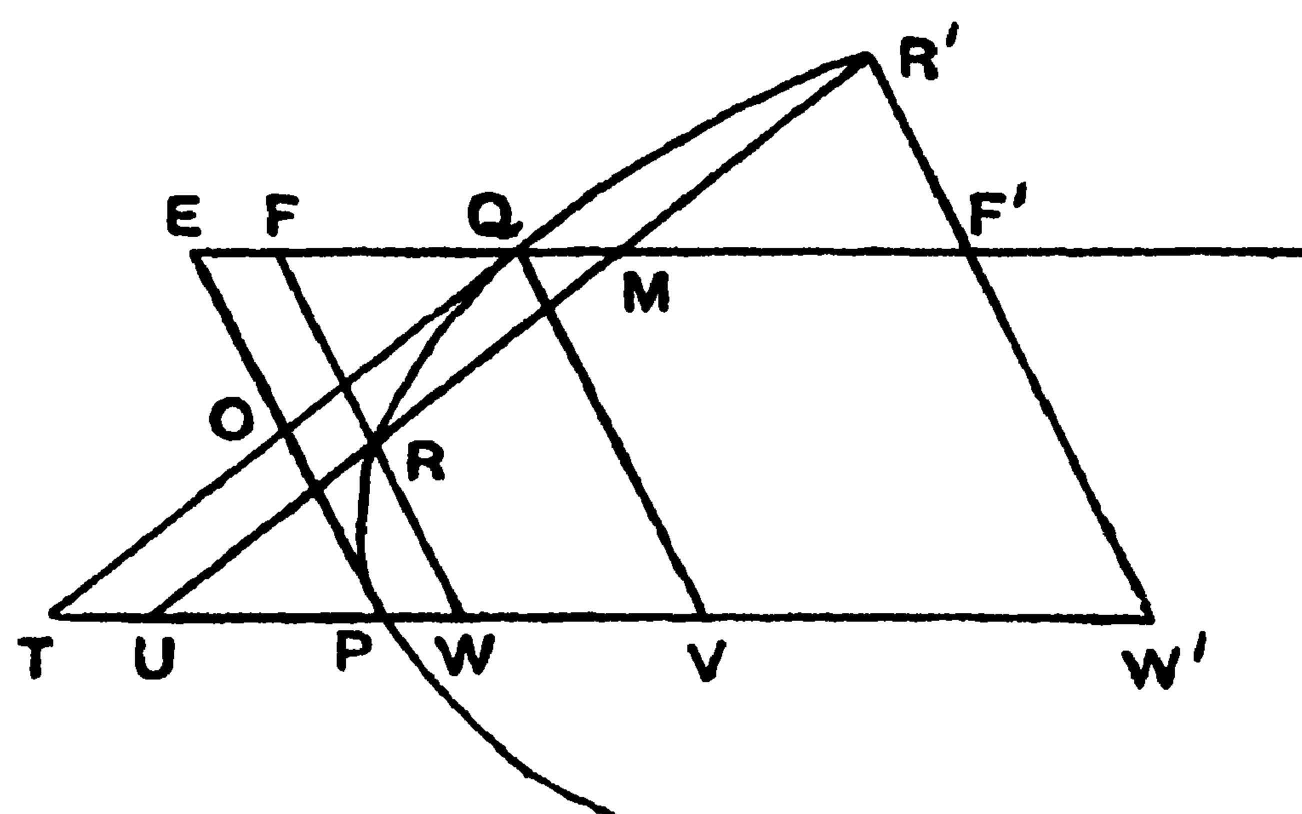
the tangent at any point of a central conic, in relation to the original diameter of reference; if Q is the point of contact, QV the ordinate to the diameter through P, and if QT, the tangent at Q, meets the diameter produced in T, then (1) for the parabola $PV = PT$, and (2) for the central conic $TP : TP' = PV : VP'$. The method of proof is to take a point T on the diameter produced satisfying the respective relations, and to prove that, if TQ be joined and produced, any point on TQ on either side of Q is outside the curve: the form of proof is by *reductio ad absurdum*, and in each case it is again proved that no other straight line can fall between TQ and the curve. The fundamental property $TP : TP' = PV : VP'$ for the central conic is then used to prove that $CV . CT = CP^2$ and $QV^2 : CV . VT = p : PP'$ (or $CD^2 : CP^2$) and the corresponding properties with reference to the diameter DD' conjugate to PP' and v, t, the points where DD' is met by the ordinate to it from Q and by the tangent at Q respectively (Props. I. 37–40).

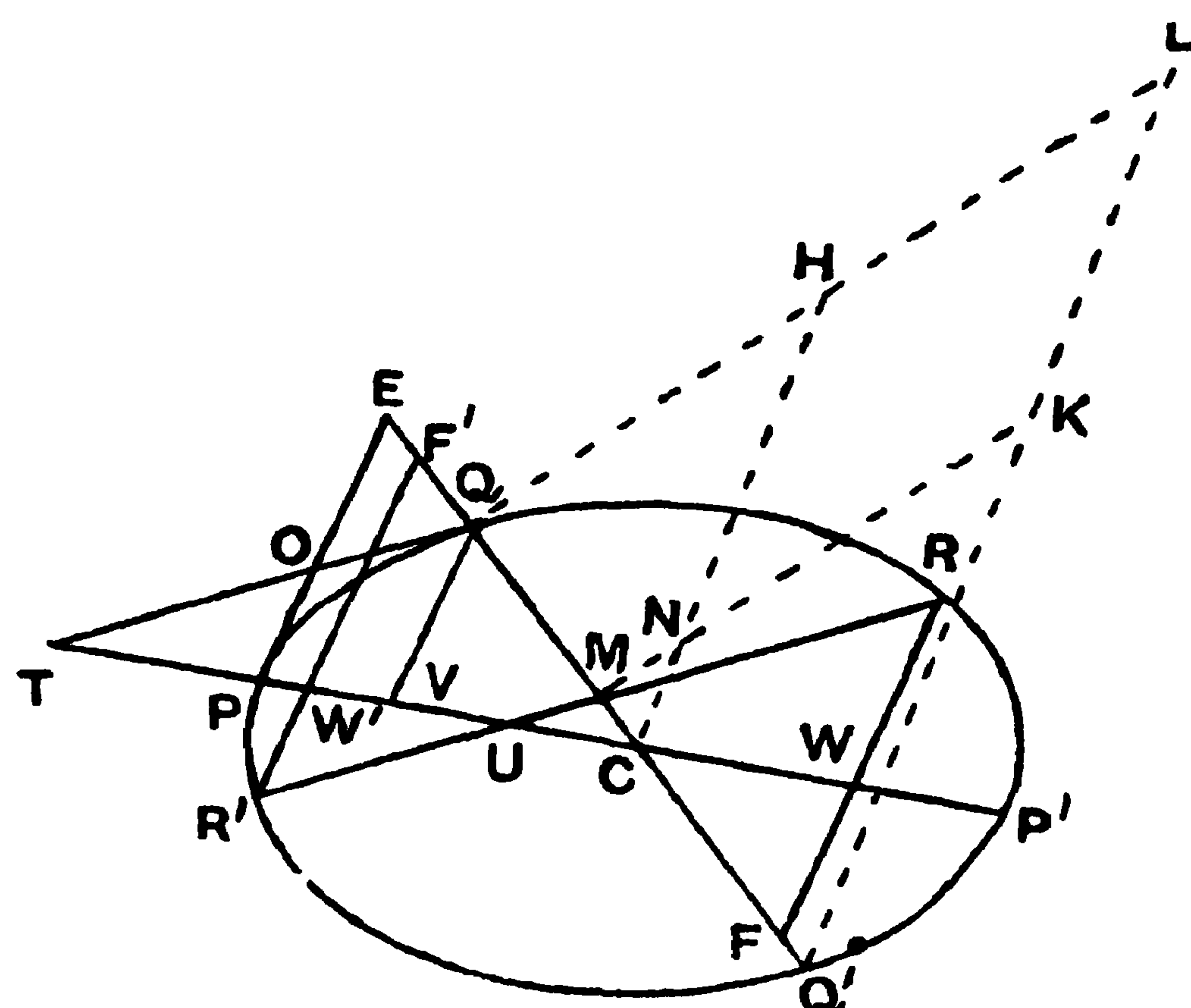
Transition to new diameter and tangent at its extremity.

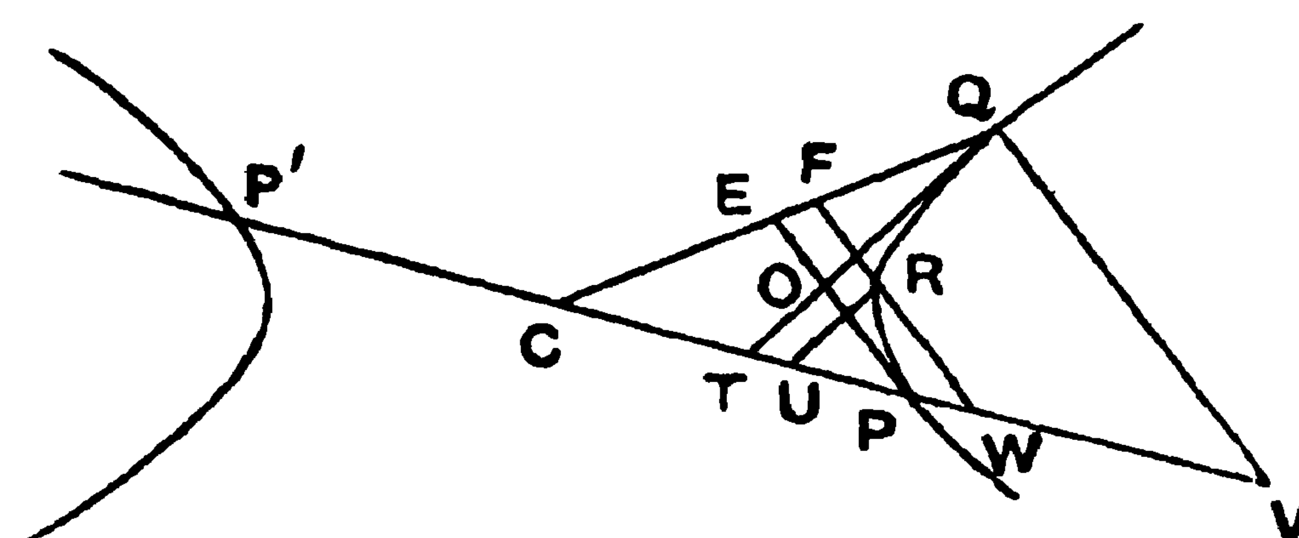
An important section of the Book follows (I. 41–50), consisting of propositions leading up to what amounts to a transformation of coordinates from the original diameter and the tangent at its extremity to *any* diameter and the tangent at its extremity; what Apollonius proves is of course that, if *any* other diameter be taken, the ordinate-property of the conic with reference to that diameter is of the same form as it is with reference to the original diameter. It is evident that this is vital to the exposition. The propositions leading up to the result in I. 50 are not usually given in our text-books of geometrical conics, but are useful and interesting.

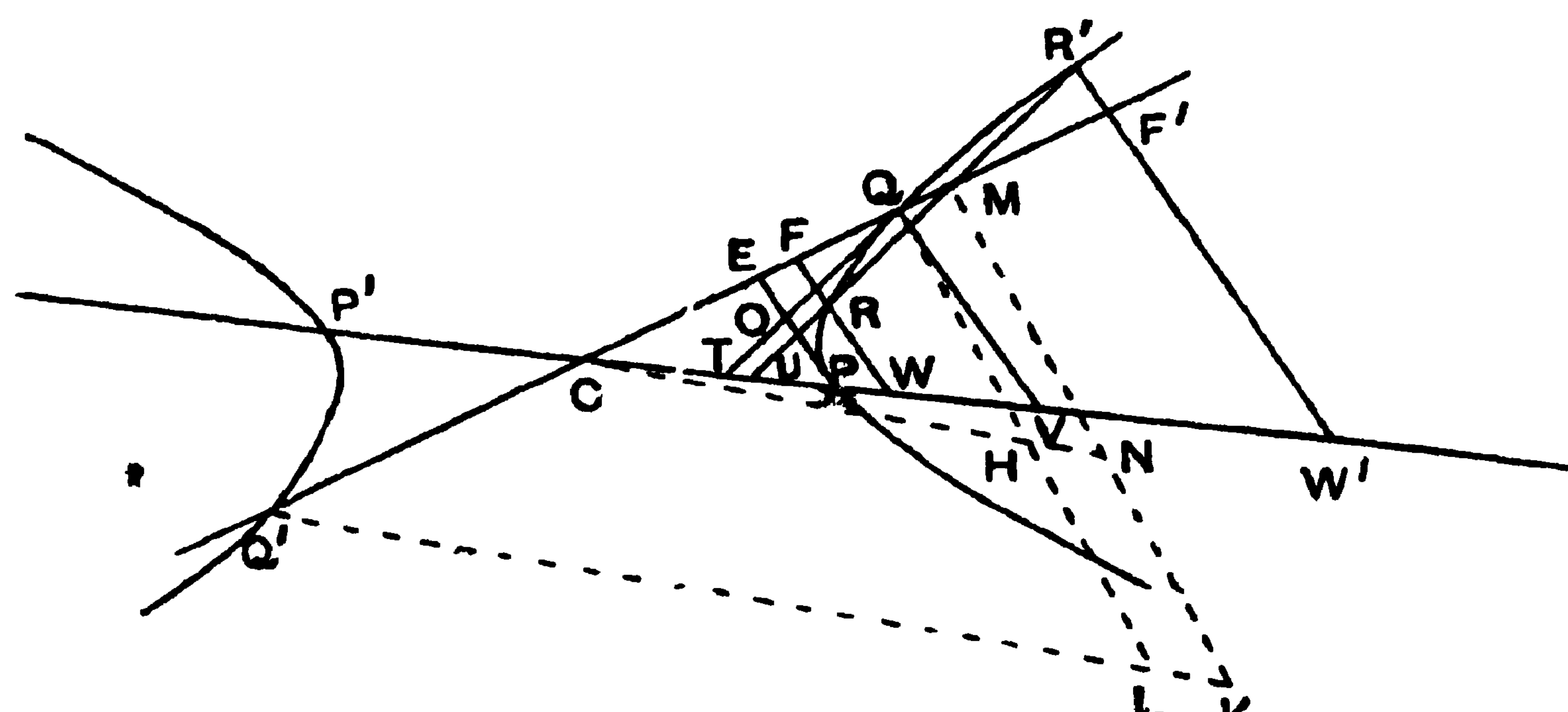
Suppose that the tangent at any point Q meets the diameter of reference PV in T, and that the tangent at P meets the diameter through Q in E. Let R be any third point on the curve; let the ordinate RW to PV meet the diameter through Q in F, and let RU parallel to the tangent at Q meet PV in U. Then

(1) in the parabola, the triangle $RUW =$ the parallelogram EW, and


R′
E
F
Q
F′
M
O
R
T
U
P
W
V
W′


L
H
E
F′
K
Q
O
R
N
M
T
V
P
W′
U
C
W
R′
P′
F
Q′


Q
F
E
P′
O
R
C
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U
P
W
V


R′
F′
Q
M
E
F
P′
O
R
T
U
P
W
C
H
V
N
W′
Q′
L
K

(2) in the hyperbola or ellipse, $\Delta RUW =$ the difference between the triangles CFW and CPE.

(1) In the parabola $\Delta RUW : \Delta QTV = RW^2 : QV^2$

$$= PW : PV$$

$$= \square EW : \square EV.$$

But, since $TV = 2PV$, $\Delta QTV = \square EV$;

therefore $\Delta RUW = \square EW$.

(2) The proof of the proposition with reference to the central conic depends on a Lemma, proved in I. 41, to the effect that, if PX, VY be similar parallelograms on CP, CV as bases, and if VZ be an equiangular parallelogram on QV as base and such that, if the ratio of CP to the other side of PX is m, the ratio of QV to the other side of VZ is $m \,.\, p/PP'$, then VZ is equal to the difference between VY and PX. The proof of the Lemma by Apollonius is difficult, but the truth of it can be easily seen thus.

By the property of the curve, $QV^2 : CV^2 \sim CP^2 = p : PP'$;

therefore $$CV^2 \sim CP^2 = \frac{PP'}{p} \,.\, QV^2.$$

Now $\square PX = \mu \,.\, CP^2/m$, where μ is a constant depending on the angle of the parallelogram.

Similarly

$$\square VY = \mu \,.\, CV^2/m, \text{ and } \square VZ = \mu \,.\, \frac{PP'}{p} QV^2/m.$$

It follows that $\square VY \sim \square PX = \square VZ$.

Taking now the triangles CFW, CPE and RUW in the ellipse or hyperbola, we see that CFW, CPE are similar, and RUW has one angle (at W) equal or supplementary to the angles at P and V in the other two triangles, while we have

$$QV^2 : CV \,.\, VT = p : PP',$$

whence $$QV : VT = (p : PP') \,.\, (CV : QV),$$

and, by parallels,

$$RW : WU = (p : PP') \,.\, (CP : PE).$$

Therefore RUW, CPE, CFW are the halves of parallelograms related as in the lemma;

therefore $$\Delta RUW = \Delta CFW \sim \Delta CPE.$$

The same property with reference to the diameter *secondary* to CPV is proved in I. 45.

It is interesting to note the exact significance of the property thus proved for the central conic. The proposition, which is the foundation of Apollonius's method of transformation of coordinates, amounts to this. If CP, CQ are fixed semi-diameters and R a variable point, the area of the quadrilateral $CFRU$ is constant for all positions of R on the conic. Suppose now that CP, CQ are taken as axes of x and y respectively. If we draw RX parallel to CQ to meet CP and RY parallel to CP to meet CQ, the proposition asserts that (subject to the proper convention as to sign)

$$\Delta RYF + \square CXRY + \Delta RXU = \text{(const.)}.$$

But since RX, RY, RF, RU are in fixed directions,

ΔRYF varies as RY^2 or x^2, $\square CXRY$ as $RX \,.\, RY$ or xy, and ΔRXU as RX^2 or y^2.

Hence, if x, y are the coordinates of R,

$$\alpha x^2 + \beta xy + \gamma y^2 = A,$$

which is the Cartesian equation of the conic referred to the centre as origin and any two diameters as axes.

The properties so obtained are next used to prove that, if UR meets the curve again in R' and the diameter through Q in M, then RR' is bisected at M. (I. 46–8).

Taking (1) the case of the parabola, we have,

$$\Delta RUW = \square EW,$$

and $$\Delta R'UW' = \square EW'.$$

By subtraction, $$(RWW'R') = \square F'W,$$

whence $$\Delta RFM = \Delta R'F'M,$$

and, since the triangles are similar, $RM = R'M$.

The same result is easily obtained for the central conic.

It follows that EQ produced in the case of the parabola,

or CQ in the case of the central conic, bisects all chords as RR' parallel to the tangent at Q. Consequently EQ and CQ are *diameters* of the respective conics.

In order to refer the conic to the new diameter and the corresponding ordinates, we have only to determine the *parameter* of these ordinates and to show that the property of the conic with reference to the new parameter and diameter is in the same form as that originally found.

The propositions I. 49, 50 do this, and show that the new parameter is in all the cases p', where (if O is the point of intersection of the tangents at P and Q)

$$OQ : QE = p' : 2QT.$$

(1) In the case of the parabola, we have $TP = PV = EQ$,

whence $$\triangle EOQ = \triangle POT.$$

Add to each the figure $POQF'W'$;

therefore $$QTW'F' = \square EW' = \triangle R'UW',$$

whence, subtracting $MUW'F'$ from both, we have

$$\triangle R'MF' = \square QU.$$

Therefore $$R'M \cdot MF' = 2QT \cdot QM.$$

But $R'M : MF' = OQ : QE = p' : 2QT$, by hypothesis;

therefore $$R'M^2 : R'M \cdot MF' = p' \cdot QM : 2QT \cdot QM.$$

And $R'M \cdot MF' = 2QT \cdot QM$, from above;

therefore $$R'M^2 = p' \cdot QM,$$

which is the desired property.[1]

[1] The proposition that, in the case of the parabola, if p be the parameter of the ordinates to the diameter through Q, then (see the first figure on p. 142)

$$OQ : QE = p : 2QT$$

has an interesting application; for it enables us to prove the proposition, assumed without proof by Archimedes (but not easy to prove otherwise), that, if in a parabola the diameter through P bisects the chord QQ' in V, and QD is drawn perpendicular to PV, then

$$QV^2 : QD^2 = p : p_a,$$

(2) In the case of the central conic, we have

$$\Delta R'UW' = \Delta CF'W' \sim \Delta CPE.$$

(Apollonius here assumes what he does not prove till III. 1, namely that $\Delta CPE = \Delta CQT$. This is proved thus.

We have $CV : CT = CV^2 : CP^2$; (I. 37, 39.)

therefore $\Delta CQV : \Delta CQT = \Delta CQV : \Delta CPE$,

so that $\Delta CQT = \Delta CPE$.)

Therefore $\Delta R'UW' = \Delta CF'W' \sim \Delta CQT$,

and it is easy to prove that in all cases

$$\Delta R'MF' = QTUM.$$

Therefore $R'M \cdot MF' = QM(QT + MU)$.

Let QL be drawn at right angles to CQ and equal to p'. Join $Q'L$ and draw MK parallel to QL to meet $Q'L$ in K. Draw CH parallel to $Q'L$ to meet QL in H and MK in N.

Now $R'M : MF' = OQ : QE$

$= QL : 2QT$, by hypothesis,

$= QH : QT$.

But $QT : MU = CQ : CM = QH : MN$,

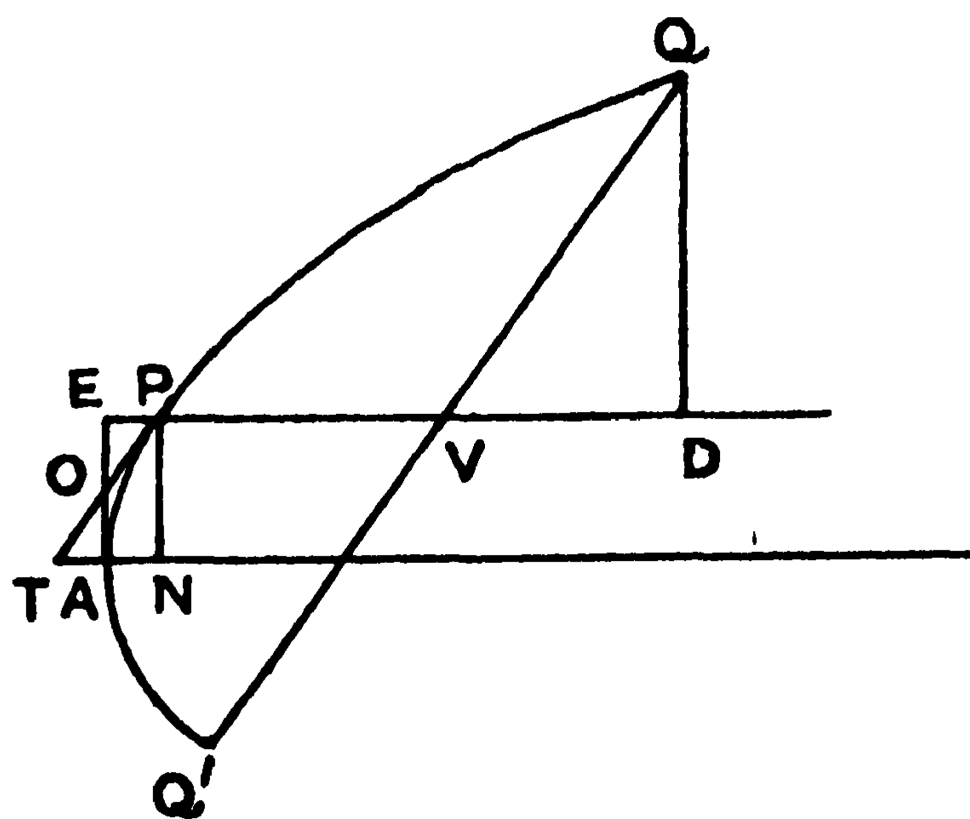
so that $(QH + MN) : (QT + MU) = QH : QT$

$= R'M : MF'$, from above.

where p_a is the parameter of the principal ordinates and p the parameter of the ordinates to the diameter PV.

If the tangent at the vertex A meets VP produced in E, and PT, the tangent at P, in O, the proposition of Apollonius proves that

$$OP : PE = p : 2PT.$$



But $OP = \frac{1}{2}PT$;

therefore $PT^2 = p \cdot PE$

$= p \cdot AN$.

Thus $QV^2 : QD^2 = PT^2 : PN^2$, by similar triangles,

$= p \cdot AN : p_a \cdot AN$

$= p : p_a$.

It follows that

$$QM(QH+MN):QM(QT+MU) = R'M^2:R'M.MF';$$

but, from above, $QM(QT+MU) = R'M.MF'$;

therefore $$R'M^2 = QM(QH+MN)$$
$$= QM.MK,$$

which is the desired property.

In the case of the hyperbola, the same property is true for the opposite branch.

These important propositions show that the ordinate property of the three conics is of the same form whatever diameter is taken as the diameter of reference. It is therefore a matter of indifference to which particular diameter and ordinates the conic is referred. This is stated by Apollonius in a summary which follows I. 50.

First appearance of principal axes.

The *axes* appear for the first time in the propositions next following (I. 52–8), where Apollonius shows how to construct each of the conics, given in each case (1) a diameter, (2) the length of the corresponding parameter, and (3) the inclination of the ordinates to the diameter. In each case Apollonius first assumes the angle between the ordinates and the diameter to be a right angle; then he reduces the case where the angle is oblique to the case where it is right by his method of transformation of coordinates; i.e. from the given diameter and parameter he *finds* the *axis* of the conic and the length of the corresponding parameter, and he then constructs the conic as in the first case where the ordinates are at right angles to the diameter. Here then we have a case of the proof of *existence* by means of *construction*. The conic is in each case constructed by finding the cone of which the given conic is a section. The problem of finding the axis of a parabola and the centre and the axes of a central conic when the conic (and not merely the elements, as here) is given comes later (in II. 44–7), where it is also proved (II. 48) that no central conic can have more than two axes.

It has been my object, by means of the above detailed account of Book I, to show not merely what results are obtained by Apollonius, but the way in which he went to work; and it will have been realized how entirely scientific and general the method is. When the foundation is thus laid, and the fundamental properties established, Apollonius is able to develop the rest of the subject on lines more similar to those followed in our text-books. My description of the rest of the work can therefore for the most part be confined to a summary of the contents.

Book II begins with a section devoted to the properties of the asymptotes. They are constructed in II. 1 in this way. Beginning, as usual, with *any* diameter of reference and the corresponding parameter and inclination of ordinates, Apollonius draws at P the vertex (the extremity of the diameter) a tangent to the hyperbola and sets off along it lengths PL, PL' on either side of P such that $PL^2 = PL'^2 = \frac{1}{4}p \, . \, PP'$ $[= CD^2]$, where p is the parameter. He then proves that CL, CL' produced will not meet the curve in any finite point and are therefore *asymptotes*. II. 2 proves further that no straight line through C within the angle between the asymptotes can itself be an asymptote. II. 3 proves that the intercept made by the asymptotes on the tangent at any point P is bisected at P, and that the square on each half of the intercept is equal to one-fourth of the 'figure' corresponding to the diameter through P (i.e. one-fourth of the rectangle contained by the 'erect' side, the *latus rectum* or parameter corresponding to the diameter, and the diameter itself); this property is used as a means of drawing a hyperbola when the asymptotes and one point on the curve are given (II. 4). II. 5–7 are propositions about a tangent at the extremity of a diameter being parallel to the chords bisected by it. Apollonius returns to the asymptotes in II. 8, and II. 8–14 give the other ordinary properties with reference to the asymptotes (II. 9 is a converse of II. 3), the equality of the intercepts between the asymptotes and the curve of any chord (II. 8), the equality of the rectangle contained by the distances between either point in which the chord meets the curve and the points of intersection with the asymptotes to the square on the parallel semi-diameter (II. 10), the latter property with reference to

the portions of the asymptotes which include between them a branch of the conjugate hyperbola (II. 11), the constancy of the rectangle contained by the straight lines drawn from any point of the curve in fixed directions to meet the asymptotes (equivalent to the Cartesian equation with reference to the asymptotes, $xy =$ const.) (II. 12), and the fact that the curve and the asymptotes proceed to infinity and approach continually nearer to one another, so that the distance separating them can be made smaller than any given length (II. 14). II. 15 proves that the two opposite branches of a hyperbola have the same asymptotes and II. 16 proves for the chord connecting points on two branches the property of II. 8. II. 17 shows that 'conjugate opposites' (two conjugate double-branch hyperbolas) have the same asymptotes. Propositions follow about conjugate hyperbolas; any tangent to the conjugate hyperbola will meet both branches of the original hyperbola and will be bisected at the point of contact (II. 19); if Q be any point on a hyperbola, and CE parallel to the tangent at Q meets the conjugate hyperbola in E, the tangent at E will be parallel to CQ and CQ, CE will be conjugate diameters (II. 20), while the tangents at Q, E will meet on one of the asymptotes (II. 21); if a chord Qq in one branch of a hyperbola meet the asymptotes in R, r and the conjugate hyperbola in Q', q', then $Q'Q \cdot Qq' = 2CD^2$ (II. 23). Of the rest of the propositions in this part of the Book the following may be mentioned: if TQ, TQ' are two tangents to a conic and V is the middle point of QQ', TV is a diameter (II. 29, 30, 38); if tQ, tQ' be tangents to opposite branches of a hyperbola, RR' the chord through t parallel to QQ', v the middle point of QQ', then vR, vR' are tangents to the hyperbola (II. 40); in a conic, or a circle, or in conjugate hyperbolas, if two chords not passing through the centre intersect, they do not bisect each other (II. 26, 41, 42). II. 44–7 show how to find a diameter of a conic and the centre of a central conic, the axis of a parabola and the axes of a central conic. The Book concludes with problems of drawing tangents to conics in certain ways, through any point on or outside the curve (II. 49), making with the axis an angle equal to a given acute angle (II. 50), making a given angle with the diameter through the point of contact (II. 51, 53); II. 52 contains a διορισμός for

the last problem, proving that, if the tangent to an ellipse at any point P meets the major axis in T, the angle CPT is not greater than the angle ABA', where B is one extremity of the minor axis.

Book III begins with a series of propositions about the equality of certain areas, propositions of the same kind as, and easily derived from, the propositions (I. 41–50) by means of which, as already shown, the transformation of coordinates is effected. We have first the proposition that, if the tangents at any points P, Q of a conic meet in O, and if they meet the diameters through Q, P respectively in E, T, then $\Delta OPT = \Delta OQE$ (III. 1, 4); and, if P, Q be points on adjacent branches of conjugate hyperbolas, $\Delta CPE = \Delta CQT$ (III. 13). With the same notation, if R be any other point on the conic, and if we draw RU parallel to the tangent at Q meeting the diameter through P in U and the diameter through Q in M, and RW parallel to the tangent at P meeting QT in H and the diameters through Q, P in F, W, then $\Delta HQF =$ quadrilateral $HTUR$ (III. 2. 6); this is proved at once from the fact that $\Delta RMF =$ quadrilateral $QTUM$ (see I. 49, 50, or pp. 145–6 above) by subtracting or adding the area $HRMQ$ on each side. Next take any other point R', and draw $R'U'$, $F'H'R'W'$ in the same way as before; it is then proved that, if RU, $R'W'$ meet in I and $R'U'$, RW in J, the quadrilaterals $F'IRF$, $IUU'R'$ are equal, and also the quadrilaterals $FJR'F'$, $JU'UR$ (III. 3, 7, 9, 10). The proof varies according to the actual positions of the points in the figures.

In Figs. 1, 2 $\quad \Delta HFQ =$ quadrilateral $HTUR$,

$$\Delta H'F'Q = H'TU'R'.$$

By subtraction, $FHH'F' = IUU'R' \mp (IH)$;

whence, if IH be added or subtracted, $F'IRF = IUU'R'$,

and again, if IJ be added to both, $FJR'F' = JU'UR$.

In Fig. 3 $\quad \Delta R'U'W' = \Delta CF'W' - \Delta CQT$,

so that $\quad \Delta CQT = CU'R'F'$.

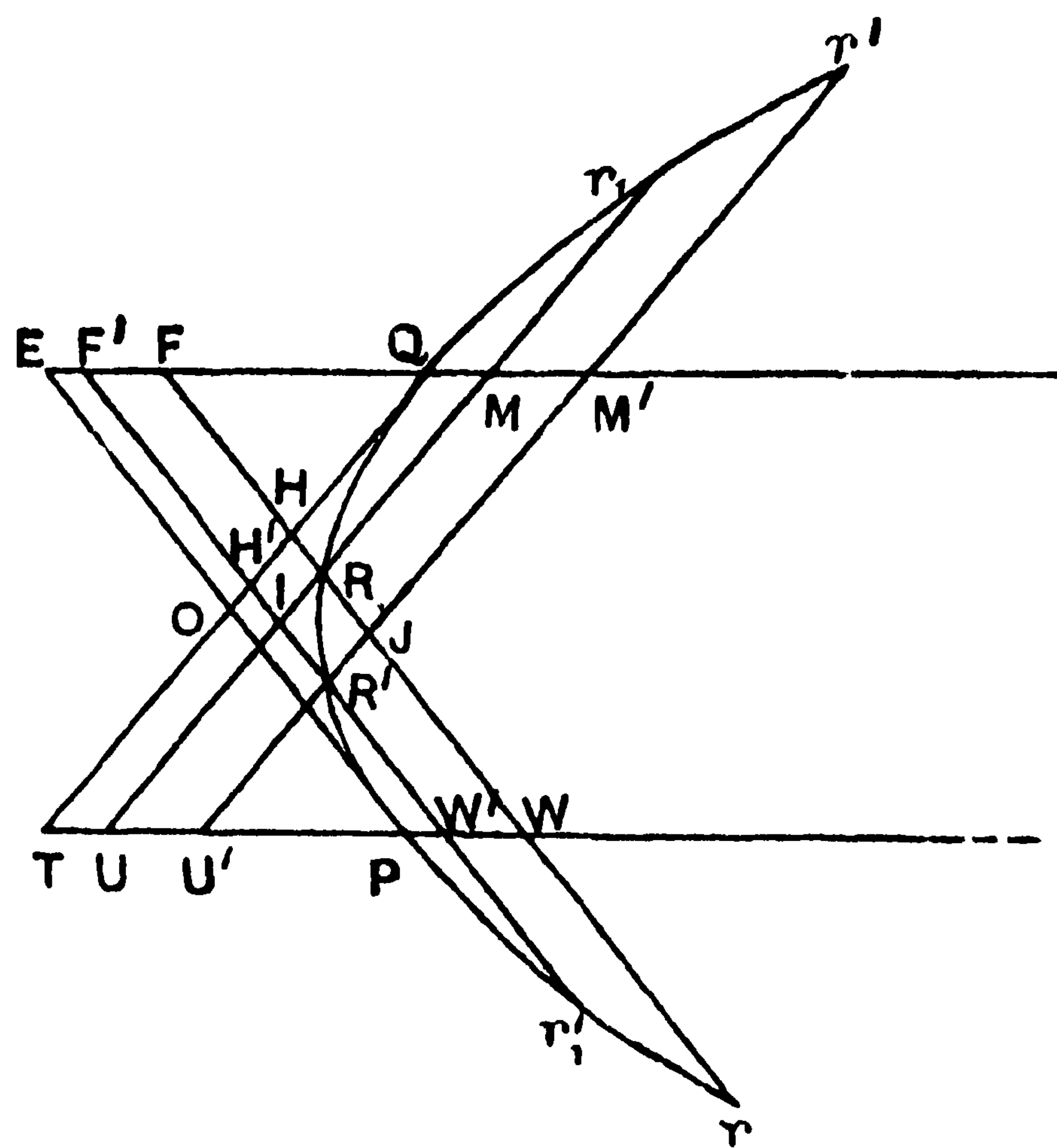


Fig. 1.

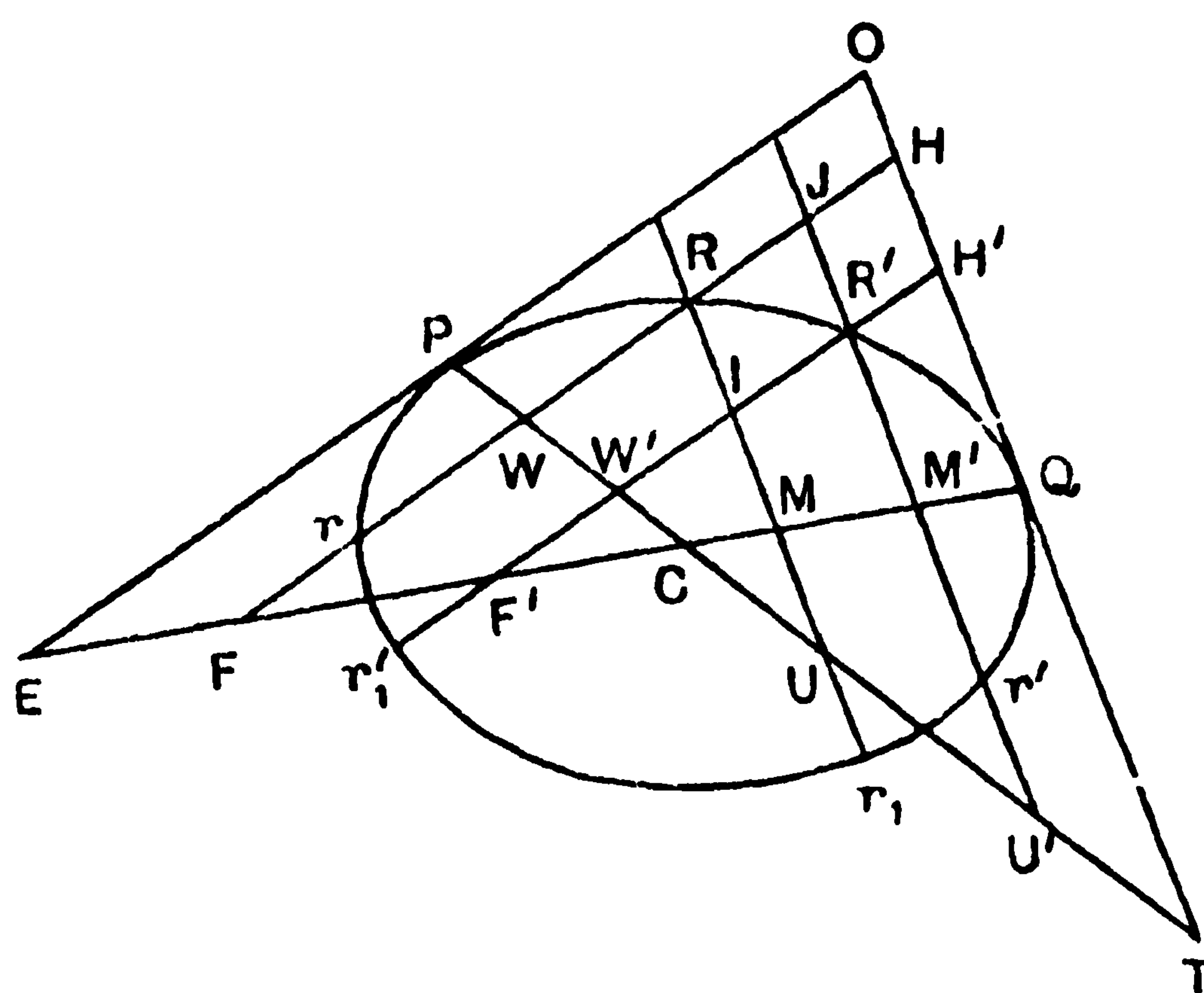


Fig. 2.

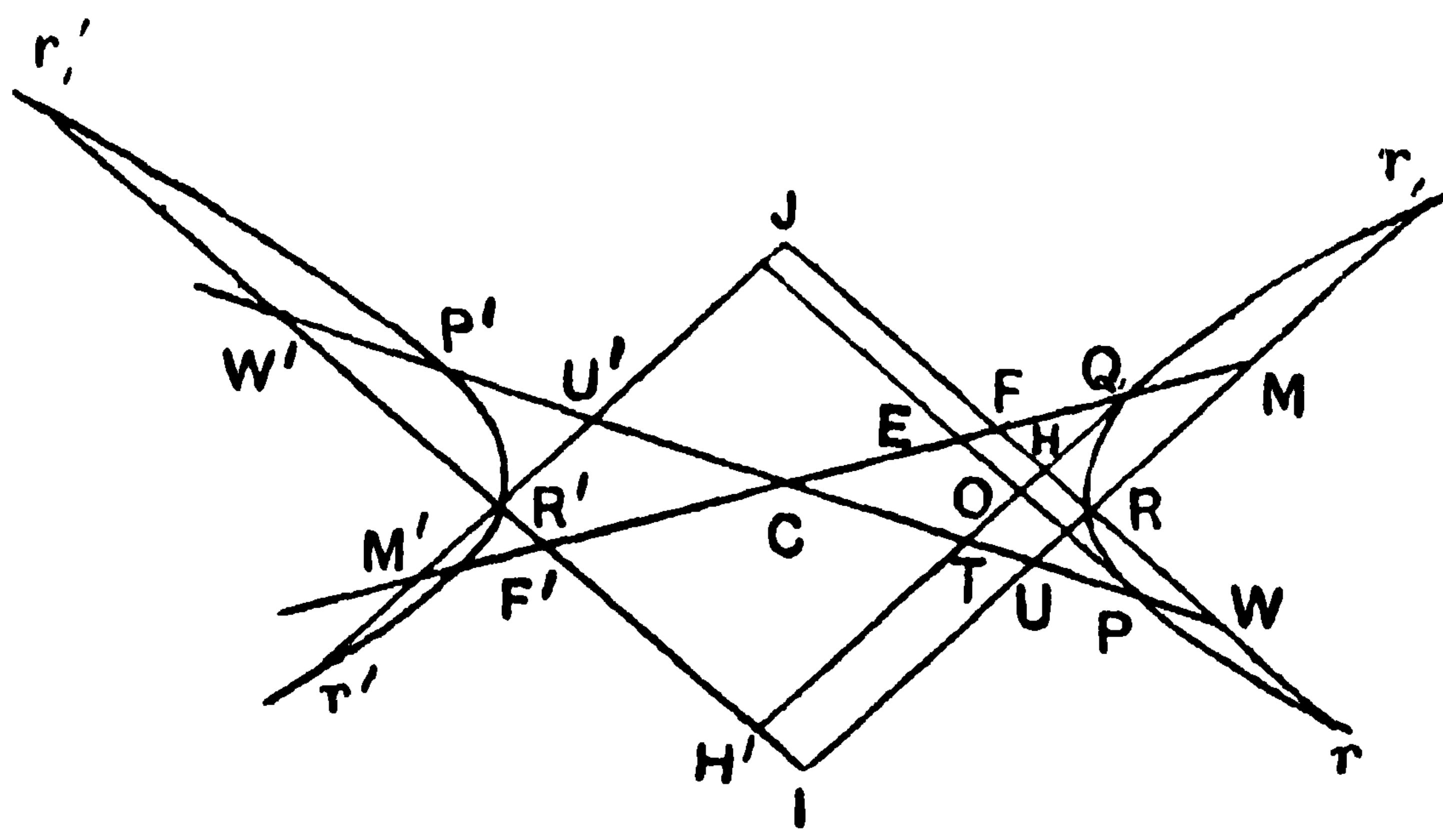


Fig. 3.

Adding the quadrilateral $CF'H'T$, we have

$$\Delta H'F'Q = H'TU'R',$$

and similarly $\Delta HFQ = HTUR$.

By subtraction, $F'H'HF = H'TU'R' - HTUR$.

Adding $H'IRH$ to each side, we have

$$F'IRF = IUU'R'.$$

If each of these quadrilaterals is subtracted from IJ,

$$FJR'F' = JU'UR.$$

The corresponding results are proved in III. 5, 11, 12, 14 for the case where the ordinates through RR' are drawn to a *secondary* diameter, and in III. 15 for the case where P, Q are on the original hyperbola and R, R' on the conjugate hyperbola.

The importance of these propositions lies in the fact that they are immediately used to prove the well-known theorems about the rectangles contained by the segments of intersecting chords and the harmonic properties of the pole and polar. The former question is dealt with in III. 16–23, which give a great variety of particular cases. We will give the proof of one case, to the effect that, if OP, OQ be two tangents to any conic and Rr, $R'r'$ be any two chords parallel to them respectively and intersecting in J, an internal or external point,

then $RJ \,.\, Jr : R'J \,.\, Jr' = OP^2 : OQ^2 = \text{(const.)}$.

We have

$RJ \,.\, Jr = RW^2 \sim JW^2$, and $RW^2 : JW^2 = \Delta RUW : \Delta JU'W$;

therefore

$$RJ \,.\, Jr : RW^2 = (RW^2 \sim JW^2) : RW^2 = JU'UR : \Delta RUW.$$

But $RW^2 : OP^2 = \Delta RUW : \Delta OPT$;

therefore, *ex aequali*, $RJ \,.\, Jr : OP^2 = JU'UR : \Delta OPT$.

Similarly $R'M'^2 : JM'^2 = \Delta R'F'M' : \Delta JFM'$,

whence $R'J \,.\, Jr' : R'M'^2 = FJR'F' : \Delta R'F'M'$.

But $R'M'^2 : OQ^2 = \Delta R'F'M' : \Delta OQE$;

therefore, *ex aequali*, $R'J \,.\, Jr' : OQ^2 = FJR'F' : \Delta OQE$.

It follows, since $FJR'F' = JU'UR$, and $\Delta OPT = \Delta OQE$,

that $RJ \,.\, Jr : OP^2 = R'J \,.\, Jr' : OQ^2$,

or $RJ \,.\, Jr : R'J \,.\, Jr' = OP^2 : OQ^2$.

If we had taken chords Rr_1, $R'r_1'$ parallel respectively to OQ, OP and intersecting in I, an internal or external point, we should have in like manner

$$RI \,.\, Ir_1 : R'I \,.\, Ir_1' = OQ^2 : OP^2.$$

As a particular case, if PP' be a diameter, and Rr, $R'r'$ be chords parallel respectively to the tangent at P and the diameter PP' and intersecting in I, then (as is separately proved)

$$RI \,.\, Ir : R'I \,.\, Ir' = p : PP'.$$

The corresponding results are proved in the cases where certain of the points lie on the conjugate hyperbola.

The six following propositions about the segments of intersecting chords (III. 24-9) refer to two chords in conjugate hyperbolas or in an ellipse drawn parallel respectively to two conjugate diameters PP', DD', and the results in modern form are perhaps worth quoting. If Rr, $R'r'$ be two chords so drawn and intersecting in O, then

(*a*) in the conjugate hyperbolas

$$\frac{RO \,.\, Or}{CP^2} \pm \frac{R'O \,.\, Or'}{CD^2} = 2,$$

and $(RO^2 + Or^2) : (R'O^2 + Or'^2) = CP^2 : CD^2$;

(*b*) in the ellipse

$$\frac{RO^2 + Or^2}{CP^2} + \frac{R'O^2 + Or'^2}{CD^2} = 4.$$

The general propositions containing the harmonic properties of the pole and polar of a conic are III. 37–40, which prove that in any conic, if TQ, Tq be tangents, and if Qq the chord of contact be bisected in V, then

(1) if any straight line through T meet the conic in R', R and Qq in I, then (Fig. 1) $RT : TR' = RI : IR'$;

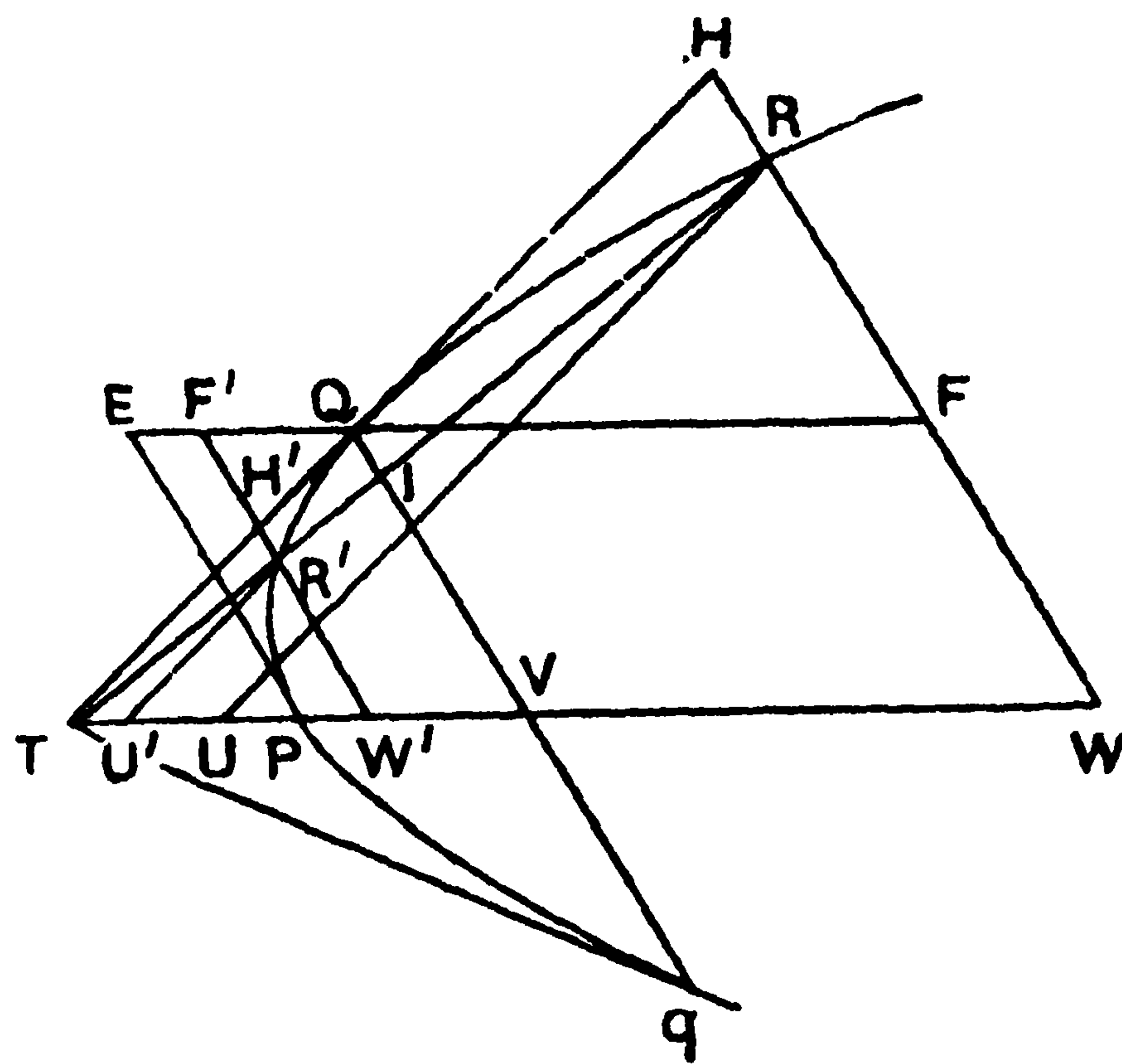


Fig. 1.

(2) if any straight line through V meet the conic in R, R' and the parallel through T to Qq in O, then (Fig. 2)

$$RO : OR' = RV : VR'.$$

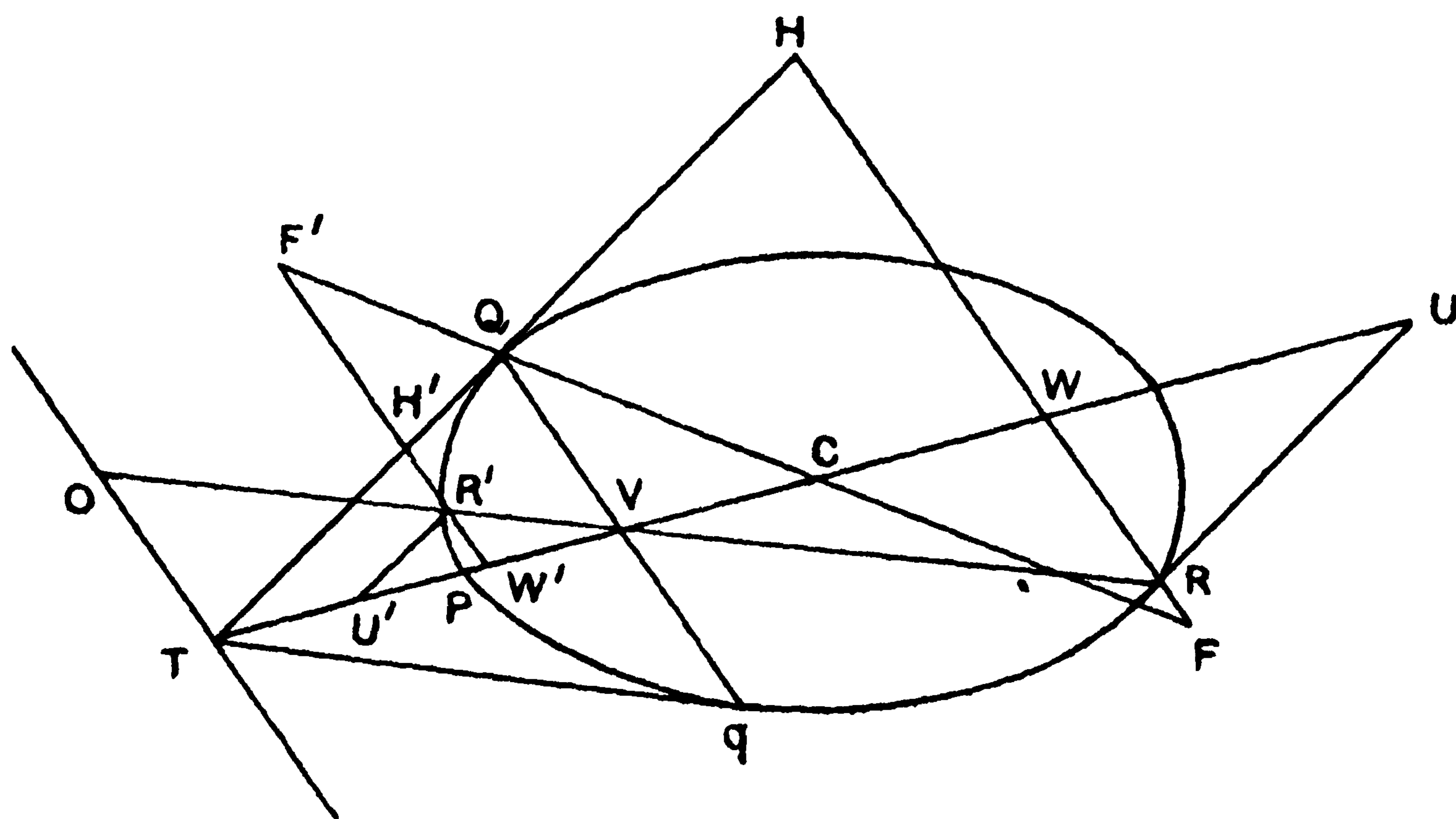


Fig. 2.

The above figures represent theorem (1) for the parabola and theorem (2) for the ellipse.

To prove (1) we have

$$R'I^2 : IR^2 = H'Q^2 : QH^2 = \Delta H'F'Q : \Delta HFQ = H'TU'R' : HTUR$$

(III. 2, 3, &c.).

Also $R'T^2 : TR^2 = R'U'^2 : UR^2 = \Delta R'U'W' : \Delta RUW$,

and, $R'T^2 : TR^2 = TW'^2 : TW^2 = \Delta TH'W' : \Delta THW$,

so that $R'T^2 : TR^2 = \Delta TH'W' \sim \Delta R'U'W' : \Delta THW \sim \Delta RUW$

$= H'TU'R' : HTUR$

$= R'I^2 : IR^2$, from above.

To prove (2) we have

$$RV^2 : VR'^2 = RU^2 : R'U'^2 = \Delta RUW : \Delta R'U'W',$$

and also

$$= HQ^2 : QH'^2 = \Delta HFQ : \Delta H'F'Q = HTUR* : H'TU'R',$$

so that

$$RV^2 : VR'^2 = HTUR \underset{\sim}{+} \Delta RUW : H'TU'R' \underset{\sim}{+} \Delta R'U'W'$$

$$= \Delta THW : \Delta TH'W'$$

$$= TW^2 : TW'^2$$

$$= RO^2 : OR'^2.$$

Props. III. 30–6 deal separately with the particular cases in which (*a*) the transversal is parallel to an asymptote of the hyperbola or (*b*) the chord of contact is parallel to an asymptote, i.e. where one of the tangents is an asymptote, which is the tangent at infinity.

Next we have propositions about intercepts made by two tangents on a third: If the tangents at three points of a parabola form a triangle, all three tangents will be cut by the points of contact in the same proportion (III. 41); if the tangents at the extremities of a diameter PP' of a central conic are cut in r, r' by any other tangent, $Pr . P'r' = CD^2$ (III. 42); if the tangents at P, Q to a hyperbola meet the asymptotes in

* Where a quadrilateral, as *HTUR* in the figure, is a cross-quadrilateral, the area is of course the difference between the two triangles which it forms, as $HTW \sim RUW$.

L, L' and M, M' respectively, then $L'M$, LM' are both parallel to PQ (III. 44).

The first of these propositions asserts that, if the tangents at three points P, Q, R of a parabola form a triangle pqr, then

$$Pr : rq = rQ : Qp = qp : pR.$$

From this property it is easy to deduce the Cartesian equation of a parabola referred to two fixed tangents as coordinate axes. Taking qR, qP as fixed coordinate axes, we find the locus of Q thus. Let x, y be the coordinates of Q. Then, if $qp = x_1$, $qr = y_1$, $qR = h$, $qP = k$, we have

$$\frac{x}{x_1 - x} = \frac{rQ}{Qp} = \frac{y_1 - y}{y} = \frac{k - y_1}{y_1} = \frac{x_1}{h - x_1}.$$

From these equations we derive

$$x_1^2 = hx, \; y_1^2 = ky;$$

also, since $\frac{x_1}{x} = \frac{y_1}{y_1 - y}$, we have $\frac{x}{x_1} + \frac{y}{y_1} = 1$.

By substituting for x_1, y_1 the values $\sqrt{(hx)}$, $\sqrt{(ky)}$ we obtain

$$\left(\frac{x}{h}\right)^{\frac{1}{2}} + \left(\frac{y}{k}\right)^{\frac{1}{2}} = 1.$$

The focal properties of central conics are proved in III. 45–52 without any reference to the directrix; there is no mention of the focus of a parabola. The foci are called 'the points arising out of the application' (τὰ ἐκ τῆς παραβολῆς γινόμενα σημεῖα), the meaning being that S, S' are taken on the axis AA' such that $AS . SA' = AS' . S'A' = \frac{1}{4}p_a . AA'$ or CB^2, that is, in the phraseology of application of areas, a rectangle is applied to AA' as base equal to one-fourth part of the 'figure', and in the case of the hyperbola exceeding, but in the case of the ellipse falling short, by a square figure. The foci being thus found, it is proved that, if the tangents Ar, $A'r'$ at the extremities of the axis are met by the tangent at any point P in r, r' respectively, rr' subtends a right angle at S, S', and the angles $rr'S$, $A'r'S'$ are equal, as also are the angles $r'rS'$, ArS (III. 45, 46). It is next shown that, if O be the intersection of rS', $r'S$, then OP is perpendicular to the tangent at P (III. 47). These propositions are

used to prove that the focal distances of P make equal angles with the tangent at P (III. 48). In III. 49–52 follow the other ordinary properties, that, if SY be perpendicular to the tangent at P, the locus of Y is the circle on AA' as diameter, that the lines from C drawn parallel to the focal distances to meet the tangent at P are equal to CA, and that the sum or difference of the focal distances of any point is equal to AA'.

The last propositions of Book III are of use with reference to the locus with respect to three or four lines. They are as follows.

1. If PP' be a diameter of a central conic, and if PQ, $P'Q$ drawn to any other point Q of the conic meet the tangents at P', P in R', R respectively, then $PR \cdot P'R' = 4CD^2$ (III. 53).

2. If TQ, TQ' be two tangents to a conic, V the middle point of QQ', P the point of contact of the tangent parallel to QQ', and R any other point on the conic, let Qr parallel to TQ' meet $Q'R$ in r, and $Q'r'$ parallel to TQ meet QR in r'; then

$$Qr \cdot Q'r' : QQ'^2 = (PV^2 : PT^2) \cdot (TQ \cdot TQ' : QV^2). \quad \text{(III. 54, 56.)}$$

3. If the tangents are tangents to opposite branches of a hyperbola and meet in t, and if R, r, r' are taken as before, while tq is half the chord through t parallel to QQ', then

$$Qr \cdot Q'r' : QQ'^2 = tQ \cdot tQ' : tq^2. \quad \text{(III. 55.)}$$

The second of these propositions leads at once to the three-line locus, and from this we easily obtain the Cartesian equation to a conic with reference to two fixed tangents as axes, where the lengths of the tangents are h, k, viz.

$$\left(\frac{x}{h} + \frac{y}{k} - 1\right)^2 = 2\lambda \left(\frac{xy}{hk}\right)^{\frac{1}{2}}.$$

Book IV is on the whole dull, and need not be noticed at length. Props. 1–23 prove the converse of the propositions in Book III about the harmonic properties of the pole and polar for a large number of particular cases. One of the propositions (IV. 9) gives a method of drawing two tangents to a conic from an external point T. Draw any two straight lines through T cutting the conic in Q, Q' and in R, R' respec-

tively. Take O on QQ' and O' on RR' so that TQ', TR' are harmonically divided. The intersections of OO' produced with the conic give the two points of contact required.

The remainder of the Book (IV. 24–57) deals with intersecting conics, and the number of points in which, in particular cases, they can intersect or touch. IV. 24 proves that no two conics can meet in such a way that part of one of them is common to both, while the rest is not. The rest of the propositions can be divided into five groups, three of which can be brought under one general enunciation. Group I consists of particular cases depending on the more elementary considerations affecting conics: e.g. two conics having their concavities in opposite directions will not meet in more than two points (IV. 35); if a conic meet one branch of a hyperbola, it will not meet the other branch in more points than two (IV. 37); a conic touching one branch of a hyperbola with its concave side will not meet the opposite branch (IV. 39). IV. 36, 41, 42, 45, 54 belong to this group. Group II contains propositions (IV. 25, 38, 43, 44, 46, 55) showing that no two conics (including in the term the double-branch hyperbola) can intersect in more than four points. Group III (IV. 26, 47, 48, 49, 50, 56) are particular cases of the proposition that two conics which touch at one point cannot intersect at more than two other points. Group IV (IV. 27, 28, 29, 40, 51, 52, 53, 57) are cases of the proposition that no two conics which touch each other at two points can intersect at any other point. Group V consists of propositions about double contact. A parabola cannot touch another parabola in more points than one (IV. 30); this follows from the property $TP = PV$. A parabola, if it fall outside a hyperbola, cannot have double contact with it (IV. 31); it is shown that for the hyperbola $PV > PT$, while for the parabola $P'V = P'T$; therefore the hyperbola would fall outside the parabola, which is impossible. A parabola cannot have internal double contact with an ellipse or circle (IV. 32). A hyperbola cannot have double contact with another hyperbola having the same centre (IV. 33); proved by means of $CV \cdot CT = CP^2$. If an ellipse have double contact with an ellipse or a circle, the chord of contact will pass through the centre (IV. 34).

Book V is of an entirely different order, indeed it is the

most remarkable of the extant Books. It deals with normals to conics regarded as *maximum* and *minimum* straight lines drawn from particular points to the curve. Included in it are a series of propositions which, though worked out by the purest geometrical methods, actually lead immediately to the determination of the evolute of each of the three conics; that is to say, the Cartesian equations to the evolutes can be easily deduced from the results obtained by Apollonius. There can be no doubt that the Book is almost wholly original, and it is a veritable geometrical *tour de force*.

Apollonius in this Book considers various points and classes of points with reference to the maximum or minimum straight lines which it is possible to draw from them to the conics, i.e. as the feet of normals to the curve. He begins naturally with points on the axis, and he takes first the point E where AE measured along the axis from the vertex A is $\frac{1}{2}p$, p being the principal parameter. The first three propositions prove generally and for certain particular cases that, if in an ellipse or a hyperbola AM be drawn at right angles to AA' and equal to $\frac{1}{2}p$, and if CM meet the ordinate PN of any point P of the curve in H, then $PN^2 = 2$ (quadrilateral $MANH$); this is a lemma used in the proofs of later propositions, V. 5, 6, &c. Next, in V. 4, 5, 6, he proves that, if $AE = \frac{1}{2}p$, then AE is the *minimum* straight line from E to the curve, and if P be any other point on it, PE increases as P moves farther away from A on either side; he proves in fact that, if PN be the ordinate from P,

(1) in the case of the parabola $PE^2 = AE^2 + AN^2$,

(2) in the case of the hyperbola or ellipse

$$PE^2 = AE^2 + AN^2 \cdot \frac{AA' \pm p}{AA'},$$

where of course $p = BB'^2/AA'$, and therefore $(AA' \pm p)/AA'$ is equivalent to what we call e^2, the square of the eccentricity. It is also proved that EA' is the *maximum* straight line from E to the curve. It is next proved that, if O be any point on the axis between A and E, OA is the minimum straight line from O to the curve and, if P is any other point on the curve, OP increases as P moves farther from A (V. 7).

Next Apollonius takes points G on the axis at a distance from A greater than $\frac{1}{2}p$, and he proves that the *minimum* straight line from G to the curve (i.e. the normal) is GP, where P is such a point that

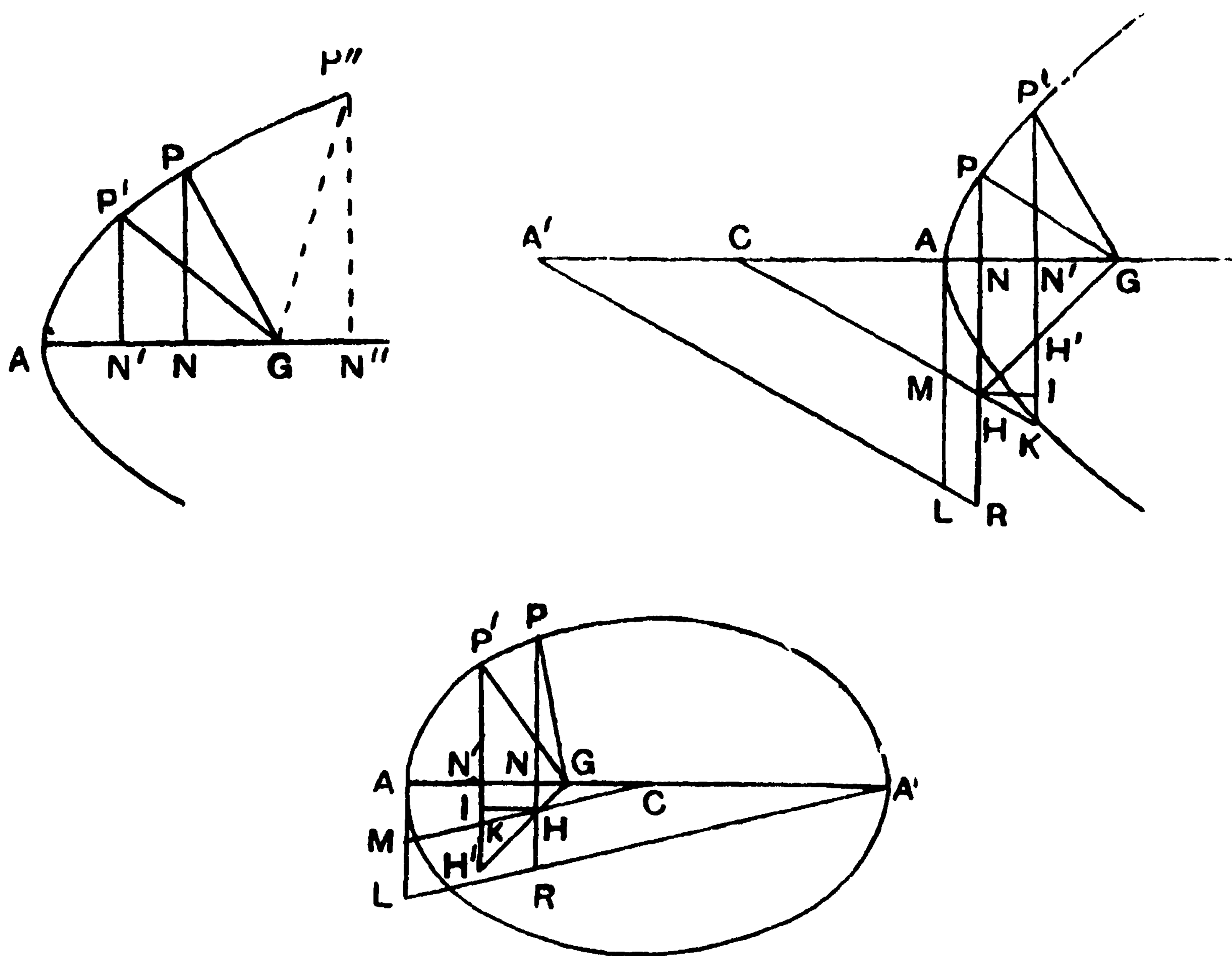
(1) in the case of the parabola $NG = \frac{1}{2}p$;

(2) in the case of the central conic $NG : CN = p : AA'$;

and, if P' is any other point on the conic, $P'G$ increases as P' moves away from P on either side; this is proved by showing that

(1) for the parabola $P'G^2 = PG^2 + NN'^2$;

(2) for the central conic $P'G^2 = PG^2 + NN'^2 . \dfrac{AA' \pm p}{AA'}$.



As these propositions contain the fundamental properties of the subnormals, it is worth while to reproduce Apollonius's proofs.

(1) In the parabola, if G be any point on the axis such that $AG > \frac{1}{2}p$, measure GN towards A equal to $\frac{1}{2}p$. Let PN be the ordinate through N, P' any other point on the curve. Then shall PG be the minimum line from G to the curve, &c.

We have $P'N'^2 = p \,.\, AN' = 2NG \,.\, AN'$;

and $N'G^2 = NN'^2 + NG^2 \pm 2NG \,.\, NN'$,

according to the position of N'.

Therefore $P'G^2 = 2NG \,.\, AN + NG^2 + NN'^2$

$$= PN^2 + NG^2 + NN'^2$$

$$= PG^2 + NN'^2;$$

and the proposition is proved.

(2) In the case of the central conic, take G on the axis such that $AG > \frac{1}{2}p$, and measure GN towards A such that

$$NG : CN = p : AA'.$$

Draw the ordinate PN through N, and also the ordinate $P'N'$ from any other point P'.

We have first to prove the lemma (V. 1, 2, 3) that, if AM be drawn perpendicular to AA' and equal to $\frac{1}{2}p$, and if CM, produced if necessary, meet PN in H, then

$$PN^2 = 2(\text{quadrilateral } MANH).$$

This is easy, for, if $AL\,(= 2AM)$ be the parameter, and $A'L$ meet PN in R, then, by the property of the curve,

$$PN^2 = AN \,.\, NR$$

$$= AN(NH + AM)$$

$$= 2\,(\text{quadrilateral } MANH).$$

Let GH, produced if necessary, meet $P'N'$ in H'. From H draw HI perpendicular to $P'H'$.

Now, since, by hypothesis, $NG : CN = p : AA'$

$$= AM : AC$$

$$= HN : NC,$$

$NH = NG$, whence also $H'N' = N'G$.

Therefore $NG^2 = 2\Delta HNG$, $N'G^2 = 2\Delta H'N'G$.

And $PN^2 = 2(MANH)$;

therefore $PG^2 = NG^2 + PN^2 = 2(AMHG)$.

Similarly, if CM meets $P'N'$ in K,

$$P'G^2 = N'G^2 + P'N'^2$$
$$= 2\Delta H'N'G + 2(AMKN')$$
$$= 2(AMHG) + 2\Delta HH'K.$$

Therefore, by subtraction,

$$P'G^2 - PG^2 = 2\Delta HH'K$$
$$= HI.(H'I \pm IK)$$
$$= HI.(HI \pm IK)$$
$$= HI^2.\frac{CA \pm AM}{CA}$$
$$= NN'^2.\frac{AA' \pm p}{AA'};$$

which proves the proposition.

If O be any point on PG, OP is the minimum straight line from O to the curve, and OP' increases as P' moves away from P on either side; this is proved in V. 12. (Since $P'G > PG$, $\angle GPP' > \angle GP'P$; therefore, *a fortiori*, $\angle OPP' > \angle OP'P$, and $OP' > OP$.)

Apollonius next proves the corresponding propositions with reference to points on the *minor* axis of an ellipse. If p' be the parameter of the ordinates to the minor axis, $p' = AA'^2/BB'$, or $\frac{1}{2}p' = CA^2/CB$. If now E' be so taken that $BE' = \frac{1}{2}p'$, then BE' is the *maximum* straight line from E' to the curve and, if P be any other point on it, $E'P$ diminishes as P moves farther from B on either side, and $E'B'$ is the *minimum* straight line from E' to the curve. It is, in fact, proved that $E'B^2 - E'P^2 = Bn^2.\frac{p' - BB'}{BB'}$, where Bn is the abscissa of P (V. 16–18). If O be any point on the minor axis such that $BO > BE'$, then OB is the *maximum* straight line from O to the curve, &c. (V. 19).

If g be a point on the minor axis such that $Bg > BC$, but $Bg < \frac{1}{2}p'$, and if Cn be measured towards B so that

$$Cn : ng = BB' : p',$$

then n is the foot of the ordinates of two points P such that Pg is the *maximum* straight line from g to the curve. Also,

if P' be any other point on it, $P'g$ diminishes as P' moves farther from P on either side to B or B', and

$$Pg^2 - P'g^2 = nn'^2 \cdot \frac{p' - BB'}{BB'} \text{ or } nn'^2 \cdot \frac{CA^2 - CB^2}{CB^2}.$$

If O be any point on Pg produced beyond the minor axis, PO is the *maximum* straight line from O to the same part of the ellipse for which Pg is a maximum, i.e. the semi-ellipse BPB', &c. (V. 20–2).

In V. 23 it is proved that, if g is on the minor axis, and gP a maximum straight line to the curve, and if Pg meets AA' in G, then GP is the *minimum* straight line from G to the curve; this is proved by similar triangles. Only one normal can be drawn from any one point on a conic (V. 24–6). The normal at any point P of a conic, whether regarded as a minimum straight line from G on the major axis or (in the case of the ellipse) as a *maximum* straight line from g on the minor axis, is perpendicular to the tangent at P (V. 27–30); in general (1) if O be any point within a conic, and OP be a maximum or a minimum straight line from O to the conic, the straight line through P perpendicular to PO touches the conic, and (2) if O' be any point on OP produced outside the conic, $O'P$ is the minimum straight line from O' to the conic, &c. (V. 31–4).

Number of normals from a point.

We now come to propositions about two or more normals meeting at a point. If the normal at P meet the axis of a parabola or the axis AA' of a hyperbola or ellipse in G, the angle PGA increases as P or G moves farther away from A, but in the case of the hyperbola the angle will always be less than the complement of half the angle between the asymptotes. Two normals at points on the same side of AA' will meet on the opposite side of that axis; and two normals at points on the same quadrant of an ellipse as AB will meet at a point within the angle ACB' (V. 35–40). In a parabola or an ellipse any normal PG will meet the curve again; in the hyperbola, (1) if AA' be not greater than p, no normal can meet the curve at a second point on the same branch, but

(2) if $AA' > p$, some normals will meet the same branch again and others not (V. 41–3).

If P_1G_1, P_2G_2 be normals at points on one side of the axis of a conic meeting in O, and if O be joined to any other point P on the conic (it being further supposed in the case of the ellipse that all three lines OP_1, OP_2, OP cut the same half of the axis), then

(1) OP cannot be a normal to the curve;

(2) if OP meet the axis in K, and PG be the normal at P, AG is less or greater than AK according as P does or does not lie between P_1 and P_2.

From this proposition it is proved that (1) three normals at points on one quadrant of an ellipse cannot meet at one point, and (2) four normals at points on one semi-ellipse bounded by the major axis cannot meet at one point (V. 44–8).

In any conic, if M be any point on the axis such that AM is not greater than $\frac{1}{2}p$, and if O be any point on the double ordinate through M, then no straight line drawn to any point on the curve on the other side of the axis from O and meeting the axis between A and M can be a normal (V. 49, 50).

Propositions leading immediately to the determination of the evolute *of a conic.*

These great propositions are V. 51, 52, to the following effect:

If AM measured along the axis be greater than $\frac{1}{2}p$ (but in the case of the ellipse less than AC), and if MO be drawn perpendicular to the axis, then a certain length (y, say) can be assigned such that

(a) if $OM > y$, no normal can be drawn through O which cuts the axis; but, if OP be any straight line drawn to the curve cutting the axis in K, $NK < NG$, where PN is the ordinate and PG the normal at P;

(b) if $OM = y$, only one normal can be so drawn through O, and, if OP be any other straight line drawn to the curve and cutting the axis in K, $NK < NG$, as before;

(c) if $OM < y$, two normals can be so drawn through O, and, if OP be any other straight line drawn to the curve, NK is

greater or less than NG according as OP is or is not intermediate between the two normals (V. 51, 52).

The proofs are of course long and complicated. The length y is determined in this way:

(1) In the case of the parabola, measure MH towards the vertex equal to $\frac{1}{2}p$, and divide AH at N_1 so that $HN_1 = 2N_1A$. The length y is then taken such that

$$y : P_1N_1 = N_1H : HM,$$

where P_1N_1 is the ordinate passing through N_1;

(2) In the case of the hyperbola and ellipse, we have $AM > \frac{1}{2}p$, so that $CA : AM < AA' : p$; therefore, if H be taken on AM such that $CH : HM = AA' : p$, H will fall between A and M.

Take two mean proportionals CN_1, CI between CA and CH, and let P_1N_1 be the ordinate through N_1.

The length y is then taken such that

$$y : P_1N_1 = (CM : MH) . (HN_1 : N_1C).$$

In the case (*b*), where $OM = y$, O is the point of intersection of consecutive normals, i.e. O is the centre of curvature at the point P; and, by considering the coordinates of O with reference to two coordinate axes, we can derive the Cartesian equations of the evolutes. E.g. (1) in the case of the parabola let the coordinate axes be the axis and the tangent at the vertex. Then $AM = x$, $OM = y$. Let $p = 4a$; then

$$HM = 2a,\quad N_1H = \tfrac{2}{3}(x-2a),\ \text{and}\ AN_1 = \tfrac{1}{3}(x-2a).$$

But $\quad y^2 : P_1N_1^2 = N_1H^2 : HM^2$, by hypothesis,

or $\quad y^2 : 4a . AN_1 = N_1H^2 : 4a^2;$

therefore $\quad ay^2 = AN_1 . N_1H^2,$

$\quad = \frac{4}{27}(x-2a)^3,$

or $\quad 27ay^2 = 4(x-2a)^3.$

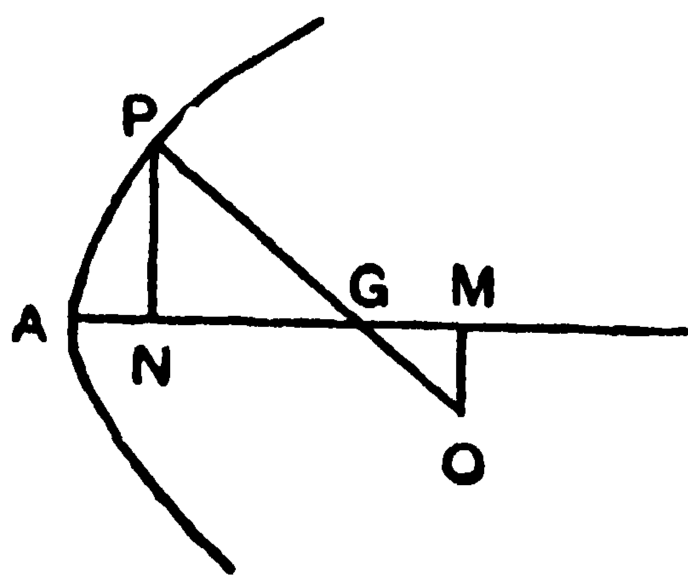
(2) In the case of the hyperbola or ellipse we naturally take CA, CB as axes of x and y. The work is here rather more complicated, but there is no difficulty in obtaining, as the locus of O, the curve

$$(ax)^{\frac{2}{3}} \mp (by)^{\frac{2}{3}} = (a^2 \pm b^2)^{\frac{2}{3}}.$$

The propositions V. 53, 54 are particular cases of the preceding propositions.

Construction of normals.

The next section of the Book (V. 55–63) relates to the construction of normals through various points according to their position within or without the conic and in relation to the axes. It is proved that one normal can be drawn through any internal point and through any external point which is not on the axis through the vertex A. In particular, if O is any point below the axis AA' of an ellipse, and OM is perpendicular to AA', then, if $AM > AC$, one normal can always be drawn through O cutting the axis between A and C, but never more than one such normal (V. 55–7). The points on the curve at which the straight lines through O are normals are determined as the intersections of the conic with a certain rectangular hyperbola. The procedure of Apollonius is equivalent to the following analytical method. Let AM be the axis of a conic, PGO one of the normals which passes through the given point O, PN the ordinate at P; and let OM be drawn perpendicular to the axis.



Take as axes of coordinates the axes in the central conic and, in the case of the parabola, the axis and the tangent at the vertex.

If then (x, y) be the coordinates of P and (x_1, y_1) those of O we have

$$\frac{y}{-y_1} = \frac{NG}{x_1 - x - NG}.$$

Therefore (1) for the parabola

$$\frac{y}{-y_1} = \frac{\frac{1}{2}p}{x_1 - x - \frac{1}{2}p},$$

or

$$xy - (x_1 - \tfrac{1}{2}p)\,y - y_1 \cdot \tfrac{1}{2}p = 0; \qquad (1)$$

(2) in the ellipse or hyperbola

$$xy\left(1 \mp \frac{b^2}{a^2}\right) - x_1 y \pm \frac{b^2}{a^2} \cdot y_1 x = 0. \qquad (2)$$

The intersections of these rectangular hyperbolas respec-

tively with the conics give the points at which the normals passing through O are normals.

Pappus criticizes the use of the rectangular hyperbola in the case of the parabola as an unnecessary resort to a '*solid* locus'; the meaning evidently is that the same points of intersection can be got by means of a certain circle taking the place of the rectangular hyperbola. We can, in fact, from the equation (1) above combined with $y^2 = px$, obtain the circle

$$(x^2+y^2)-(x_1+\tfrac{1}{2}p)x-\tfrac{1}{2}y_1y = 0.$$

The Book concludes with other propositions about maxima and minima. In particular V. 68–71 compare the lengths of tangents TQ, TQ', where Q is nearer to the axis than Q'. V. 72, 74 compare the lengths of two normals from a point O from which only two can be drawn and the lengths of other straight lines from O to the curve; V. 75–7 compare the lengths of three normals to an ellipse drawn from a point O below the major axis, in relation to the lengths of other straight lines from O to the curve.

Book VI is of much less interest. The first part (VI. 1–27) relates to equal (i.e. congruent) or similar conics and segments of conics; it is naturally preceded by some definitions including those of 'equal' and 'similar' as applied to conics and segments of conics. Conics are said to be similar if, the same number of ordinates being drawn to the axis at proportional distances from the vertices, all the ordinates are respectively proportional to the corresponding abscissae. The definition of similar segments is the same with diameter substituted for axis, and with the additional condition that the angles between the base and diameter in each are equal. Two parabolas are equal if the ordinates to a diameter in each are inclined to the respective diameters at equal angles and the corresponding parameters are equal; two ellipses or hyperbolas are equal if the ordinates to a diameter in each are equally inclined to the respective diameters and the diameters as well as the corresponding parameters are equal (VI. 1. 2). Hyperbolas or ellipses are similar when the 'figure' on a diameter of one is similar (instead of equal) to the 'figure' on a diameter of the other, and the ordinates to the diameters in

each make equal angles with them; all parabolas are similar (VI. 11, 12, 13). No conic of one of the three kinds (parabolas, hyperbolas or ellipses) can be equal or similar to a conic of either of the other two kinds (VI. 3, 14, 15). Let QPQ', qpq' be two segments of similar conics in which QQ', qq' are the bases and PV, pv are the diameters bisecting them; then, if PT, pt be the tangents at P, p and meet the axes at T, t at equal angles, and if $PV:PT = pv:pt$, the segments are similar and similarly situated, and conversely (VI. 17, 18). If two ordinates be drawn to the axes of two parabolas, or the major or conjugate axes of two similar central conics, as PN, $P'N'$ and pn, $p'n'$ respectively, such that the ratios $AN:an$ and $AN':an'$ are each equal to the ratio of the respective *latera recta*, the segments PP', pp' will be similar; also PP' will not be similar to any segment in the other conic cut off by two ordinates other than pn, $p'n'$, and conversely (VI. 21, 22). If any cone be cut by two parallel planes making hyperbolic or elliptic sections, the sections will be similar but not equal (VI. 26, 27).

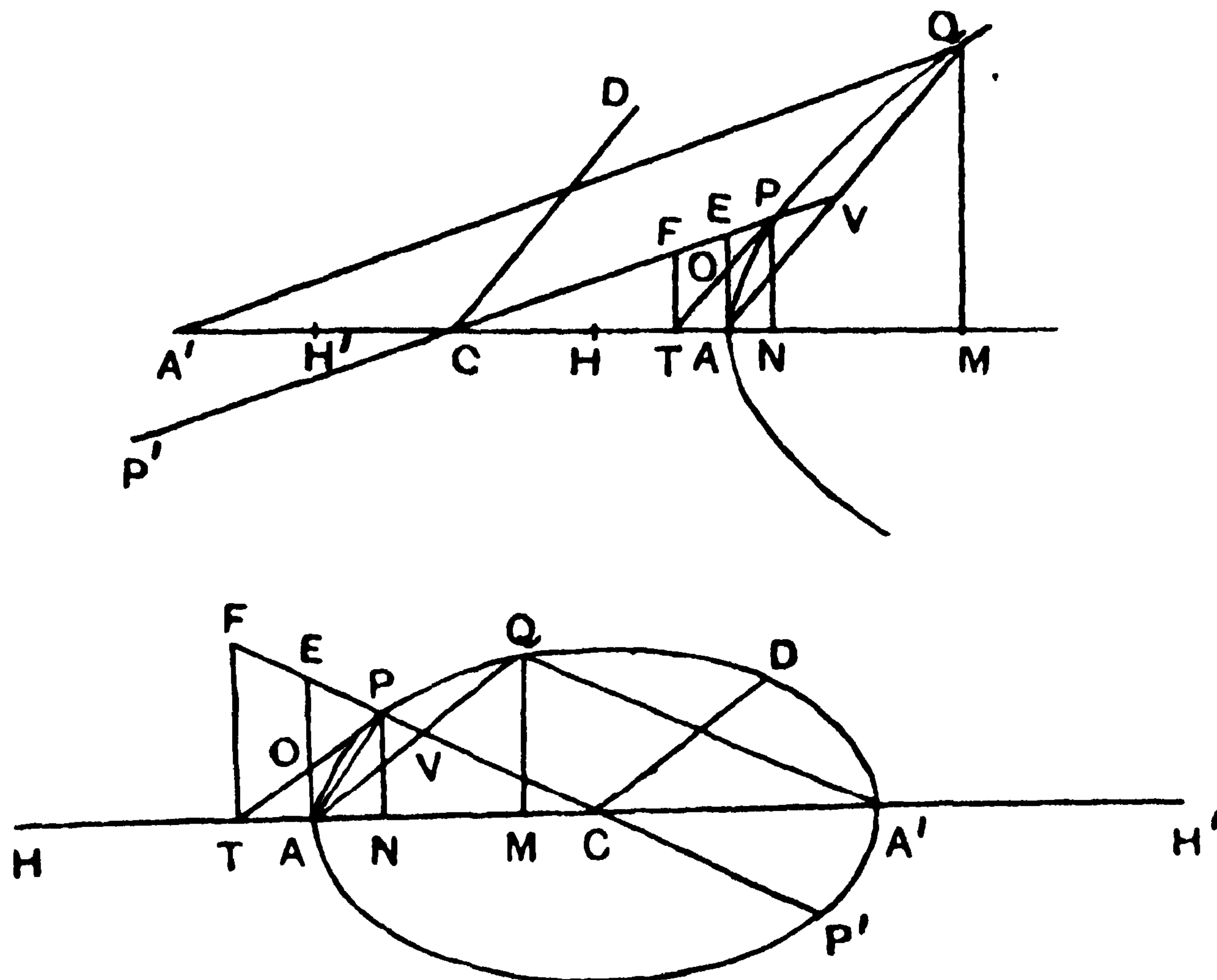
The remainder of the Book consists of problems of construction; we are shown how in a given right cone to find a parabolic, hyperbolic or elliptic section equal to a given parabola, hyperbola or ellipse, subject in the case of the hyperbola to a certain *διορισμός* or condition of possibility (VI. 28–30); also how to find a right cone similar to a given cone and containing a given parabola, hyperbola or ellipse as a section of it, subject again in the case of the hyperbola to a certain *διορισμός* (VI. 31–3). These problems recall the somewhat similar problems in I. 51–9.

Book VII begins with three propositions giving expressions for AP^2 $(= AN^2 + PN^2)$ in the same form as those for PN^2 in the statement of the ordinary property. In the parabola AH is measured along the axis produced (i.e. in the opposite direction to AN) and of length equal to the *latus rectum*, and it is proved that, for any point P, $AP^2 = AN.NH$ (VII. 1). In the case of the central conics AA' is divided at H, internally for the hyperbola and externally for the ellipse (AH being the segment adjacent to A) so that $AH:A'H = p:AA'$, where p is the parameter corresponding to AA', or $p = BB'^2/AA'$, and it is proved that

$$AP^2 : AN.NH = AA' : A'H.$$

The same is true if AA' is the minor axis of an ellipse and p the corresponding parameter (VII. 2, 3).

If AA' be divided at H' as well as H (internally for the hyperbola and externally for the ellipse) so that H is adjacent to A and H' to A', and if $A'H : AH = AH' : A'H' = AA' : p$, the lines AH, $A'H'$ (corresponding to p in the proportion) are called by Apollonius *homologues*, and he makes considerable



use of the auxiliary points H, H' in later propositions from VII. 6 onwards. Meantime he proves two more propositions, which, like VII. 1–3, are by way of lemmas. First, if CD be the semi-diameter parallel to the tangent at P to a central conic, and if the tangent meet the axis AA' in T, then

$$PT^2 : CD^2 = NT : CN. \qquad \text{(VII. 4.)}$$

Draw AE, TF at right angles to CA to meet CP, and let AE meet PT in O. Then, if p' be the parameter of the ordinates to CP, we have

$$\tfrac{1}{2}p' : PT = OP : PE \qquad \text{(I. 49, 50.)}$$

$$= PT : PF,$$

or $$\tfrac{1}{2}p' . PF = PT^2.$$

Therefore $$PT^2 : CD^2 = \tfrac{1}{2}p' . PF : \tfrac{1}{2}p' . CP$$

$$= PF : CP$$

$$= NT : CN.$$

Secondly, Apollonius proves that, if PN be a principal ordinate in a parabola, p the principal parameter, p' the parameter of the ordinates to the diameter through P, then $p' = p + 4AN$ (VII. 5); this is proved by means of the same property as VII. 4, namely $\frac{1}{2}p' : PT = OP : PE$.

Much use is made in the remainder of the Book of two points Q and M, where AQ is drawn parallel to the conjugate diameter CD to meet the curve in Q, and M is the foot of the principal ordinate at Q; since the diameter CP bisects both AA' and QA, it follows that $A'Q$ is parallel to CP. Many ratios between functions of PP', DD' are expressed in terms of AM, $A'M$, MH, MH', AH, $A'H$, &c. The first propositions of the Book proper (VII. 6, 7) prove, for instance, that $PP'^2 : DD'^2 = MH' : MH$.

For $PT^2 : CD^2 = NT : CN = AM : A'M$, by similar triangles.

Also $CP^2 : PT^2 = A'Q^2 : AQ^2$.

Therefore, *ex aequali*,

$$\begin{aligned} CP^2 : CD^2 &= (AM : A'M) \times (A'Q^2 : AQ^2) \\ &= (AM : A'M) \times (A'Q^2 : A'M \,.\, MH') \\ &\qquad \times (A'M \,.\, MH' : AM \,.\, MH) \times (AM \,.\, MH : AQ^2) \\ &= (AM : A'M) \times (AA' : AH') \times (A'M : AM) \\ &\qquad \times (MH' : MH) \times (A'H : AA'), \text{ by aid of VII. 2, 3.} \end{aligned}$$

Therefore $\qquad PP'^2 : DD'^2 = MH' : MH.$

Next (VII. 8, 9, 10, 11) the following relations are proved, namely

(1) $AA'^2 : (PP' \pm DD')^2 = A'H \,.\, MH' : \{MH' \pm \sqrt{(MH \,.\, MH')}\}^2$,

(2) $AA'^2 : PP' \,.\, DD' = A'H : \sqrt{(MH \,.\, MH')}$,

(3) $AA'^2 : (PP'^2 \pm DD'^2) = A'H : MH \pm MH'$.

The steps by which these results are obtained are as follows.

First, $$AA'^2 : PP'^2 = A'H : MH' \qquad (\alpha)$$
$$= A'H \,.\, MH' : MH'^2.$$

(This is proved thus:

$$\begin{aligned} AA'^2 : PP'^2 &= CA^2 : CP^2 \\ &= CN \,.\, CT : CP^2 \\ &= A'M \,.\, A'A : A'Q^2. \end{aligned}$$

But $$A'Q^2 : A'M . MH' = AA' : AH' \quad \text{(VII. 2, 3)}$$
$$= AA' : A'H$$
$$= A'M . AA' : A'M . A'H,$$

so that, alternately,

$$A'M . AA' : A'Q^2 = A'M . A'H : A'M . MH'$$
$$= A'H : MH'.)$$

Next, $$PP'^2 : DD'^2 = MH' : MH, \text{ as above,} \quad (\beta)$$
$$= MH'^2 : MH . MH',$$

whence $$PP' : DD' = MH' : \sqrt{(MH . MH')}, \quad (\gamma)$$

and $$PP'^2 : (PP' \pm DD')^2 = MH'^2 : \{MH' \pm \sqrt{(MH . MH')}\}^2;$$

(1) above follows from this relation and (α) *ex aequali*;

(2) follows from (α) and (γ) *ex aequali*, and (3) from (α) and (β).

We now obtain immediately the important proposition that $PP'^2 \pm DD'^2$ is constant, whatever be the position of P on an ellipse or hyperbola (the upper sign referring to the ellipse), and is equal to $AA'^2 \pm BB'^2$ (VII. 12, 13, 29, 30).

For $$AA'^2 : BB'^2 = AA' : p = A'H : AH = A'H : A'H',$$

by construction:

therefore $$AA'^2 : AA'^2 \pm BB'^2 = A'H : HH';$$

also, from (α) above,

$$AA'^2 : PP'^2 = A'H : MH';$$

and, by means of (β),

$$PP'^2 : (PP'^2 \pm DD'^2) = MH' : MH' \pm MH$$
$$= MH' : HH'.$$

Ex aequali, from the last two relations, we have

$$AA'^2 : (PP'^2 \pm DD'^2) = A'H : HH'$$
$$= AA'^2 : AA'^2 \pm BB'^2, \text{ from above,}$$

whence $$PP'^2 \pm DD'^2 = AA'^2 \pm BB'^2.$$

A number of other ratios are expressed in terms of the straight lines terminating at A, A', H, H', M, M' as follows (VII. 14–20).

In the ellipse $AA'^2 : PP'^2 \sim DD'^2 = A'H : 2CM$,

and in the hyperbola or ellipse (if p be the parameter of the ordinates to PP')

$$AA'^2 : p^2 = A'H \,.\, MH' : MH^2,$$

$$AA'^2 : (PP' \pm p)^2 = A'H \,.\, MH' : (MH \pm MH')^2,$$

$$AA'^2 : PP' \,.\, p = A'H : MH,$$

and $AA'^2 : (PP'^2 \pm p^2) = A'H \,.\, MH' : (MH'^2 \pm MH^2)$.

Apollonius is now in a position, by means of all these relations, resting on the use of the auxiliary points H, H', M, to compare different functions of any conjugate diameters with the same functions of the axes, and to show how the former vary (by way of increase or diminution) as P moves away from A. The following is a list of the functions compared, where for brevity I shall use a, b to represent AA', BB'; a', b' to represent PP', DD'; and p, p' to represent the parameters of the ordinates to AA', PP' respectively.

In a hyperbola, according as $a >$ or $< b$, $a' >$ or $< b'$, and the ratio $a' : b'$ decreases or increases as P moves from A on either side; also, if $a = b$, $a' = b'$ (VII. 21–3); in an ellipse $a : b > a' : b'$, and the latter ratio diminishes as P moves from A to B (VII. 24).

In a hyperbola or ellipse $a + b < a' + b'$, and $a' + b'$ in the hyperbola increases continually as P moves farther from A, but in the ellipse increases till a', b' take the position of the equal conjugate diameters when it is a *maximum* (VII. 25, 26).

In a hyperbola in which a, b are unequal, or in an ellipse, $a \sim b > a' \sim b'$, and $a' \sim b'$ diminishes as P moves away from A, in the hyperbola continually, and in the ellipse till a', b' are the equal conjugate diameters (VII. 27).

$ab < a'b'$, and $a'b'$ increases as P moves away from A, in the hyperbola continually, and in the ellipse till a', b' coincide with the equal conjugate diameters (VII. 28).

VII. 31 is the important proposition that, if PP', DD' are

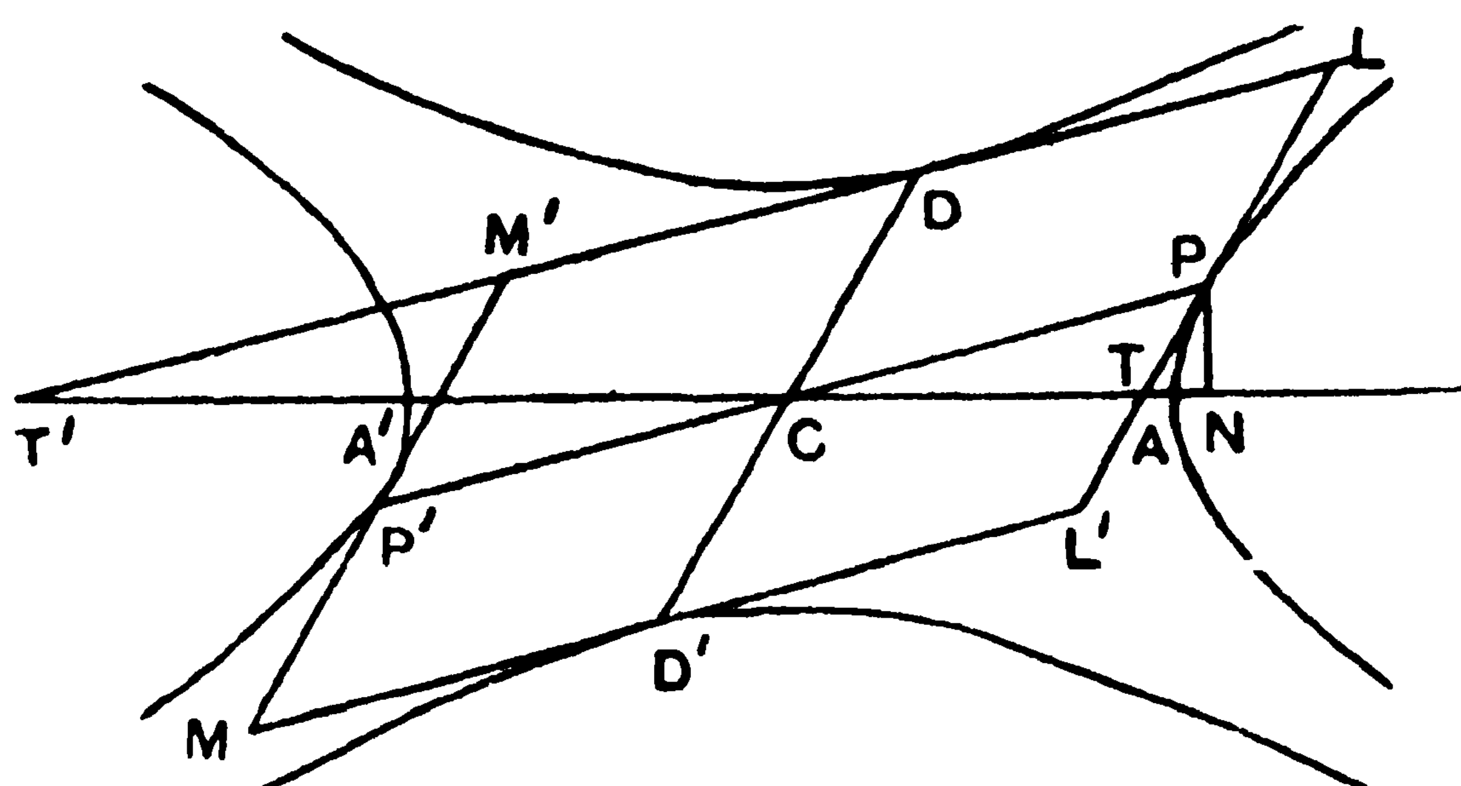
conjugate diameters in an ellipse or conjugate hyperbolas, and if the tangents at their extremities form the parallelogram $LL'MM'$, then

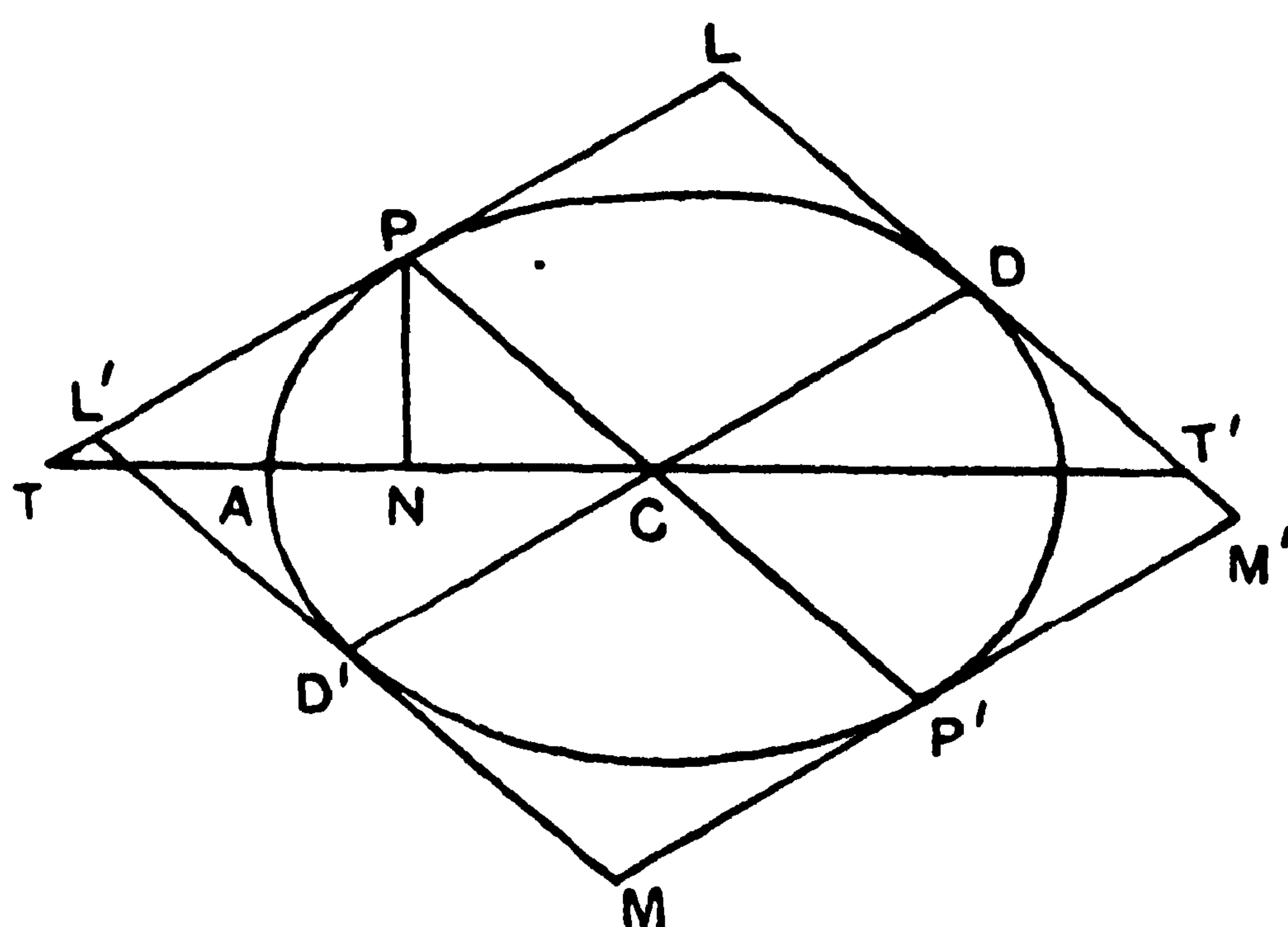
the parallelogram $LL'MM'$ = rect. $AA'.BB'$.

The proof is interesting. Let the tangents at P, D respectively meet the major or transverse axis in T, T'.

Now (by VII. 4) $PT^2 : CD^2 = NT : CN$;

therefore $2\triangle CPT : 2\triangle T'DC = NT : CN$.





But $2\triangle CPT : (CL) = PT : CD$,

$= CP : DT'$, by similar triangles,

$= (CL) : 2\triangle T'DC$.

That is, (CL) is a mean proportional between $2\triangle CPT$ and $2\triangle T'DC$.

Therefore, since $\sqrt{(NT.CN)}$ is a mean proportional between NT and CN,

$$2\Delta CPT:(CL) = \sqrt{(CN.NT)}:CN$$
$$= PN.\frac{CA}{CB}:CN \qquad \text{(I. 37, 39)}$$
$$= PN.CT:CT.CN.\frac{CB}{CA}$$
$$= 2\Delta CPT:CA.CB;$$

therefore $$(CL) = CA.CB.$$

The remaining propositions of the Book trace the variations of different functions of the conjugate diameters, distinguishing the maximum values, &c. The functions treated are the following:

p', the parameter of the ordinates to PP' in the hyperbola, according as AA' is (1) not less than p, the parameter corresponding to AA', (2) less than p but not less than $\frac{1}{2}p$, (3) less than $\frac{1}{2}p$ (VII. 33-5).

$PP' \sim p'$, as compared with $AA' \sim p$ in the hyperbola (VII. 36) or the ellipse (VII. 37).

$PP' + p'$ „ „ $AA' + p$ in the hyperbola (VII. 38-40) or the ellipse (VII. 41).

$PP'.p'$ „ „ $AA'.p$ in the hyperbola (VII. 42) or the ellipse (VII. 43).

$PP'^2 + p'^2$ „ „ $AA'^2 + p^2$ in the hyperbola, according as (1) AA' is not less than p, or (2) $AA' < p$, but AA'^2 not less than $\frac{1}{2}(AA' \sim p)^2$, or (3) $AA'^2 < \frac{1}{2}(AA' \sim p)^2$ (VII. 44-6).

$PP'^2 + p'^2$ „ „ $AA'^2 + p^2$ in the ellipse, according as AA'^2 is not greater, or is greater, than $(AA' + p)^2$ (VII. 47, 48).

$PP'^2 \sim p'^2$ „ „ $AA'^2 \sim p^2$ in the hyperbola, according as $AA' >$ or $< p$ (VII. 49, 50).

$PP'^2 \sim p'^2$ „ „ $AA'^2 \sim p^2$ or $BB'^2 \sim p_b{}^2$ in the ellipse, according as $PP' >$ or $< p'$ (VII. 51).

As we have said, Book VIII is lost. The nature of its contents can only be conjectured from Apollonius's own remark that it contained determinate conic problems for which Book VII was useful, particularly in determining limits of possibility. Unfortunately, the lemmas of Pappus do not enable us to form any clearer idea. But it is probable enough that the Book contained a number of problems having for their object the finding of conjugate diameters in a given conic such that certain functions of their lengths have given values. It was on this assumption that Halley attempted a restoration of the Book.

If it be thought that the above account of the *Conics* is disproportionately long for a work of this kind, it must be remembered that the treatise is a great classic which deserves to be more known than it is. What militates against its being read in its original form is the great extent of the exposition (it contains 387 separate propositions), due partly to the Greek habit of proving particular cases of a general proposition separately from the proposition itself, but more to the cumbrousness of the enunciations of complicated propositions in general terms (without the help of letters to denote particular points) and to the elaborateness of the Euclidean form, to which Apollonius adheres throughout.

Other works by Apollonius.

Pappus mentions and gives a short indication of the contents of six other works of Apollonius which formed part of the *Treasury of Analysis*.[1] Three of these should be mentioned in close connexion with the *Conics*.

(a) *On the Cutting-off of a Ratio* (λόγου ἀποτομή), two Books.

This work alone of the six mentioned has survived, and that only in the Arabic; it was published in a Latin translation by Edmund Halley in 1706. It deals with the general problem, '*Given two straight lines, parallel to one another or intersecting, and a fixed point on each line, to draw through*

[1] Pappus, vii, pp. 640–8, 660–72.

a given point a straight line which shall cut off segments from each line (measured from the fixed points) bearing a given ratio to one another.' Thus, let A, B be fixed points on the two given straight lines AC, BK, and let O be the given point. It is required to draw through O a straight line cutting the given straight lines in points M, N respectively such that AM is to BN in a given ratio. The two Books of the treatise discussed the various possible cases of this problem which arise according to the relative positions of the given straight lines and points, and also the necessary conditions and limits of possibility in cases where a solution is not always possible. The first Book begins by supposing the given lines to be parallel, and discusses the different cases which arise; Apollonius then passes to the cases in which the straight lines intersect, but one of the given points, A or B, is at the intersection of the two lines. Book II proceeds to the general case shown in the above figure, and first proves that the general case can be reduced to the case in Book I where one of the given points, A or B, is at the intersection of the two lines. The reduction is easy. For join OB meeting AC in B', and draw $B'N'$ parallel to BN to meet OM in N'. Then the ratio $B'N' : BN$, being equal to the ratio $OB' : OB$, is constant. Since, therefore, $BN : AM$ is a given ratio, the ratio $B'N' : AM$ is also given.

Apollonius proceeds in all cases by the orthodox method of analysis and synthesis. Suppose the problem solved and OMN drawn through O in such a way that $B'N' : AM$ is a given ratio $= \lambda$, say.

Draw OC parallel to BN or $B'N'$ to meet AM in C. Take D on AM such that $OC : AD = \lambda = B'N' : AM$.

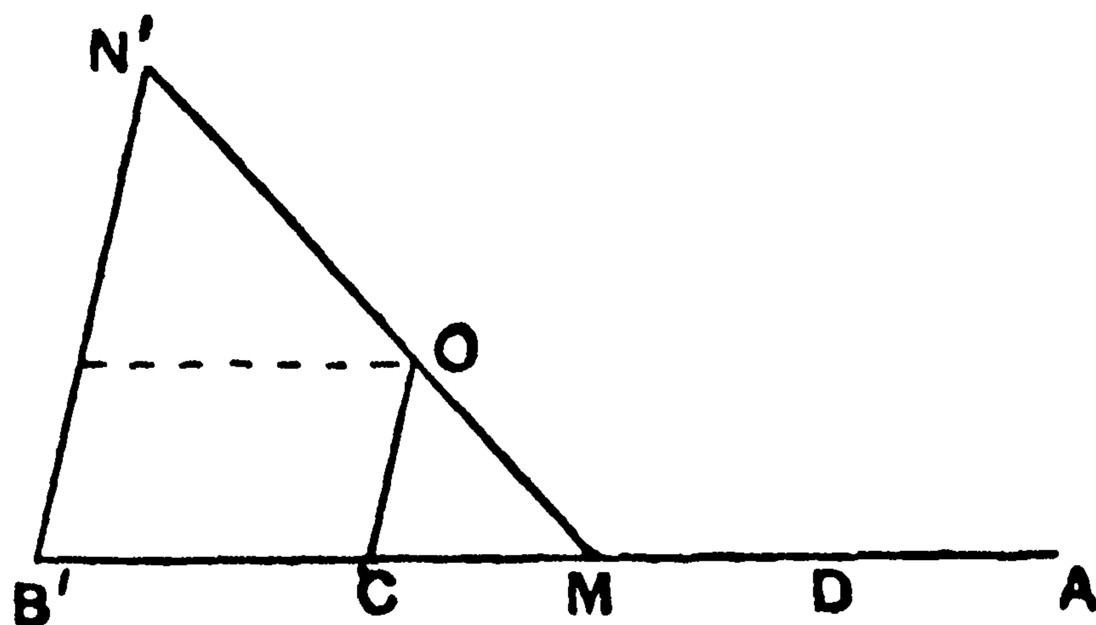
Then $$AM : AD = B'N' : OC$$
$$= B'M : CM;$$
therefore $$MD : AD = B'C : CM,$$
or $$CM . MD = AD . B'C, \text{ a given rectangle.}$$

Hence the problem is reduced to one of *applying to CD a rectangle ($CM . MD$) equal to a given rectangle ($AD . B'C$) but falling short by a square figure.* In the case as drawn, whatever be the value of λ, the solution is always possible because the given rectangle $AD . CB'$ is always less than $CA . AD$, and therefore always less than $\frac{1}{4}CD^2$; one of the positions of M falls between A and D because $CM . MD < CA . AD$.

The proposition III. 41 of the *Conics* about the intercepts made on two tangents to a parabola by a third tangent (pp. 155-6 above) suggests an obvious application of our problem. We had, with the notation of that proposition,

$$Pr : rq = rQ : Qp = qp : pR.$$

Suppose that the two tangents qP, qR are given as fixed tangents with their points of contact P, R. Then we can draw another tangent if we can draw a straight line intersecting qP, qR in such a way that $Pr : rq = qp : pR$ or $Pq : qr = qR : pR$, i.e. $qr : pR = Pq : qR$ (a constant ratio); i.e. we have to draw a straight line such that the intercept by it on qP measured from q has a given ratio to the intercept by it on qR measured from R. This is a particular case of our problem to which, as a matter of fact, Apollonius devotes special attention. In the annexed figure the letters have the



same meaning as before, and $N'M$ has to be drawn through O such that $B'N' : AM = \lambda$. In this case there are limits to

the value of λ in order that the solution may be possible. Apollonius begins by stating the limiting case, saying that we obtain a solution in a special manner in the case where M is the middle point of CD, so that the rectangle $CM.MD$ or $CB'.AD$ has its maximum value.

The corresponding limiting value of λ is determined by finding the corresponding position of D or M.

We have $$B'C : MD = CM : AD, \text{ as before,}$$
$$= B'M : MA;$$

whence, since $MD = CM$,

$$B'C : B'M = CM : MA$$
$$= B'M : B'A,$$

so that $$B'M^2 = B'C . B'A.$$

Thus M is found and therefore D also.

According, therefore, as λ is less or greater than the particular value of $OC : AD$ thus determined, Apollonius finds no solution or two solutions.

Further, we have

$$AD = B'A + B'C - (B'D + B'C)$$
$$= B'A + B'C - 2B'M$$
$$= B'A + B'C - 2\sqrt{B'A . B'C}.$$

If then we refer the various points to a system of co ordinates in which $B'A$, $B'N'$ are the axes of x and y, and i[f] we denote O by (x, y) and the length $B'A$ by h,

$$\lambda = OC/AD = y/(h + x - 2\sqrt{hx}).$$

If we suppose Apollonius to have used these results for th[e] parabola, he cannot have failed to observe that the limitin[g] case described is that in which O is on the parabola, whil[e] $N'OM$ is the tangent at O; for, as above,

$$B'M : B'A = B'C : B'M = N'O : N'M, \text{ by parallels,}$$

so that $B'A$, $N'M$ are divided at M, O respectively in the sam[e] proportion.

Further, if we put for λ the ratio between the lengths of the two fixed tangents, then if h, k be those lengths,

$$\frac{k}{h} = \frac{y}{h + x - 2\sqrt{hx}},$$

which can easily be reduced to

$$\left(\frac{x}{h}\right)^{\frac{1}{2}} + \left(\frac{y}{k}\right)^{\frac{1}{2}} = 1,$$

the equation of the parabola referred to the two fixed tangents as axes.

(β) *On the cutting-off of an area* (χωρίου ἀποτομή), two Books.

This work, also in two Books, dealt with a similar problem, with the difference that the intercepts on the given straight lines measured from the given points are required, not to have a given ratio, but to contain a given rectangle. Halley included an attempted restoration of this work in his edition of the *De sectione rationis*.

The general case can here again be reduced to the more special one in which one of the fixed points is at the intersection of the two given straight lines. Using the same figure as before, but with D taking the position shown by (D) in the figure, we take that point such that

$$OC \cdot AD = \text{the given rectangle.}$$

We have then to draw $ON'M$ through O such that

$$B'N' \cdot AM = OC \cdot AD,$$

or $$B'N' : OC = AD : AM.$$

But, by parallels, $$B'N' : OC = B'M : CM;$$

therefore $$AM : CM = AD : B'M$$

$$= MD : B'C,$$

so that $$B'M \cdot MD = AD \cdot B'C.$$

Hence, as before, the problem is reduced to an application of a rectangle in the well-known manner. The complete

treatment of this problem in all its particular cases with their *διορισμοί* could present no difficulty to Apollonius.

If the two straight lines are parallel, the solution of the problem gives a means of drawing any number of tangents to an ellipse when two parallel tangents, their points of contact, and the length of the parallel semi-diameter are given (see *Conics*, III. 42). In the case of the hyperbola (III. 43) the intercepts made by any tangent on the asymptotes contain a constant rectangle. Accordingly the drawing of tangents depends upon the particular case of our problem in which both fixed points are the intersection of the two fixed lines.

(γ) *On determinate section* (*διωρισμένη τομή*), two Books.

The general problem here is, Given four points A, B, C, D on a straight line, to determine another point P on the same straight line such that the ratio $AP.CP : BP.DP$ has a given value. It is clear from Pappus's account[1] of the contents of this work, and from his extensive collection of lemmas to the different propositions in it, that the question was very exhaustively discussed. To determine P by means of the equation

$$AP.CP = \lambda.BP.DP,$$

where A, B, C, D, λ are given, is in itself an easy matter since the problem can at once be put into the form of a quadratic equation, and the Greeks would have no difficulty in reducing it to the usual *application of areas*. If, however (as we may fairly suppose), it was intended for application in further investigations, the complete discussion of it would naturally include not only the finding of a solution, but also the determination of the limits of possibility and the number of possible solutions for different positions of the point-pairs A, C and B, D, for the cases in which the points in either pair coincide, or in which one of the points is infinitely distant, and so on. This agrees with what we find in Pappus, who makes it clear that, though we do not meet with any express mention of *series* of point-pairs determined by the equation for different values of λ, yet the treatise contained what amounts to a com-

[1] Pappus, vii, pp. 642–4.

plete *Theory of Involution*. Pappus says that the separate cases were dealt with in which the given ratio was that of either (1) the square of one abscissa measured from the required point or (2) the rectangle contained by two such abscissae to any one of the following: (1) the square of one abscissa, (2) the rectangle contained by one abscissa and another separate line of given length independent of the position of the required point, (3) the rectangle contained by two abscissae. We learn also that maxima and minima were investigated. From the lemmas, too, we may draw other conclusions, e. g.

(1) that, in the case where $\lambda = 1$, or $AP.CP = BP.DP$, Apollonius used the relation $BP:DP = AB.BC:AD.DC$,

(2) that Apollonius probably obtained a double point E of the involution determined by the point-pairs A, C and B, D by means of the relation

$$AB.BC:AD.DC = BE^2:DE^2.$$

A possible application of the problem was the determination of the points of intersection of the given straight line with a conic determined as a four-line locus, since A, B, C, D are in fact the points of intersection of the given straight line with the four lines to which the locus has reference.

(δ) *On Contacts* or *Tangencies* (ἐπαφαί), two Books.

Pappus again comprehends in one enunciation the varieties of problems dealt with in the treatise, which we may reproduce as follows: *Given three things, each of which may be either a point, a straight line or a circle, to draw a circle which shall pass through each of the given points (so far as it is points that are given) and touch the straight lines or circles.*[1] The possibilities as regards the different data are ten. We may have any one of the following: (1) three points, (2) three straight lines, (3) two points and a straight line, (4) two straight lines and a point, (5) two points and a circle, (6) two circles and a point, (7) two straight lines and

[1] Pappus, vii, p. 644, 25-8.

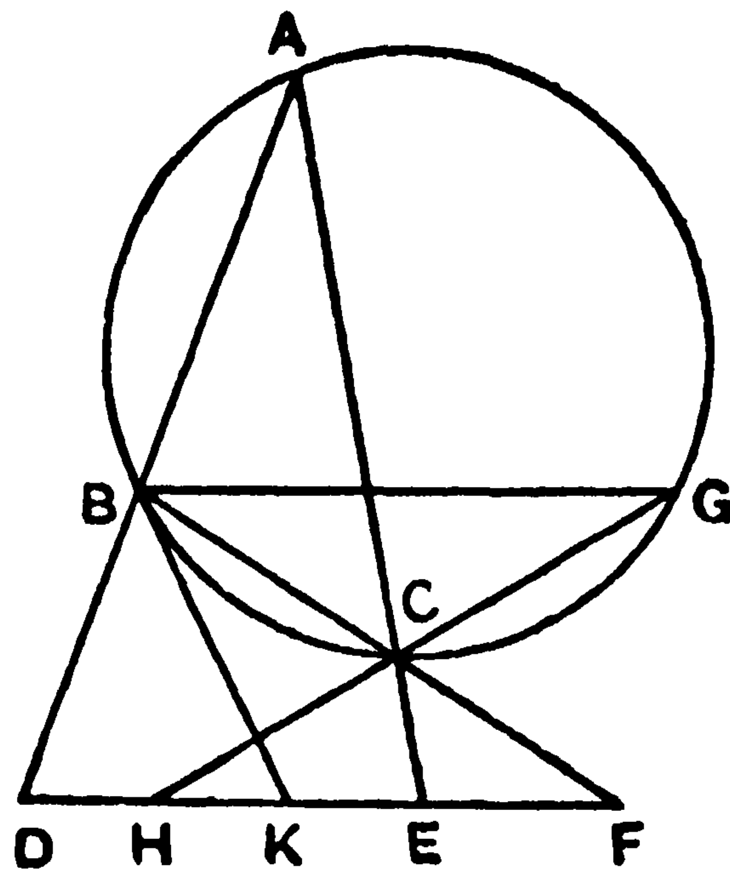
a circle, (8) two circles and a straight line, (9) a point, a circle and a straight line, (10) three circles. Of these varieties the first two are treated in Eucl. IV; Book I of Apollonius's treatise treated of (3), (4), (5), (6), (8), (9), while (7), the case of two straight lines and a circle, and (10), that of the three circles, occupied the whole of Book II.

The last problem (10), where the data are three circles, has exercised the ingenuity of many distinguished geometers, including Vieta and Newton. Vieta (1540–1603) set the problem to Adrianus Romanus (van Roomen, 1561–1615) who solved it by means of a hyperbola. Vieta was not satisfied with this, and rejoined with his *Apollonius Gallus* (1600) in which he solved the problem by plane methods. A solution of the same kind is given by Newton in his *Arithmetica Universalis* (Prob. xlvii), while an equivalent problem is solved by means of two hyperbolas in the *Principia*, Lemma xvi. The problem is quite capable of a 'plane' solution, and, as a matter of fact, it is not difficult to restore the actual solution of Apollonius (which of course used the 'plane' method depending on the straight line and circle only), by means of the lemmas given by Pappus. Three things are necessary to the solution. (1) A proposition, used by Pappus elsewhere[1] and easily proved, that, if two circles touch internally or externally, any straight line through the point of contact divides the circles into segments respectively similar. (2) The proposition that, given three circles, their six centres of similitude (external and internal) lie three by three on four straight lines. This proposition, though not proved in Pappus, was certainly known to the ancient geometers; it is even possible that Pappus omitted to prove it because it was actually proved by Apollonius in his treatise. (3) An auxiliary problem solved by Pappus and enunciated by him as follows.[2] Given a circle ABC, and given three points D, E, F in a straight line, to inflect (the broken line) DAE (to the circle) so as to make BC in a straight line with CF; in other words, to inscribe in the circle a triangle the sides of which, when produced, pass respectively through three given points lying in a straight line. This problem is interesting as a typical example of the ancient analysis followed by synthesis. Suppose the problem

[1] Pappus, iv, pp. 194–6. [2] *Ib.* vii, p. 848.

solved, i.e. suppose DA, EA drawn to the circle cutting it in points B, C such that BC produced passes through F.

Draw BG parallel to DF; join GC and produce it to meet DE in H.



Then

$$\angle BAC = \angle BGC$$
$$= \angle CHF$$
$$= \text{supplement of } \angle CHD;$$

therefore A, D, H, C lie on a circle, and

$$DE \,.\, EH = AE \,.\, EC.$$

Now $AE \,.\, EC$ is given, being equal to the square on the tangent from E to the circle; and DE is given; therefore HE is given, and therefore the point H.

But F is also given; therefore the problem is reduced to drawing HC, FC to meet the circle in such a way that, if HC, FC produced meet the circle again in G, B, the straight line BG is parallel to HF: a problem which Pappus has previously solved.[1]

Suppose this done, and draw BK the tangent at B meeting HF in K. Then

$$\angle KBC = \angle BGC, \text{ in the alternate segment,}$$
$$= \angle CHF.$$

Also the angle CFK is common to the two triangles KBF, CHF; therefore the triangles are similar, and

$$CF : FH = KF : FB,$$

or

$$HF \,.\, FK = BF \,.\, FC.$$

Now $BF \,.\, FC$ is given, and so is HF;
therefore FK is given, and therefore K is given.

The synthesis is as follows. Take a point H on DE such that $DE \,.\, EH$ is equal to the square on the tangent from E to the circle.

Next take K on HF such that $HF \,.\, FK =$ the square on the tangent from F to the circle.

Draw the tangent to the circle from K, and let B be the point of contact. Join BF meeting the circle in C, and join

[1] Pappus, vii, pp. 830–2.

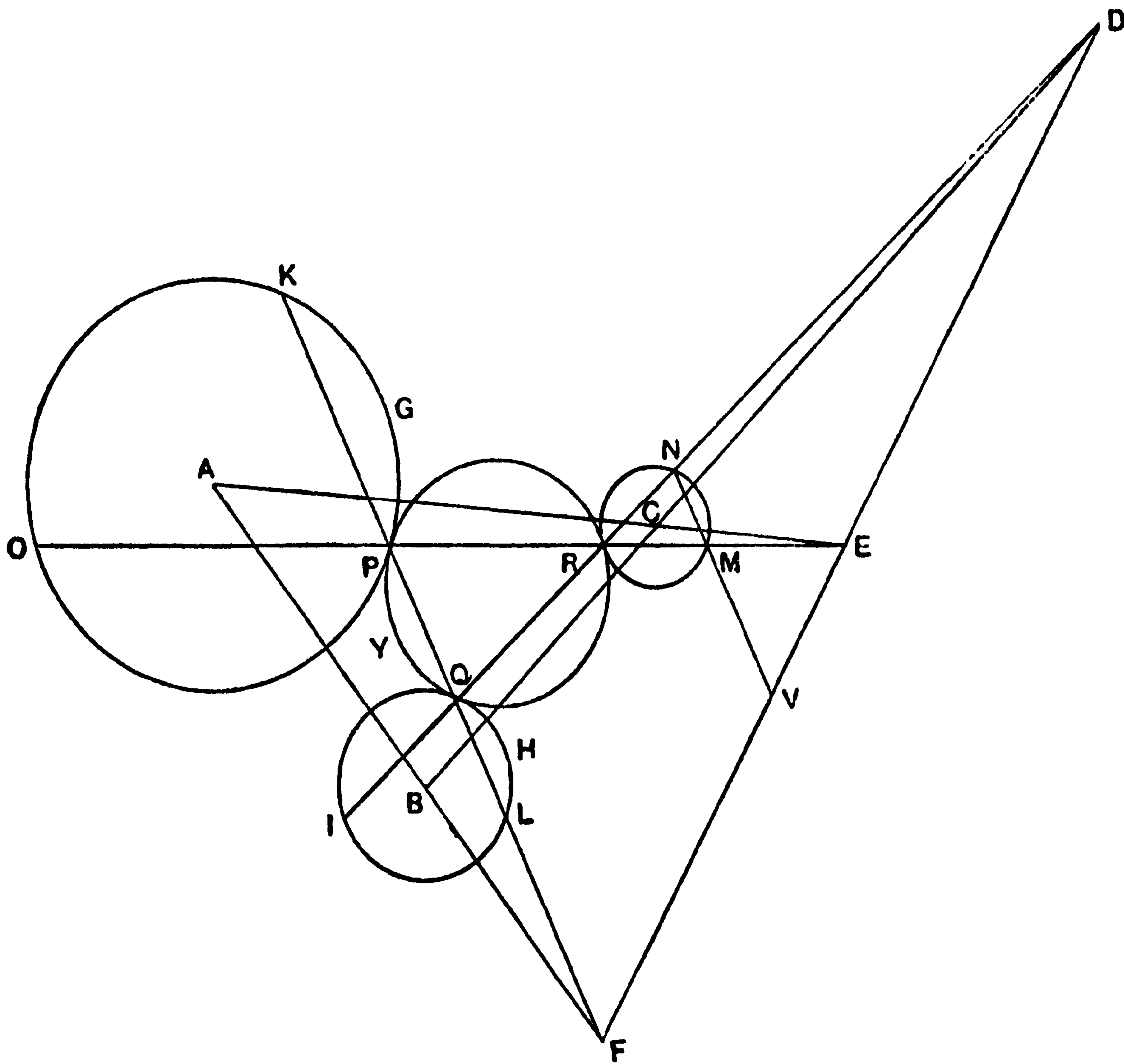
HC meeting the circle again in G. It is then easy to prove that BG is parallel to DF.

Now join EC, and produce it to meet the circle again at A; join AB.

We have only to prove that AB, BD are in one straight line.

Since $DE.EH = AE.EC$, the points A, D, H, C are concyclic.

Now the angle CHF, which is the supplement of the angle



CHD, is equal to the angle BGC, and therefore to the angle BAC.

Therefore the angle BAC is equal to the supplement of angle DHC, so that the angle BAC is equal to the angle DAC, and AB, BD are in a straight line.

The problem of Apollonius is now easy. We will take the case in which the required circle touches all the three given circles externally as shown in the figure. Let the radii of the

given circles be a, b, c and their centres A, B, C. Let D, E, F be the external centres of similitude so that $BD : DC = b : c$, &c.

Suppose the problem solved, and let P, Q, R be the points of contact. Let PQ produced meet the circles with centres A, B again in K, L. Then, by the proposition (1) above, the segments KGP, QHL are both similar to the segment PYQ; therefore they are similar to one another. It follows that PQ produced beyond L passes through F. Similarly QR, PR produced pass respectively through D, E.

Let PE, QD meet the circle with centre C again in M, N. Then, the segments PQR, RNM being similar, the angles PQR, RNM are equal, and therefore MN is parallel to PQ. Produce NM to meet EF in V.

Then $$EV : EF = EM : EP = EC : EA = c : a;$$
therefore the point V is given.

Accordingly the problem reduces itself to this: Given three points V, E, D in a straight line, it is required to draw DR, ER to a point R on the circle with centre C so that, if DR, ER meet the circle again in N, M, NM produced shall pass through V. This is the problem of Pappus just solved.

Thus R is found, and DR, ER produced meet the circles with centres B and A in the other required points Q, P respectively.

(ϵ) *Plane loci*, two Books.

Pappus gives a pretty full account of the contents of this work, which has sufficed to enable restorations of it to be made by three distinguished geometers, Fermat, van Schooten, and (most completely) by Robert Simson. Pappus prefaces his account by a classification of loci on two different plans. Under the first classification loci are of three kinds: (1) ἐφεκτικοί, *holding-in* or *fixed*; in this case the locus of a point is a point, of a line a line, and of a solid a solid, where presumably the line or solid can only move on itself so that it does not change its position: (2) διεξοδικοί, *passing-along*: this is the ordinary sense of a locus, where the locus of a point is a line, and of a line a solid: (3) ἀναστροφικοί, *moving backwards and forwards*, as it were, in which sense a plane may be the locus of a point and a solid

of a line.[1] The second classification is the familiar division into *plane, solid,* and *linear* loci, *plane* loci being straight lines and circles only, *solid* loci conic sections only, and *linear* loci those which are not straight lines nor circles nor any of the conic sections. The loci dealt with in our treatise are accordingly all straight lines or circles. The proof of the propositions is of course enormously facilitated by the use of Cartesian coordinates, and many of the loci are really the geometrical equivalent of fundamental theorems in analytical or algebraical geometry. Pappus begins with a composite enunciation, including a number of propositions, in these terms, which, though apparently confused, are not difficult to follow out:

'If two straight lines be drawn, from one given point or from two, which are (*a*) in a straight line or (*b*) parallel or (*c*) include a given angle, and either (*α*) bear a given ratio to one another or (*β*) contain a given rectangle, then, if the locus of the extremity of one of the lines is a plane locus given in position, the locus of the extremity of the other will also be a plane locus given in position, which will sometimes be of the same kind as the former, sometimes of the other kind, and will sometimes be similarly situated with reference to the straight line, and sometimes contrarily, according to the particular differences in the suppositions.'[2]

(The words 'with reference to *the straight line*' are obscure, but the straight line is presumably some obvious straight line in each figure, e.g., when there are two given points, the straight line joining them.) After quoting three obvious loci 'added by Charmandrus', Pappus gives three loci which, though containing an unnecessary restriction in the third case, amount to the statement that any equation of the first degree between coordinates inclined at fixed angles to (*a*) two axes perpendicular or oblique, (*b*) to any number of axes, represents a straight line. The enunciations (5–7) are as follows.[3]

5. 'If, when a straight line is given in magnitude and is moved so as always to be parallel to a certain straight line given in position, one of the extremities (of the moving straight line), lies on a straight line given in position, the

[1] Pappus, vii, pp. 660. 18–662. 5. [2] *Ib.* vii, pp. 662. 25–664. 7.
[3] *Ib.*, pp. 664. 20–666. 6.

other extremity will also lie on a straight line given in position.'

(That is, $x = a$ or $y = b$ in Cartesian coordinates represents a straight line.)

6. 'If from any point straight lines be drawn to meet at given angles two straight lines either parallel or intersecting, and if the straight lines so drawn have a given ratio to one another or if the sum of one of them and a line to which the other has a given ratio be given (in length), then the point will lie on a straight line given in position.'

(This includes the equivalent of saying that, if x, y be the coordinates of the point, each of the equations $x = my$, $x + my = c$ represents a straight line.)

7. 'If any number of straight lines be given in position, and straight lines be drawn from a point to meet them at given angles, and if the straight lines so drawn be such that the rectangle contained by one of them and a given straight line added to the rectangle contained by another of them and (another) given straight line is equal to the rectangle contained by a third and a (third) given straight line, and similarly with the others, the point will lie on a straight line given in position.'

(Here we have trilinear or multilinear coordinates proportional to the distances of the variable point from each of the three or more fixed lines. When there are three fixed lines, the statement is that $ax + by = cz$ represents a straight line. The precise meaning of the words 'and similarly with the the others' or 'of the others'—καὶ τῶν λοιπῶν ὁμοίως—is uncertain; the words seem to imply that, when there were more than three rectangles ax, by, cz ..., two of them were taken to be equal to the sum of all the others; but it is quite possible that Pappus meant that any linear equation between these rectangles represented a straight line. Precisely how far Apollonius went in generality we are not in a position to judge.)

The last enunciation (8) of Pappus referring to Book I states that,

'If from any point (two) straight lines be drawn to meet (two) parallel straight lines given in position at given angles, and

cut off from the parallels straight lines measured from given points on them such that (*a*) they have a given ratio or (*b*) they contain a given rectangle or (*c*) the sum or difference of figures of given species described on them respectively is equal to a given area, the point will lie on a straight line given in position.'[1]

The contents of Book II are equally interesting. Some of the enunciations shall for brevity be given by means of letters instead of in general terms. If from two given points A, B two straight lines be 'inflected' (κλασθῶσιν) to a point P, then (1), if $AP^2 \sim BP^2$ is given, the locus of P is a straight line; (2) if AP, BP are in a given ratio, the locus is a straight line or a circle [this is the proposition quoted by Eutocius in his commentary on the *Conics*, but already known to Aristotle]; (4) if AP^2 is 'greater by a given area than in a given ratio' to BP^2, i.e. if $AP^2 = a^2 + m \, . \, BP^2$, the locus is a circle given in position. An interesting proposition is (5) that, 'If from any number of given points whatever straight lines be inflected to one point, and the figures (given in species) described on all of them be together equal to a given area, the point will lie on a circumference (circle) given in position'; that is to say, if $\alpha \, . \, AP^2 + \beta \, . \, BP^2 + \gamma \, . \, CP^2 + \ldots =$ a given area (where $\alpha, \beta, \gamma \ldots$ are constants), the locus of P is a circle. (3) states that, if AN be a fixed straight line and A a fixed point on it, and if AP be any straight line drawn to a point P such that, if PN is perpendicular to AN, $AP^2 = a \, . \, AN$ or $a \, . \, BN$, where a is a given length and B is another fixed point on AN, then the locus of P is a circle given in position; this is equivalent to the fact that, if A be the origin, AN the axis of x, and $x = AN$, $y = PN$ be the coordinates of P, the locus $x^2 + y^2 = ax$ or $x^2 + y^2 = a(x - b)$ is a circle. (6) is somewhat obscurely enunciated: 'If from two given points straight lines be inflected (to a point), and from the point (of concourse) a straight line be drawn parallel to a straight line given in position and cutting off from another straight line given in position an intercept measured from a given point on it, and if the sum of figures (given in species) described on the two inflected lines be equal to the rectangle contained by a given straight line and the intercept, the point at which the straight lines are

[1] Pappus, vii, p. 666. 7–13.

inflected lies on a circle given in position.' The meaning seems to be this: Given two fixed points A, B, a length a, a straight line OX with a point O fixed upon it, and a direction represented, say, by any straight line OZ through O, then, if AP, BP be drawn to P, and PM parallel to OZ meets OX in M, the locus of P will be a circle given in position if

$$\alpha . AP^2 + \beta . BP^2 = a . OM,$$

where α, β are constants. The last two loci are again obscurely expressed, but the sense is this: (7) If PQ be any chord of a circle passing through a fixed internal point O, and R be an external point on PQ produced such that either (a) $OR^2 = PR . RQ$ or (b) $OR^2 + PO . OQ = PR . RQ$, the locus of R is a straight line given in position. (8) is the reciprocal of this: Given the fixed point O, the straight line which is the locus of R, and also the relation (a) or (b), the locus of P, Q is a circle.

(ζ) Νεύσεις (*Vergings* or *Inclinations*), two Books.

As we have seen, the problem in a νεῦσις is to place between two straight lines, a straight line and a curve, or two curves, a straight line of given length in such a way that it *verges* towards a fixed point, i.e. it will, if produced, pass through a fixed point. Pappus observes that, when we come to particular cases, the problem will be 'plane', 'solid' or 'linear', according to the nature of the particular hypotheses; but a selection had been made from the class which could be solved by plane methods, i.e. by means of the straight line and circle, the object being to give those which were more generally useful in geometry. The following were the cases thus selected and proved.[1]

I. Given (a) a semicircle and a straight line at right angles to the base, or (b) two semicircles with their bases in a straight line, to insert a straight line of given length verging to an angle of the semicircle [or of one of the semicircles].

II. Given a rhombus with one side produced, to insert a straight line of given length in the external angle so that it verges to the opposite angle.

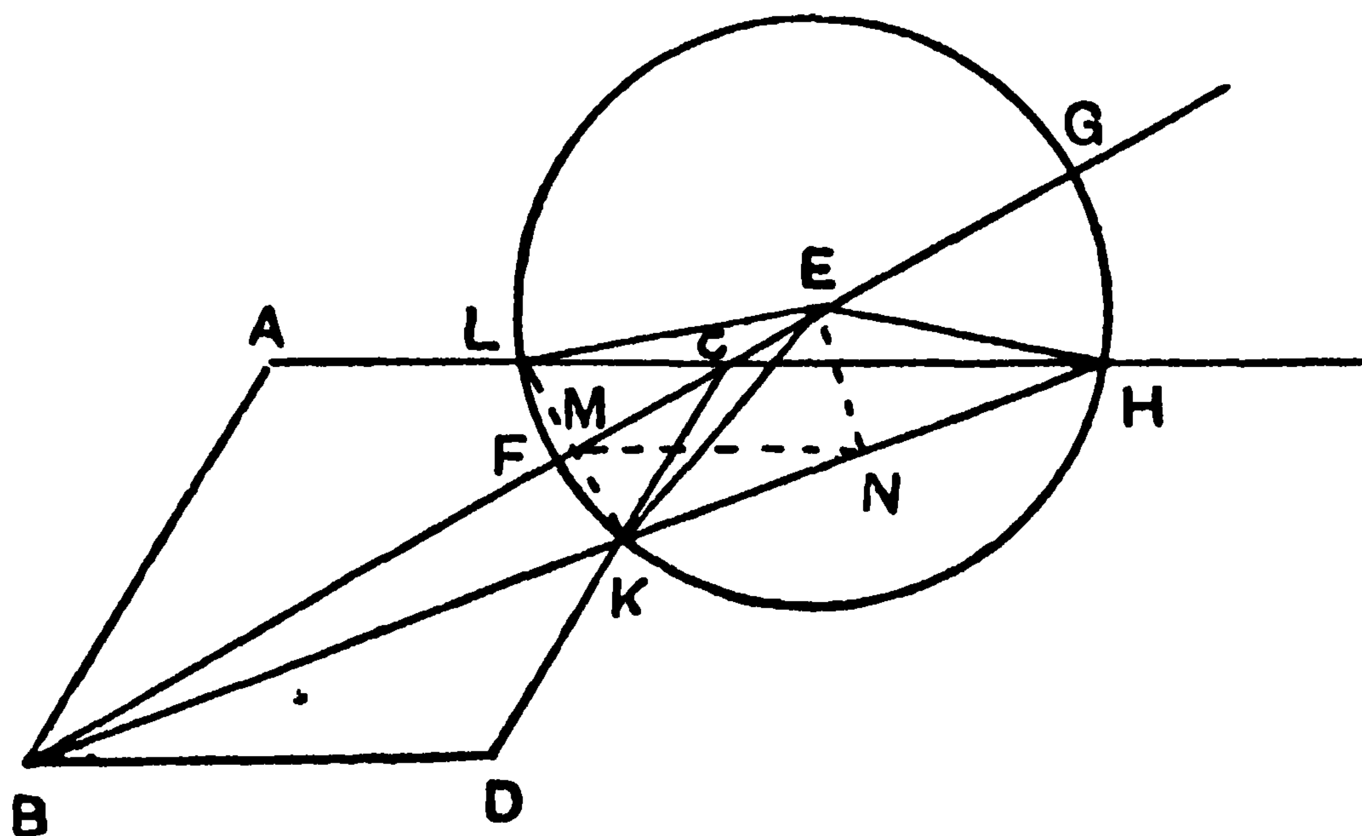
[1] Pappus, vii, pp. 670–2.

III. Given a circle, to insert a chord of given length verging to a given point.

In Book I of Apollonius's work there were four cases of I (*a*), two cases of III, and two of II; the second Book contained ten cases of I (*b*).

Restorations were attempted by Marino Ghetaldi (*Apollonius redivivus*, Venice, 1607, and *Apollonius redivivus ... Liber secundus*, Venice, 1613), Alexander Anderson (in a *Supplementum Apollonii redivivi*, 1612), and Samuel Horsley (Oxford, 1770); the last is much the most complete.

In the case of the rhombus (II) the construction of Apollonius can be restored with certainty. It depends on a lemma given by Pappus, which is as follows: Given a rhombus *AD* with diagonal *BC* produced to *E*, if *F* be taken on *BC* such that *EF* is a mean proportional between *BE* and *EC*, and if a circle be



described with *E* as centre and *EF* as radius cutting *CD* in *K* and *AC* produced in *H*, then shall *B*, *K*, *H* be in one straight line.[1]

Let the circle cut *AC* in *L*, join *LK* meeting *BC* in *M*, and join *HE*, *LE*, *KE*.

Since now *CL*, *CK* are equally inclined to the diameter of the circle, $CL = CK$. Also $EL = EK$, and it follows that the triangles *ECK*, *ECL* are equal in all respects, so that

$$\angle CKE = \angle CLE = \angle CHE.$$

By hypothesis, $EB : EF = EF : EC$,

or $EB : EK = EK : EC$.

[1] Pappus, vii, pp. 778-80.

Therefore the triangles BEK, KEC, which have the angle BEK common, are similar, and

$$\angle CBK = \angle CKE = \angle CHE \text{ (from above).}$$

But $$\angle HCE = \angle ACB = \angle BCK.$$

Therefore in the triangles CBK, CHE two angles are respectively equal, so that $\angle CEH = \angle CKB$ also.

But since $\angle CKE = \angle CHE$ (from above), K, C, E, H are concyclic.

Hence $$\angle CEH + \angle CKH = \text{(two right angles)};$$

therefore, since $$\angle CEH = \angle CKB,$$

$$\angle CKB + \angle CKH = \text{(two right angles)},$$

and BKH is a straight line.

It is certain, from the nature of this lemma, that Apollonius made his construction by drawing the circle shown in the figure.

He would no doubt arrive at it by analysis somewhat as follows.

Suppose the problem solved, and HK inserted as required $(= k)$.

Bisect HK in N, and draw NE at right angles to KH meeting BC produced in E. Draw KM perpendicular to BC, and produce it to meet AC in L. Then, by the property of the rhombus, $LM = MK$, and, since $KN = NH$ also, MN is parallel to LH.

Now, since the angles at M, N are right, M, K, N, E are concyclic.

Therefore $\angle CEK = \angle MNK = \angle CHK$, so that C, K, H, E are concyclic.

Therefore $\angle BCD =$ supplement of $KCE = \angle EHK = \angle EKH$, and the triangles EKH, DCB are similar.

Lastly,

$$\angle EBK = \angle EKH - \angle CEK = \angle EHK - \angle CHK = \angle EHC = \angle EKC;$$

therefore the triangles EBK, EKC are similar, and

$$BE : EK = EK : EC,$$

or $$BE \, . \, EC = EK^2.$$

But, by similar triangles *EKH*, *DCB*,

$$EK : KH = DC : CB,$$

and, since the ratio $DC : CB$, as well as *KH*, is given, *EK* is given.

The construction then is as follows.

If k be the given length, take a straight line p such that

$$p : k = AB : BC;$$

apply to *BC* a rectangle $BE . EC$ equal to p^2 and exceeding by a square; then with *E* as centre and radius equal to p describe a circle cutting *AC* produced in *H* and *CD* in *K*. *HK* is then equal to k and, by Pappus's lemma, verges towards *B*.

Pappus adds an interesting solution of the same problem with reference to a square instead of a rhombus; the solution is by one Heraclitus and depends on a lemma which Pappus also gives.[1]

We hear of yet other lost works by Apollonius.

(η) A *Comparison of the dodecahedron with the icosahedron.* This is mentioned by Hypsicles in the preface to the so-called Book XIV of Euclid. Like the *Conics*, it appeared in two editions, the second of which contained the proposition that, if there be a dodecahedron and an icosahedron inscribed in one and the same sphere, the surfaces of the solids are in the same ratio as their volumes; this was established by showing that the perpendiculars from the centre of the sphere to a pentagonal face of the dodecahedron and to a triangular face of the icosahedron are equal.

(θ) Marinus on Euclid's *Data* speaks of a *General Treatise* (καθόλου πραγματεία) in which Apollonius used the word *assigned* (τεταγμένον) as a comprehensive term to describe the *datum* in general. It would appear that this work must have dealt with the fundamental principles of mathematics, definitions, axioms, &c., and that to it must be referred the various remarks on such subjects attributed to Apollonius by Proclus, the elucidation of the notion of a line, the definition

[1] Pappus, vii, pp. 780-4.

of plane and solid angles, and his attempts to prove the axioms; it must also have included the three definitions (13–15) in Euclid's *Data* which, according to a scholium, were due to Apollonius and must therefore have been interpolated (they are definitions of κατηγμένη, ἀνηγμένη, and the elliptical phrase παρὰ θέσει, which means 'parallel to a straight line given in position'). Probably the same work also contained Apollonius's alternative constructions for the problems of Eucl. I. 10, 11 and 23 given by Proclus. Pappus speaks of a mention by Apollonius 'before his own elements' of the class of locus called ἐφεκτικός, and it may be that the treatise now in question is referred to rather than the *Plane Loci* itself.

(ι) The work *On the Cochlias* was on the cylindrical helix. It included the theoretical generation of the curve on the surface of the cylinder, and the proof that the curve is *homoeomeric* or uniform, i.e. such that any part will fit upon or coincide with any other.

(κ) A work on *Unordered Irrationals* is mentioned by Proclus, and a scholium on Eucl. X. 1 extracted from Pappus's commentary remarks that 'Euclid did not deal with all rationals and irrationals, but only with the simplest kinds by the combination of which an infinite number of irrationals are formed, of which latter Apollonius also gave some'. To a like effect is a passage of the fragment of Pappus's commentary on Eucl. X discovered in an Arabic translation by Woepcke: 'it was Apollonius who, besides the *ordered* irrational magnitudes, showed the existence of the *unordered*, and by accurate methods set forth a great number of them'. The hints given by the author of the commentary seem to imply that Apollonius's extensions of the theory of irrationals took two directions, (1) generalizing the *medial* straight line of Euclid, on the basis that, between two lines commensurable in square (only), we may take not only one sole medial line but three or four, and so on *ad infinitum*, since we can take, between any two given straight lines, as many lines as we please in continued proportion, (2) forming compound irrationals by the addition and subtraction of more than two terms of the sort composing the *binomials*, *apotomes*, &c.

1523.2

(λ) *On the burning-mirror* (περὶ τοῦ πυρίου) is the title of another work of Apollonius mentioned by the author of the *Fragmentum mathematicum Bobiense*, which is attributed by Heiberg to Anthemius but is more likely (judging by its survivals of antiquated terminology) to belong to a much earlier date. The fragment shows that Apollonius discussed the spherical form of mirror among others. Moreover, the extant fragment by Anthemius himself (on burning mirrors) proves the property of mirrors of parabolic section, using the properties of the parabola (*a*) that the tangent at any point makes equal angles with the axis and with the focal distance of the point, and (*b*) that the distance of any point on the curve from the focus is equal to its distance from a certain straight line (our 'directrix'); and we can well believe that the parabolic form of mirror was also considered in Apollonius's work, and that he was fully aware of the focal properties of the parabola, notwithstanding the omission from the *Conics* of all mention of the focus of a parabola.

(μ) In a work called ὠκυτόκιον ('quick-delivery') Apollonius is said to have found an approximation to the value of π 'by a different calculation (from that of Archimedes), bringing it within closer limits'.[1] Whatever these closer limits may have been, they were considered to be less suitable for practical use than those of Archimedes.

It is a moot question whether Apollonius's system of arithmetical notation (by tetrads) for expressing large numbers and performing the usual arithmetical operations with them, as described by Pappus, was included in this same work. Heiberg thinks it probable, but there does not seem to be any necessary reason why the notation for large numbers, classifying them into myriads, double myriads, triple myriads, &c., i.e. according to powers of 10,000, need have been connected with the calculation of the value of π, unless indeed the numbers used in the calculation were so large as to require the tetradic system for the handling of them.

We have seen that Apollonius is credited with a solution of the problem of the two mean proportionals (vol. i, pp. 262–3).

[1] v. Eutocius on Archimedes, *Measurement of a Circle*,

Astronomy.

We are told by Ptolemaeus Chennus[1] that Apollonius was famed for his astronomy, and was called ϵ (Epsilon) because the form of that letter is associated with that of the moon, to which his accurate researches principally related. Hippolytus says he made the distance of the moon's circle from the surface of the earth to be 500 myriads of stades.[2] This figure can hardly be right, for, the diameter of the earth being, according to Eratosthenes's evaluation, about eight myriads of stades, this would make the distance of the moon from the earth about 125 times the earth's radius. This is an unlikely figure, seeing that Aristarchus had given limits for the ratios between the distance of the moon and its diameter, and between the diameters of the moon and the earth, which lead to about 19 as the ratio of the moon's distance to the earth's radius. Tannery suggests that perhaps Hippolytus made a mistake in copying from his source and took the figure of 5,000,000 stades to be the length of the radius instead of the *diameter* of the moon's orbit.

But we have better evidence of the achievements of Apollonius in astronomy. In Ptolemy's *Syntaxis*[3] he appears as an authority upon the hypotheses of epicycles and eccentrics designed to account for the apparent motions of the planets. The propositions of Apollonius quoted by Ptolemy contain exact statements of the alternative hypotheses, and from this fact it was at one time concluded that Apollonius invented the two hypotheses. This, however, is not the case. The hypothesis of epicycles was already involved, though with restricted application, in the theory of Heraclides of Pontus that the two inferior planets, Mercury and Venus, revolve in circles like satellites round the sun, while the sun itself revolves in a circle round the earth; that is, the two planets describe epicycles about the material sun as moving centre. In order to explain the motions of the superior planets by means of epicycles it was necessary to conceive of an epicycle about a point as moving centre which is not a material but a mathematical point. It was some time before this extension of the theory of epicycles took place, and in the meantime

[1] *apud Photium*, Cod. cxc, p. 151 b 18, ed. Bekker.

[2] Hippol. *Refut.* iv. 8, p. 66, ed. Duncker.

[3] Ptolemy, *Syntaxis*, xii. 1.

another hypothesis, that of eccentrics, was invented to account for the movements of the superior planets only. We are at this stage when we come to Apollonius. His enunciations show that he understood the theory of epicycles in all its generality, but he states specifically that the theory of eccentrics can only be applied to the three planets which can be at any distance from the sun. The reason why he says that the eccentric hypothesis will not serve for the inferior planets is that, in order to make it serve, we should have to suppose the circle described by the centre of the eccentric circle to be greater than the eccentric circle itself. (Even this generalization was made later, at or before the time of Hipparchus.) Apollonius further says in his enunciation about the eccentric that 'the centre of the eccentric circle moves about the centre of the zodiac in the direct order of the signs and *at a speed equal to that of the sun*, while the star moves on the eccentric about its centre in the inverse order of the signs and at a speed equal to the anomaly'. It is clear from this that the theory of eccentrics was invented for the specific purpose of explaining the movements of Mars, Jupiter, and Saturn about the sun and for that purpose alone. This explanation, combined with the use of epicycles about the sun as centre to account for the motions of Venus and Mercury, amounted to the system of Tycho Brahe; that system was therefore anticipated by some one intermediate in date between Heraclides and Apollonius and probably nearer to the latter, or it may have been Apollonius himself who took this important step. If it was, then Apollonius, coming after Aristarchus of Samos, would be exactly the Tycho Brahe of the Copernicus of antiquity. The actual propositions quoted by Ptolemy as proved by Apollonius among others show mathematically at what points, under each of the two hypotheses, the apparent forward motion changes into apparent retrogradation and vice versa, or the planet appears to be *stationary*.

XV

THE SUCCESSORS OF THE GREAT GEOMETERS

WITH Archimedes and Apollonius Greek geometry reached its culminating point. There remained details to be filled in, and no doubt in a work such as, for instance, the *Conics* geometers of the requisite calibre could have found propositions containing the germ of theories which were capable of independent development. But, speaking generally, the further progress of geometry on general lines was practically barred by the restrictions of method and form which were inseparable from the classical Greek geometry. True, it was open to geometers to discover and investigate curves of a higher order than conics, such as spirals, conchoids, and the like. But the Greeks could not get very far even on these lines in the absence of some system of coordinates and without freer means of manipulation such as are afforded by modern algebra, in contrast to the geometrical algebra, which could only deal with equations connecting lines, areas, and volumes, but involving no higher dimensions than three, except in so far as the use of proportions allowed a very partial exemption from this limitation. The theoretical methods available enabled quadratic, cubic and bi-quadratic equations or their equivalents to be solved. But all the solutions were *geometrical*; in other words, quantities could only be represented by lines, areas and volumes, or ratios between them. There was nothing corresponding to operations with general algebraical quantities irrespective of what they represented. There were no *symbols* for such quantities. In particular, the irrational was discovered in the form of incommensurable *lines*; hence irrationals came to be represented by straight lines as they are in Euclid, Book X, and the Greeks had no other way of representing them. It followed that a product of two irrationals could only be represented by a *rectangle*, and so on. Even when Diophantus came to use a symbol for an unknown

quantity, it was only an abbreviation for the word ἀριθμός, with the meaning of 'an undetermined multitude of units', not a general quantity. The restriction then of the algebra employed by geometers to the geometrical form of algebra operated as an insuperable obstacle to any really new departure in theoretical geometry.

It might be thought that there was room for further extensions in the region of solid geometry. But the fundamental principles of solid geometry had also been laid down in Euclid, Books XI–XIII; the theoretical geometry of the sphere had been fully treated in the ancient *sphaeric*; and any further application of solid geometry, or of loci in three dimensions, was hampered by the same restrictions of method which hindered the further progress of plane geometry.

Theoretical geometry being thus practically at the end of its resources, it was natural that mathematicians, seeking for an opening, should turn to the *applications* of geometry. One obvious branch remaining to be worked out was the geometry of measurement, or *mensuration* in its widest sense, which of course had to wait on pure theory and to be based on its results. One species of mensuration was immediately required for astronomy, namely the measurement of triangles, especially spherical triangles; in other words, trigonometry plane and spherical. Another species of mensuration was that in which an example had already been set by Archimedes, namely the measurement of areas and volumes of different shapes, and arithmetical approximations to their true values in cases where they involved surds or the ratio (π) between the circumference of a circle and its diameter; the object of such mensuration was largely practical. Of these two kinds of mensuration, the first (trigonometry) is represented by Hipparchus, Menelaus and Ptolemy; the second by Heron of Alexandria. These mathematicians will be dealt with in later chapters; this chapter will be devoted to the successors of the great geometers who worked on the same lines as the latter.

Unfortunately we have only very meagre information as to what these geometers actually accomplished in keeping up the tradition. No geometrical works by them have come down to us in their entirety, and we are dependent on isolated extracts or scraps of information furnished by commen-

tators, and especially by Pappus and Eutocius. Some of these are very interesting, and it is evident from the extracts from the works of such writers as Diocles and Dionysodorus that, for some time after Archimedes and Apollonius, mathematicians had a thorough grasp of the contents of the works of the great geometers, and were able to use the principles and methods laid down therein with ease and skill.

Two geometers properly belonging to this chapter have already been dealt with. The first is NICOMEDES, the inventor of the conchoid, who was about intermediate in date between Eratosthenes and Apollonius. The conchoid has already been described above (vol. i, pp. 238–40). It gave a general method of solving any νεῦσις where one of the lines which cut off an intercept of given length on the line verging to a given point is a straight line; and it was used both for the finding of two mean proportionals and for the trisection of any angle, these problems being alike reducible to a νεῦσις of this kind. How far Nicomedes discussed the properties of the curve in itself is uncertain; we only know from Pappus that he proved two properties, (1) that the so-called 'ruler' in the instrument for constructing the curve is an asymptote, (2) that any straight line drawn in the space between the 'ruler' or asymptote and the conchoid must, if produced, be cut by the conchoid.[1] The equation of the curve referred to polar coordinates is, as we have seen, $r = a + b \sec \theta$. According to Eutocius, Nicomedes prided himself inordinately on his discovery of this curve, contrasting it with Eratosthenes's mechanism for finding any number of mean proportionals, to which he objected formally and at length on the ground that it was impracticable and entirely outside the spirit of geometry.[2]

Nicomedes is associated by Pappus with Dinostratus, the brother of Menaechmus, and others as having applied to the squaring of the circle the curve invented by Hippias and known as the *quadratrix*,[3] which was originally intended for the purpose of trisecting any angle. These facts are all that we know of Nicomedes's achievements.

[1] Pappus, iv, p. 244. 21–8.
[2] Eutoc. on Archimedes, *On the Sphere and Cylinder*, Archimedes, vol. iii, p. 98.
[3] Pappus, iv, pp. 250. 33–252. 4. Cf. vol. i, p. 225 sq.

The second name is that of DIOCLES. We have already (vol. i, pp. 264–6) seen him as the discoverer of the curve known as the *cissoid*, which he used to solve the problem of the two mean proportionals, and also (pp. 47–9 above) as the author of a method of solving the equivalent of a certain cubic equation by means of the intersection of an ellipse and a hyperbola. We are indebted for our information on both these subjects to Eutocius,[1] who tells us that the fragments which he quotes came from Diocles's work περὶ πυρείων, *On burning-mirrors*. The connexion of the two things with the subject of this treatise is not obvious, and we may perhaps infer that it was a work of considerable scope. What exactly were the forms of the burning-mirrors discussed in the treatise it is not possible to say, but it is probably safe to assume that among them were concave mirrors in the forms (1) of a sphere, (2) of a paraboloid, and (3) of the surface described by the revolution of an ellipse about its major axis. The author of the *Fragmentum mathematicum Bobiense* says that Apollonius in his book *On the burning-mirror* discussed the case of the concave spherical mirror, showing about what point ignition would take place; and it is certain that Apollonius was aware that an ellipse has the property of reflecting all rays through one focus to the other focus. Nor is it likely that the corresponding property of a parabola with reference to rays parallel to the axis was unknown to Apollonius. Diocles therefore, writing a century or more later than Apollonius, could hardly have failed to deal with all three cases. True, Anthemius (died about A.D. 534) in his fragment on burning-mirrors says that the ancients, while mentioning the usual burning-mirrors and saying that such figures are conic sections, omitted to specify which conic sections, and how produced, and gave no geometrical proofs of their properties. But if the properties were commonly known and quoted, it is obvious that they must have been proved by the ancients, and the explanation of Anthemius's remark is presumably that the original works in which they were proved (e.g. those of Apollonius and Diocles) were already lost when he wrote. There appears to be no trace of Diocles's work left either in Greek or Arabic,

[1] Eutocius, *loc. cit.*, p. 66. 8 sq., p. 160. 3 sq.

unless we have a fragment from it in the *Fragmentum mathematicum Bobiense*. But Moslem writers regarded Diocles as the discoverer of the parabolic burning-mirror; 'the ancients', says al Singārī (Sachāwī, Anṣārī), 'made mirrors of plane surfaces. Some made them concave (i.e. spherical) until Diocles (Diūklis) showed and proved that, if the surface of these mirrors has its curvature in the form of a parabola, they then have the greatest power and burn most strongly. There is a work on this subject composed by Ibn al-Haitham.' This work survives in Arabic and in Latin translations, and is reproduced by Heiberg and Wiedemann[1]: it does not, however, mention the name of Diocles, but only those of Archimedes and Anthemius. Ibn al-Haitham says that famous men like Archimedes and Anthemius had used mirrors made up of a number of spherical rings; afterwards, he adds, they considered the form of curves which would reflect rays to one point, and found that the concave surface of a paraboloid of revolution has this property. It is curious to find Ibn al-Haitham saying that the ancients had not set out the proofs sufficiently, nor the method by which they discovered them, words which almost exactly recall those of Anthemius himself. Nevertheless the whole course of Ibn al-Haitham's proofs is on the Greek model, Apollonius being actually quoted by name in the proof of the main property of the parabola required, namely that the tangent at any point of the curve makes equal angles with the focal distance of the point and the straight line drawn through it parallel to the axis. A proof of the property actually survives in the Greek *Fragmentum mathematicum Bobiense*, which evidently came from some treatise on the parabolic burning-mirror; but Ibn al-Haitham does not seem to have had even this fragment at his disposal, since his proof takes a different course, distinguishing three different cases, reducing the property by analysis to the known property $AN = AT$, and then working out the synthesis. The proof in the *Fragmentum* is worth giving. It is substantially as follows, beginning with the preliminary lemma that, if PT, the tangent at any point P, meets the axis at T, and if AS be measured along the axis from the vertex A equal to $\frac{1}{4}AL$, where AL is the parameter, then $SP = ST$.

[1] *Bibliotheca mathematica*, x_3, 1910, pp. 201–37.

Let PN be the ordinate from P; draw AY at right angles to the axis meeting PT in Y, and join SY.

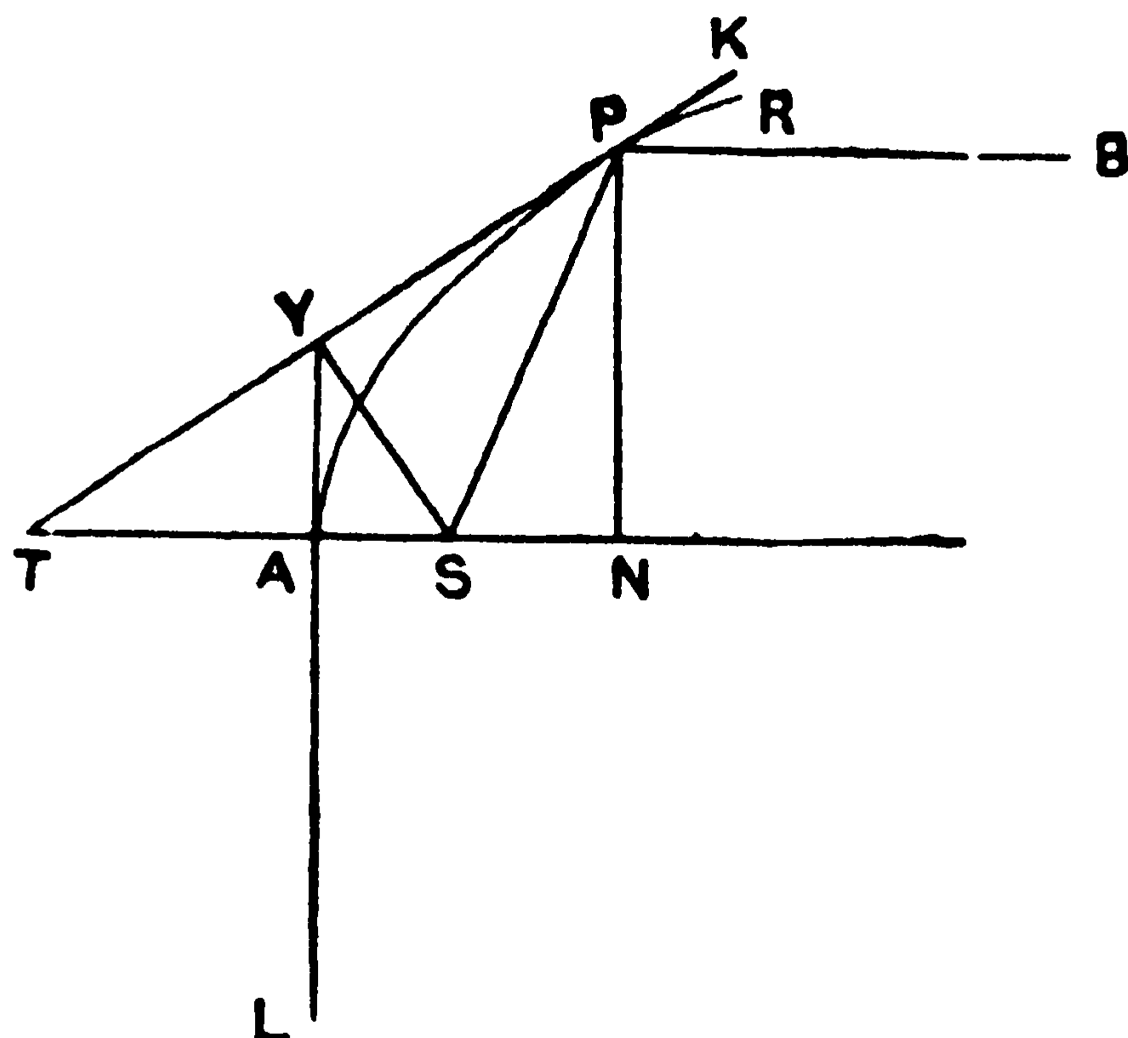
Now $$PN^2 = AL . AN$$
$$= 4AS . AN$$
$$= 4AS . AT \text{ (since } AN = AT\text{)}.$$

But $PN = 2AY$ (since $AN = AT$);

therefore $$AY^2 = TA . AS,$$

and the angle TYS is right.

The triangles SYT, SYP being right-angled, and TY being equal to YP, it follows that $SP = ST$.



With the same figure, let BP be a ray parallel to AN impinging on the curve at P. It is required to prove that the angles of incidence and reflection (to S) are equal.

We have $SP = ST$, so that 'the angles at the points T, P are equal. So', says the author, 'are the angles TPA, KPR [the angles between the tangent and the *curve* on each side of the point of contact]. Let the difference between the angles be taken; therefore the angles SPA, RPB which remain [again 'mixed' angles] are equal. Similarly we shall show that all the lines drawn parallel to AS will be reflected at equal angles to the point S.'

The author then proceeds: 'Thus burning-mirrors constructed with the surface of impact (in the form) of the *section of a right-angled cone* may easily, in the manner

above shown, be proved to bring about ignition at the point indicated.'

Heiberg held that the style of this fragment is Byzantine and that it is probably by Anthemius. Cantor conjectured that here we might, after all, have an extract from Diocles's work. Heiberg's supposition seems to me untenable because of the author's use (1) of the ancient terms 'section of a right-angled cone' for parabola and 'diameter' for axis (to say nothing of the use of the parameter, of which there is no word in the genuine fragment of Anthemius), and (2) of the mixed 'angles of contact'. Nor does it seem likely that even Diocles, living a century after Apollonius, would have spoken of the 'section of a right-angled cone' instead of a parabola, or used the 'mixed' angle of which there is only the merest survival in Euclid. The assumption of the equality of the two angles made by the curve with the tangent on both sides of the point of contact reminds us of Aristotle's assumption of the equality of the angles '*of* a segment' of a circle as prior to the truth proved in Eucl. I. 5. I am inclined, therefore, to date the fragment much earlier even than Diocles. Zeuthen suggested that the property of the paraboloidal mirror may have been discovered by Archimedes, who, according to a Greek tradition, wrote *Catoptrica*. This, however, does not receive any confirmation in Ibn al-Haitham or in Anthemius, and we can only say that the fragment at least goes back to an original which was probably not later than Apollonius.

PERSEUS is only known from allusions to him in Proclus,[1] as the discoverer and investigator of the *spiric sections*. They are classed by Proclus among curves obtained by cutting solids, and in this respect they are associated with the conic sections. We may safely infer that they were discovered after the conic sections, and only after the theory of conics had been considerably developed. This was already the case in Euclid's time, and it is probable, therefore, that Perseus was not earlier than Euclid. On the other hand, by that time the investigation of conics had brought the exponents of the subject such fame that it would be natural for mathematicians to see whether there was not an opportunity for winning a

[1] Proclus on Eucl. I, pp. 111. 23–112. 8, 356. 12. Cf. vol. i, p. 226.

like renown as discoverers of other curves to be obtained by cutting well-known solid figures other than the cone and cylinder. A particular case of one such solid figure, the *σπεῖρα*, had already been employed by Archytas, and the more general form of it would not unnaturally be thought of as likely to give sections worthy of investigation. Since Geminus is Proclus's authority, Perseus may have lived at any date from Euclid's time to (say) 75 B.C., but the most probable supposition seems to be that he came before Apollonius and near to Euclid in date.

The *spire* in one of its forms is what we call a *tore*, or an anchor-ring. It is generated by the revolution of a circle about a straight line in its plane in such a way that the plane of the circle always passes through the axis of revolution. It takes three forms according as the axis of revolution is (*a*) altogether outside the circle, when the spire is *open* (*διεχής*), (*b*) a tangent to the circle, when the surface is *continuous* (*συνεχής*), or (*c*) a chord of the circle, when it is *interlaced* (*ἐμπεπλεγμένη*), or *crossing-itself* (*ἐπαλλάττουσα*); an alternative name for the surface was *κρίκος*, a *ring*. Perseus celebrated his discovery in an epigram to the effect that 'Perseus on his discovery of three lines (curves) upon five sections gave thanks to the gods therefor'.[1] There is some doubt about the meaning of 'three lines *upon* five sections' (*τρεῖς γραμμὰς ἐπὶ πέντε τομαῖς*). We gather from Proclus's account of three sections distinguished by Perseus that the plane of section was always parallel to the axis of revolution or perpendicular to the plane which cuts the tore symmetrically like the division in a split-ring. It is difficult to interpret the phrase if it means three curves made by five different sections. Proclus indeed implies that the three curves were sections of the three kinds of tore respectively (the open, the closed, and the interlaced), but this is evidently a slip. Tannery interprets the phrase as meaning 'three curves *in addition to* five sections'.[2] Of these the five sections belong to the open tore, in which the distance (c) of the centre of the generating circle from the axis of revolution is greater than the radius (a) of the generating circle. If d be the perpen-

[1] Proclus on Eucl. I, p. 112. 2.
[2] See Tannery, *Mémoires scientifiques*, II, pp. 24–8.

dicular distance of the plane of section from the axis of rotation, we can distinguish the following cases:

(1) $c+a>d>c$. Here the curve is an oval.

(2) $d = c$: transition from case (1) to the next case.

(3) $c>d>c-a$. The curve is now a closed curve narrowest in the middle.

(4) $d = c-a$. In this case the curve is the *hippopede* (horse-fetter), a curve in the shape of the figure of 8. The lemniscate of Bernoulli is a particular case of this curve, that namely in which $c = 2a$.

(5) $c-a>d>0$. In this case the section consists of two ovals symmetrical with one another.

The three curves specified by Proclus are those corresponding to (1), (3) and (4).

When the tore is 'continuous' or closed, $c = a$, and we have sections corresponding to (1), (2) and (3) only; (4) reduces to two circles touching one another.

But Tannery finds in the third, the interlaced, form of tore three new sections corresponding to (1) (2) (3), each with an oval in the middle. This would make three curves in addition to the five sections, or eight curves in all. We cannot be certain that this is the true explanation of the phrase in the epigram: but it seems to be the best suggestion that has been made.

According to Proclus, Perseus worked out the property of his curves, as Nicomedes did that of the conchoid, Hippias that of the *quadratrix*, and Apollonius those of the three conic sections. That is, Perseus must have given, in some form, the equivalent of the Cartesian equation by which we can represent the different curves in question. If we refer the tore to three axes of coordinates at right angles to one another with the centre of the tore as origin, the axis of y being taken to be the axis of revolution, and those of z, x being perpendicular to it in the plane bisecting the tore (making it a split-ring), the equation of the tore is

$$(x^2+y^2+z^2+c^2-a^2)^2 = 4c^2(z^2+x^2),$$

where c, a have the same meaning as above. The different sections parallel to the axis of revolution are obtained by giving (say) z any value between 0 and $c+a$. For the value $z = a$ the curve is the oval of Cassini which has the property that, if r, r' be the distances of any point on the curve from two fixed points as poles, $rr' =$ const. For, if $z = a$, the equation becomes

$$(x^2+y^2+c^2)^2 = 4c^2x^2+4c^2a^2,$$

or
$$\{\overline{c-x}^2+y^2\}\{\overline{c+x}^2+y^2\} = 4c^2a^2;$$

and this is equivalent to $rr' = \pm 2ca$ if x, y are the coordinates of any point on the curve referred to Ox, Oy as axes, where O is the middle point of the line ($2c$ in length) joining the two poles, and Ox lies along that line in either direction, while Oy is perpendicular to it. Whether Perseus discussed this case and arrived at the property in relation to the two poles is of course quite uncertain.

Isoperimetric figures.

The subject of isoperimetric figures, that is to say, the comparison of the areas of figures having different shapes but the same perimeter, was one which would naturally appeal to the early Greek mathematicians. We gather from Proclus's notes on Eucl. I. 36, 37 that those theorems, proving that parallelograms or triangles on the same or equal bases and between the same parallels are equal in area, appeared to the ordinary person paradoxical because they meant that, by moving the side opposite to the base in the parallelogram, or the vertex of the triangle, to the right or left as far as we please, we may increase the perimeter of the figure to any extent while keeping the same area. Thus the perimeter in parallelograms or triangles is in itself no criterion as to their area. Misconception on this subject was rife among non-mathematicians. Proclus tells us of describers of countries who inferred the size of cities from their perimeters; he mentions also certain members of communistic societies in his own time who cheated their fellow-members by giving them land of greater perimeter but less area than the plots which they took

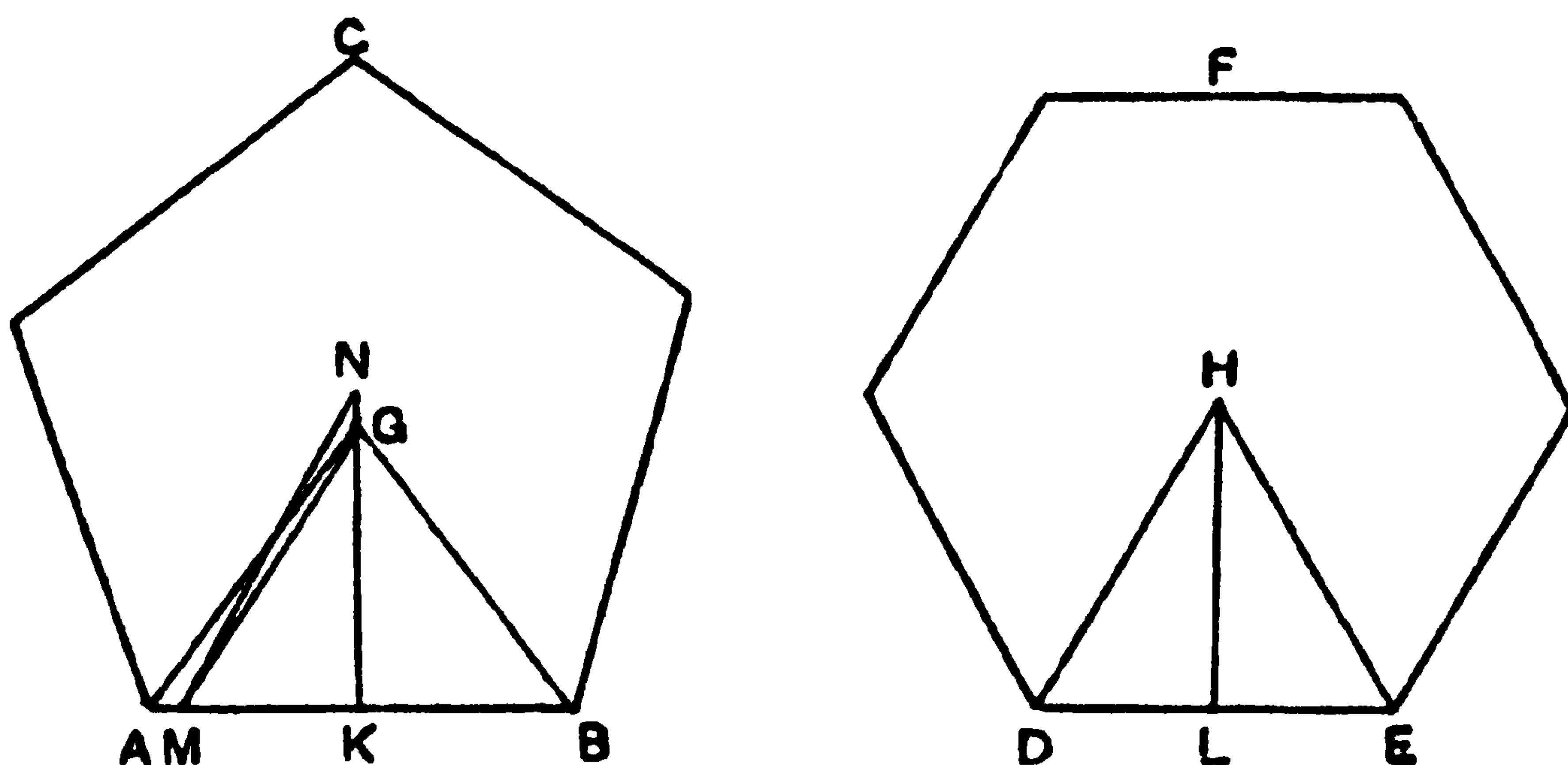
themselves, so that, while they got a reputation for greater honesty, they in fact took more than their share of the produce.[1] Several remarks by ancient authors show the prevalence of the same misconception. Thucydides estimates the size of Sicily according to the time required for circumnavigating it.[2] About 130 B.C. Polybius observed that there were people who could not understand that camps of the same periphery might have different capacities.[3] Quintilian has a similar remark, and Cantor thinks he may have had in his mind the calculations of Pliny, who compares the size of different parts of the earth by adding their lengths to their breadths.[4]

ZENODORUS wrote, at some date between (say) 200 B.C. and A.D. 90, a treatise περὶ ἰσομέτρων σχημάτων, *On isometric figures*. A number of propositions from it are preserved in the commentary of Theon of Alexandria on Book I of Ptolemy's *Syntaxis*; and they are reproduced in Latin in the third volume of Hultsch's edition of Pappus, for the purpose of comparison with Pappus's own exposition of the same propositions at the beginning of his Book V, where he appears to have followed Zenodorus pretty closely while making some changes in detail.[5] From the closeness with which the style of Zenodorus follows that of Euclid and Archimedes we may judge that his date was not much later than that of Archimedes, whom he mentions as the author of the proposition (*Measurement of a Circle*, Prop. 1) that the area of a circle is half that of the rectangle contained by the perimeter of the circle and its radius. The important propositions proved by Zenodorus and Pappus include the following: (1) *Of all regular polygons of equal perimeter, that is the greatest in area which has the most angles.* (2) *A circle is greater than any regular polygon of equal contour.* (3) *Of all polygons of the same number of sides and equal perimeter the equilateral and equiangular polygon is the greatest in area.* Pappus added the further proposition that *Of all segments of a circle having the same circumference the semicircle is the greatest in*

[1] Proclus on Eucl. I, p. 403. 5 sq.
[2] Thuc. vi. 1.
[3] Polybius, ix. 21.
[4] Pliny, *Hist. nat.* vi. 208.
[5] Pappus, v, p. 308 sq.

area. Zenodorus's treatise was not confined to propositions about plane figures, but gave also the theorem that *Of all solid figures the surfaces of which are equal, the sphere is the greatest in solid content.*

We will briefly indicate Zenodorus's method of proof. To begin with (1); let ABC, DEF be equilateral and equiangular polygons of the same perimeter, DEF having more angles than ABC. Let G, H be the centres of the circumscribing circles, GK, HL the perpendiculars from G, H to the sides AB, DE, so that K, L bisect those sides.



Since the perimeters are equal, $AB > DE$, and $AK > DL$. Make KM equal to DL and join GM.

Since AB is the same fraction of the perimeter that the angle AGB is of four right angles, and DE is the same fraction of the same perimeter that the angle DHE is of four right angles, it follows that

$$AB : DE = \angle AGB : \angle DHE,$$

that is, $$AK : MK = \angle AGK : \angle DHL.$$

But $$AK : MK > \angle AGK : \angle MGK$$

(this is easily proved in a lemma following by the usual method of drawing an arc of a circle with G as centre and GM as radius cutting GA and GK produced. The proposition is of course equivalent to $\tan\alpha / \tan\beta > \alpha/\beta$, where $\frac{1}{2}\pi > \alpha > \beta$).

Therefore $$\angle MGK > \angle DHL,$$

and consequently $$\angle GMK < \angle HDL.$$

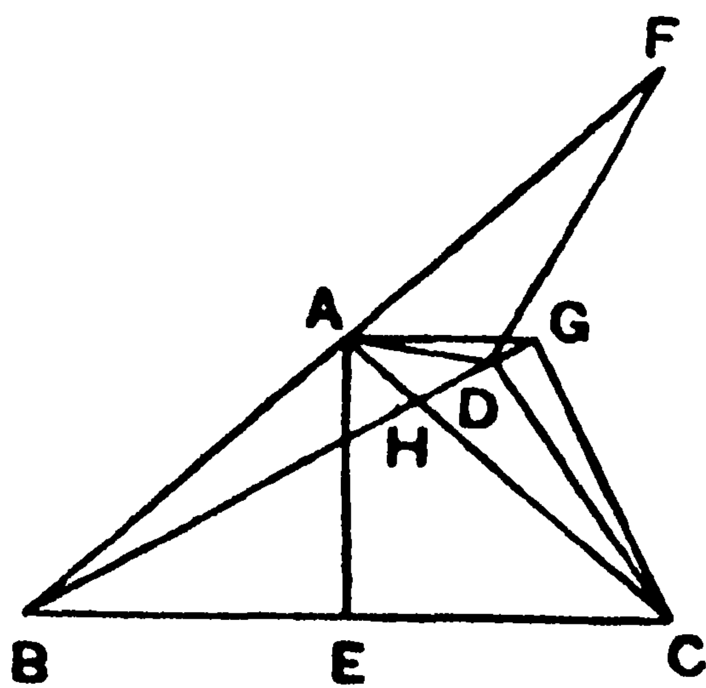
Make the angle NMK equal to the angle HDL, so that MN meets KG produced in N.

The triangles NMK, HDL are now equal in all respects, and NK is equal to HL, so that $GK < HL$.

But the area of the polygon ABC is half the rectangle contained by GK and the perimeter, while the area of the polygon DEF is half the rectangle contained by HL and the same perimeter. Therefore the area of the polygon DEF is the greater.

(2) The proof that a circle is greater than any regular polygon with the same perimeter is deduced immediately from Archimedes's proposition that the area of a circle is equal to the right-angled triangle with perpendicular side equal to the radius and base equal to the perimeter of the circle; Zenodorus inserts a proof *in extenso* of Archimedes's proposition, with preliminary lemma. The perpendicular from the centre of the circle circumscribing the polygon is easily proved to be less than the radius of the given circle with perimeter equal to that of the polygon: whence the proposition follows.

(3) The proof of this proposition depends on some preliminary lemmas. The first proves that, if there be two triangles on the same base and with the same perimeter, one being isosceles and the other scalene, the isosceles triangle has the greater area. (Given the scalene triangle BDC on the base BC, it is easy to draw on BC as base the isosceles triangle having the same perimeter. We have only to take BH equal to $\frac{1}{2}(BD+DC)$, bisect BC at E, and erect at E the perpendicular AE such that $AE^2 = BH^2 - BE^2$.)



Produce BA to F so that $BA = AF$, and join AD, DF.

Then $BD + DF > BF$, i.e. $BA + AC$, i.e. $BD + DC$, by hypothesis; therefore $DF > DC$, whence in the triangles FAD, CAD the angle $FAD >$ the angle CAD.

Therefore $\angle FAD > \frac{1}{2}\angle FAC$

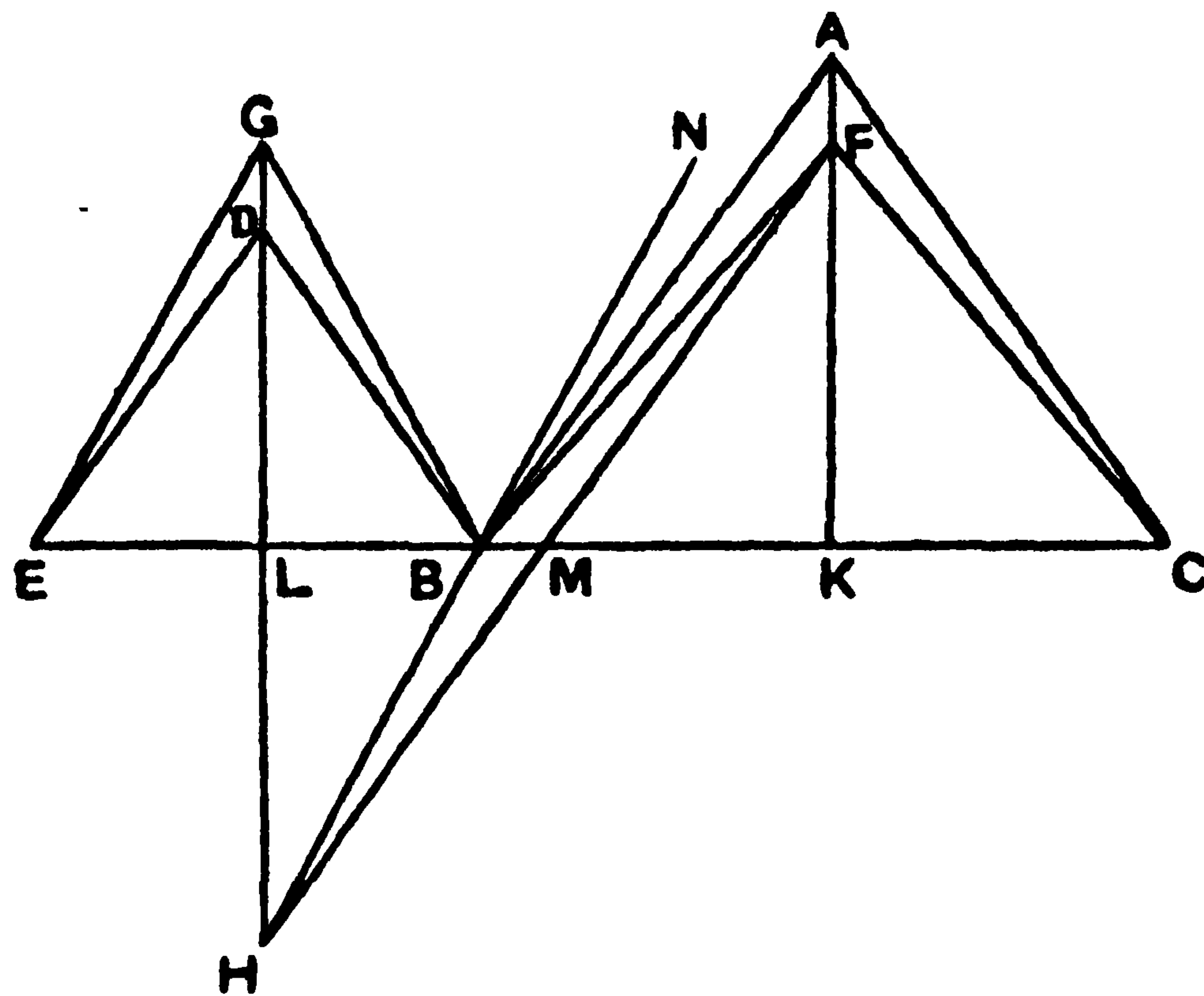
$> \angle BCA$.

Make the angle FAG equal to the angle BCA or ABC, so that AG is parallel to BC; let BD produced meet AG in G, and join GC.

Then $$\triangle ABC = \triangle GBC$$
$$> \triangle DBC.$$

The second lemma is to the effect that, given two isosceles triangles not similar to one another, if we construct on the same bases two triangles *similar to one another* such that the sum of their perimeters is equal to the sum of the perimeters of the first two triangles, then the sum of the areas of the similar triangles is greater than the sum of the areas of the non-similar triangles. (The easy construction of the similar triangles is given in a separate lemma.)

Let the bases of the isosceles triangles, EB, BC, be placed in one straight line, BC being greater than EB.



Let ABC, DEB be the similar isosceles triangles, and FBC, GEB the non-similar, the triangles being such that

$$BA + AC + ED + DB = BF + FC + EG + GB.$$

Produce AF, GD to meet the bases in K, L. Then clearly AK, GL bisect BC, EB at right angles at K, L.

Produce GL to H, making LH equal to GL.

Join HB and produce it to N; join HF.

Now, since the triangles ABC, DEB are similar, the angle ABC is equal to the angle DEB or DBE.

Therefore $\angle NBC\,(= \angle HBE = \angle GBE) > \angle DBE$ or $\angle ABC$;

therefore the angle ABH, and *a fortiori* the angle FBH, is less than two right angles, and HF meets BK in some point M.

Now, by hypothesis, $DB+BA = GB+BF$:

therefore $$DB+BA = HB+BF > HF.$$

By an easy lemma, since the triangles DEB, ABC are similar,

$$(DB+BA)^2 = (DL+AK)^2+(BL+BK)^2$$
$$= (DL+AK)^2+LK^2.$$

Therefore $$(DL+AK)^2+LK^2 > HF^2$$
$$> (GL+FK)^2+LK^2,$$

whence $$DL+AK > GL+FK,$$

and it follows that $$AF > GD.$$

But $BK > BL$; therefore $AF.BK > GD.BL$.

Hence the 'hollow-angled (figure)' (κοιλογώνιον) $ABFC$ is greater than the hollow-angled (figure) $GEDB$.

Adding $\Delta\, DEB + \Delta\, BFC$ to each, we have

$$\Delta\, DEB + \Delta\, ABC > \Delta\, GEB + \Delta\, FBC.$$

The above is the only case taken by Zenodorus. The proof still holds if $EB = BC$, so that $BK = BL$. But it fails in the case in which $EB > BC$ and the vertex G of the triangle EB belonging to the non-similar pair is still above D and not below it (as F is below A in the preceding case). This was no doubt the reason why Pappus gave a proof intended to apply to all the cases without distinction. This proof is the same as the above proof by Zenodorus up to the point where it is proved that

$$DL+AK > GL+FK,$$

but there diverges. Unfortunately the text is bad, and gives no sufficient indication of the course of the proof; but it would seem that Pappus used the relations

$$DL : GL = \Delta\, DEB : \Delta\, GEB,$$
$$AK : FK = \Delta\, ABC : \Delta\, FBC,$$

and $$AK^2 : DL^2 = \Delta\, ABC : \Delta\, DEB,$$

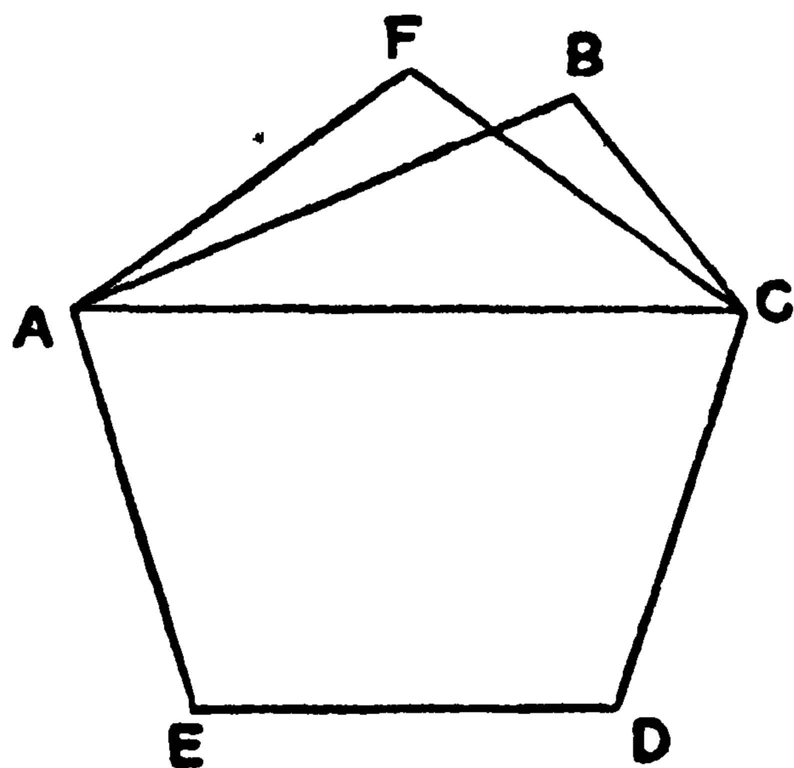
combined of course with the fact that $GB+BF = DB+BA$, in order to prove the proposition that,

according as $DL+AK >$ or $< GL+FK$,

$$\Delta\, DEB + \Delta\, ABC > \text{ or } < \Delta\, GEB + \Delta\, FBC.$$

The proof of his proposition, whatever it was, Pappus indicates that he will give later; but in the text as we have it the promise is not fulfilled.

Then follows the proof that the maximum polygon of given perimeter is both equilateral and equiangular.



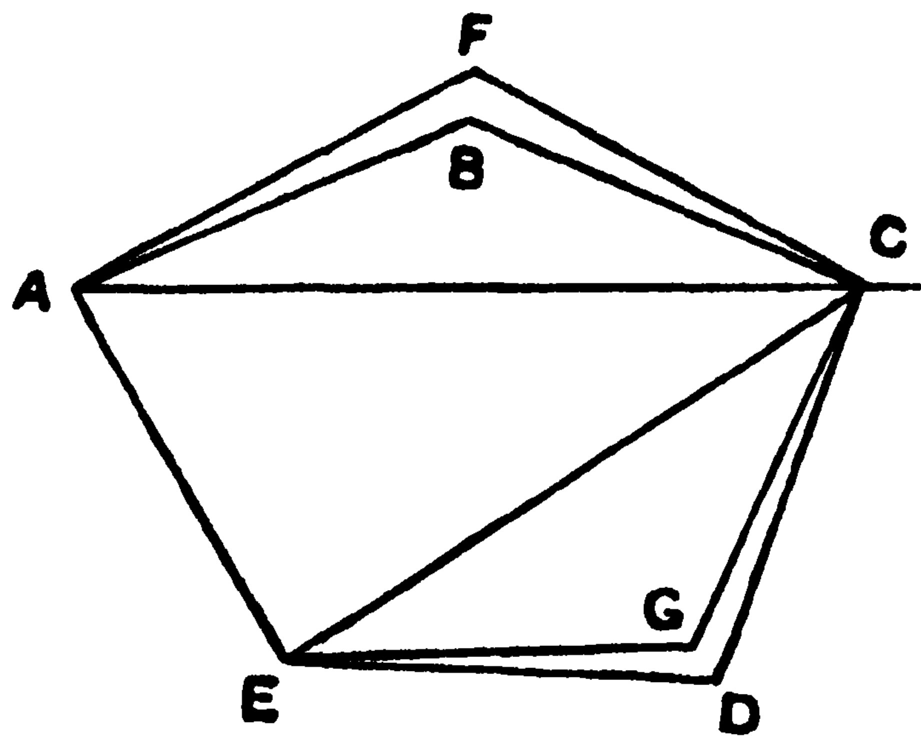
(1) It is equilateral.

For, if not, let two sides of the maximum polygon, as AB, BC, be unequal. Join AC, and on AC as base draw the isosceles triangle AFC such that $AF+FC=AB+BC$. The area of the triangle AFC is then greater than the area of the triangle ABC, and the area of the whole polygon has been increased by the construction: which is impossible, as by hypothesis the area is a maximum.

Similarly it can be proved that no other side is unequal to any other.

(2) It is also equiangular.

For, if possible, let the maximum polygon $ABCDE$ (which we have proved to be equilateral) have the angle at B greater than the angle at D. Then BAC, DEC are non-similar isosceles triangles. On AC, CE as bases describe the two isosceles triangles FAC, GEC similar to one another which have the sum of their perimeters equal to the sum of the perimeters of BAC, DEC. Then the sum of the areas of the two similar isosceles triangles is greater than the sum of the areas of the triangles BAC, DEC; the area of the polygon is therefore increased, which is contrary to the hypothesis.



Hence no two angles of the polygon can be unequal.

The maximum polygon of given perimeter is therefore both equilateral and equiangular.

Dealing with the sphere in relation to other solids having

their surfaces equal to that of the sphere, Zenodorus confined himself to proving (1) that the sphere is greater if the other solid with surface equal to that of the sphere is a solid formed by the revolution of a regular polygon about a diameter bisecting it as in Archimedes, *On the Sphere and Cylinder*, Book I, and (2) that the sphere is greater than any of the regular solids having its surface equal to that of the sphere.

Pappus's treatment of the subject is more complete in that he proves that the sphere is greater than the cone or cylinder the surface of which is equal to that of the sphere, and further that of the five regular solids which have the same surface that which has more faces is the greater.[1]

Hypsicles (second half of second century B.C.) has already been mentioned (vol. i, pp. 419 20) as the author of the continuation of the *Elements* known as Book XIV. He is quoted by Diophantus as having given a definition of a polygonal number as follows:

'If there are as many numbers as we please beginning from 1 and increasing by the same common difference, then, when the common difference is 1, the sum of all the numbers is a triangular number; when 2, a square: when 3, a pentagonal number [and so on]. And the number of angles is called after the number which exceeds the common difference by 2, and the side after the number of terms including 1.'

This definition amounts to saying that the nth a-gonal number (1 counting as the first) is $\frac{1}{2}n\{2+(n-1)(a-2)\}$. If, as is probable, Hypsicles wrote a treatise on polygonal numbers, it has not survived. On the other hand, the Ἀναφορικός (*Ascensiones*) known by his name has survived in Greek as well as in Arabic, and has been edited with translation.[2] True, the treatise (if it really be by Hypsicles, and not a clumsy effort by a beginner working from an original by Hypsicles) does no credit to its author; but it is in some respects interesting, and in particular because it is the first Greek

[1] Pappus, v, Props. 19, 38–56.
[2] Manitius, *Des Hypsikles Schrift Anaphorikos*, Dresden, Lehmannsche Buchdruckerei, 1888.

work in which we find the division of the ecliptic circle into 360 'parts' or degrees. The author says, after the preliminary propositions,

'The circle of the zodiac having been divided into 360 equal circumferences (arcs), let each of the latter be called a *degree in space* (μοῖρα τοπική, 'local' or 'spatial part'). And similarly, supposing that the time in which the zodiac circle returns to any position it has left is divided into 360 equal times, let each of these be called a *degree in time* (μοῖρα χρονική).'

From the word καλείσθω ('let it be called') we may perhaps infer that the terms were new in Greece. This brings us to the question of the origin of the division (1) of the circle of the zodiac, (2) of the circle in general, into 360 parts. On this question innumerable suggestions have been made. With reference to (1) it was suggested as long ago as 1788 (by Formaleoni) that the division was meant to correspond to the number of days in the year. Another suggestion is that it would early be discovered that, in the case of any circle the inscribed hexagon dividing the circumference into six parts has each of its sides equal to the radius, and that this would naturally lead to the circle being regularly divided into six parts; after this, the very ancient sexagesimal system would naturally come into operation and each of the parts would be divided into 60 subdivisions, giving 360 of these for the whole circle. Again, there is an explanation which is not even geometrical, namely that in the Babylonian numeral system, which combined the use of 6 and 10 as bases, the numbers 6, 60, 360, 3600 were fundamental round numbers, and these numbers were transferred from arithmetic to the heavens. The obvious objection to the first of these explanations (referring the 360 to the number of days in the solar year) is that the Babylonians were well acquainted, as far back as the monuments go, with 365·2 as the number of days in the year. A variant of the hexagon-theory is the suggestion .that a *natural* angle to be discovered, and to serve as a measure of others, is the angle of an equilateral triangle, found by drawing a star * like a six-spoked wheel without any circle. If the base of a sundial was so divided into six angles, it would be

natural to divide each of the sixth parts into either 10 or 60 parts; the former division would account for the attested division of the day into 60 hours, while the latter division on the sexagesimal system would give the 360 time-degrees (each of 4 minutes) making up the day of 24 hours. The purely arithmetical explanation is defective in that the series of numbers for which the Babylonians had special names is not 60, 360, 3600 but 60 (Soss), *600* (Ner), and 3600 or 60^2 (Sar). On the whole, after all that has been said, I know of no better suggestion than that of Tannery.[1] It is certain that both the division of the ecliptic into 360 degrees and that of the νυχθήμερον into 360 time-degrees were adopted by the Greeks from Babylon. Now the earliest division of the ecliptic was into 12 parts, the signs, and the question is, how were the signs subdivided? Tannery observes that, according to the cuneiform inscriptions, as well as the testimony of Greek authors, the sign was divided into parts one of which (*dargatu*) was double of the other (*murran*), the former being 1/30th, the other (called *stadium* by Manilius) 1/60th, of the sign; the former division would give 360 parts, the latter 720 parts for the whole circle. The latter division was more natural, in view of the long-established system of sexagesimal fractions; it also had the advantage of corresponding tolerably closely to the apparent diameter of the sun in comparison with the circumference of the sun's apparent circle. But, on the other hand, the double fraction, the 1/30th, was contained in the circle of the zodiac approximately the same number of times as there are days in the year, and consequently corresponded nearly to the distance described by the sun along the zodiac in one day. It would seem that this advantage was sufficient to turn the scale in favour of dividing each sign of the zodiac into 30 parts, giving 360 parts for the whole circle. While the Chaldaeans thus divided the ecliptic into 360 parts, it does not appear that they applied the same division to the equator or any other circle. They measured angles in general by *ells*, an ell representing 2°, so that the complete circle contained 180, not 360, parts, which they called ells. The explanation may perhaps be that the Chaldaeans divided

[1] Tannery, 'La coudée astronomique et les anciennes divisions du cercle' (*Mémoires scientifiques*, ii, pp. 256–68).

the *diameter* of the circle into 60 ells in accordance with their usual sexagesimal division, and then came to divide the circumference into 180 such ells on the ground that the circumference is roughly three times the diameter. The measurement in *ells* and *dactyli* (of which there were 24 to the ell) survives in Hipparchus (*On the Phaenomena of Eudoxus and Aratus*), and some measurements in terms of the same units are given by Ptolemy. It was Hipparchus who first divided the circle in general into 360 parts or degrees, and the introduction of this division coincides with his invention of trigonometry.

The contents of Hypsicles's tract need not detain us long. The problem is: If we know the ratio which the length of the longest day bears to the length of the shortest day at any given place, to find how many time-degrees it takes any given sign to rise; and, after this has been found, the author finds what length of time it takes any given degree in any sign to rise, i.e. the interval between the rising of one degree-point on the ecliptic and that of the next following. It is explained that the longest day is the time during which one half of the zodiac (Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius) rises, and the shortest day the time during which the other half (Capricornus, Aquarius, Pisces, Aries, Taurus, Gemini) rises. Now at Alexandria the longest day is to the shortest as 7 to 5; the longest therefore contains 210 'time-degrees', the shortest 150. The two quadrants Cancer–Virgo and Libra–Sagittarius take the same time to rise, namely 105 time-degrees, and the two quadrants Capricornus-Pisces and Aries–Gemini each take the same time, namely 75 time-degrees. It is further assumed that the times taken by Virgo, Leo, Cancer, Gemini, Taurus, Aries are in descending arithmetical progression, while the times taken by Libra, Scorpio, Sagittarius, Capricornus, Aquarius, Pisces continue the same descending arithmetical series. The following lemmas are used and proved:

I. If $a_1, a_2 \dots a_n, a_{n+1}, a_{n+2} \dots a_{2n}$ is a descending arithmetical progression of $2n$ terms with $\delta\,(= a_1 - a_2 = a_2 - a_3 = \dots)$ as common difference,

$$a_1 + a_2 + \dots + a_n - (a_{n+1} + a_{n+2} + \dots + a_{2n}) = n^2\delta.$$

II. If $a_1, a_2 \dots a_n \dots a_{2n-1}$ is a descending arithmetical progression of $2n-1$ terms with δ as common difference and a_n is the middle term, then

$$a_1 + a_2 + \dots + a_{2n-1} = (2n-1)a_n.$$

III. If $a_1, a_2 \dots a_n, a_{n+1} \dots a_{2n}$ is a descending arithmetical progression of $2n$ terms, then

$$a_1 + a_2 + \dots + a_{2n} = n(a_1 + a_{2n}) = n(a_2 + a_{2n-1}) = \dots$$
$$= n(a_n + a_{n+1}).$$

Now let A, B, C be the descending series the sum of which is 105, and D, E, F the next three terms in the same series the sum of which is 75, the common difference being δ; we then have, by (I),

$$A + B + C - (D + E + F) = 9\delta, \text{ or } 30 = 9\delta,$$

and $\delta = 3\frac{1}{3}$.

Next, by (II), $A + B + C = 3B$, or $3B = 105$, and $B = 35$:

therefore A, B, C, D, E, F are equal to $38\frac{1}{3}$, 35, $31\frac{2}{3}$, $28\frac{1}{3}$, 25, $21\frac{2}{3}$ time-degrees respectively, which the author of the tract expresses in time-degrees and minutes as $38^t\ 20'$, 35^t, $31^t\ 40'$, $28^t\ 20'$, 25^t, $21^t\ 40'$. We have now to carry through the same procedure for each degree in each sign. If the difference between the times taken to rise by one sign and the next is $3^t\ 20'$, what is the difference for each of the 30 degrees in the sign? We have here 30 terms followed by 30 other terms of the same descending arithmetical progression, and the formula (I) gives $3^t.\ 20' = (30)^2 d$, where d is the common difference; therefore $d = \frac{1}{900} \times 3^t.\ 20' = 0^t\ 0'\ 13''\ 20'''$. Lastly, take the sign corresponding to $21^t\ 40'$. This is the sum of a descending arithmetical progression of 30 terms $a_1, a_2 \dots a_{30}$ with common difference $0^t\ 0'\ 13''\ 20'''$. Therefore, by (III), $21^t\ 40' = 15(a_1 + a_{30})$, whence $a_1 + a_{30} = 1^t\ 26'\ 40''$. Now, since there are 30 terms $a_1, a_2 \dots a_{30}$, we have

$$a_1 - a_{30} = 29d = 0^t\ 6'\ 26''\ 40'''.$$

It follows that $a_{30} = 0^t\ 40'\ 6''\ 40'''$ and $a_1 = 0^t\ 46'\ 33''\ 20'''$,

and from these and the common difference $0^t\ 0'\ 13''\ 20'''$ all the times corresponding to all the degrees in the circle can be found.

The procedure was probably, as Tannery thinks, taken direct from the Babylonians, who would no doubt use it for the purpose of enabling the time to be determined at any hour of the night. Another view is that the object was astrological rather than astronomical (Manitius). In either case the method was exceedingly rough, and the assumed increases and decreases in the times of the risings of the signs in arithmetical progression are not in accordance with the facts. The book could only have been written before the invention of trigonometry by Hipparchus, for the problem of finding the times of rising of the signs is really one of spherical trigonometry, and these times were actually calculated by Hipparchus and Ptolemy by means of tables of chords.

Dionysodorus is known in the first place as the author of a solution of the cubic equation subsidiary to the problem of Archimedes, *On the Sphere and Cylinder*, II. 4, To cut a given sphere by a plane so that the volumes of the segments have to one another a given ratio (see above, p. 46). Up to recently this Dionysodorus was supposed to be Dionysodorus of Amisene in Pontus, whom Suidas describes as 'a mathematician worthy of mention in the field of education'. But we now learn from a fragment of the Herculaneum Roll, No. 1044, that 'Philonides was a pupil, first of Eudemus, and afterwards of Dionysodorus, the son of Dionysodorus the Caunian'. Now Eudemus is evidently Eudemus of Pergamum to whom Apollonius dedicated the first two Books of his *Conics*, and Apollonius actually asks him to show Book II to Philonides. In another fragment Philonides is said to have published some lectures by Dionysodorus. Hence our Dionysodorus may be Dionysodorus of Caunus and a contemporary of Apollonius, or very little later.[1] A Dionysodorus is also mentioned by Heron[2] as the author of a tract *On the Spire* (or tore) in which he proved that, if d be the diameter of the revolving circle which

[1] W. Schmidt in *Bibliotheca mathematica*, iv_3, pp. 321-5.
[2] Heron, *Metrica*, ii. 13, p. 128. 3.

generates the tore, and c the distance of its centre from the axis of revolution,

$$(\text{volume of tore}) : \pi c^2 . d = \tfrac{1}{4}\pi d^2 : \tfrac{1}{2} cd,$$

that is, $$(\text{volume of tore}) = \tfrac{1}{2}\pi^2 . cd^2,$$

which is of course the product of the area of the generating circle and the length of the path of its centre of gravity. The form in which the result is stated, namely that the tore is to the cylinder with height d and radius c as the generating circle of the tore is to half the parallelogram cd, indicates quite clearly that Dionysodorus proved his result by the same procedure as that employed by Archimedes in the *Method* and in the book *On Conoids and Spheroids*; and indeed the proof on Archimedean lines is not difficult.

Before passing to the mathematicians who are identified with the discovery and development of trigonometry, it will be convenient, I think, to dispose of two more mathematicians belonging to the last century B.C., although this involves a slight departure from chronological order; I mean Posidonius and Geminus.

POSIDONIUS, a Stoic, the teacher of Cicero, is known as Posidonius of Apamea (where he was born) or of Rhodes (where he taught); his date may be taken as approximately 135–51 B.C. In pure mathematics he is mainly quoted as the author of certain definitions, or for views on technical terms, e.g. 'theorem' and 'problem', and subjects belonging to elementary geometry. More important were his contributions to mathematical geography and astronomy. He gave his great work on geography the title *On the Ocean*, using the word which had always had such a fascination for the Greeks; its contents are known to us through the copious quotations from it in Strabo; it dealt with physical as well as mathematical geography, the zones, the tides and their connexion with the moon, ethnography and all sorts of observations made during extensive travels. His astronomical book bore the title *Meteorologica* or περὶ μετεώρων, and, while Geminus wrote a commentary on or exposition of this work, we may assign to it a number of views quoted from Posidonius in

Cleomedes's work *De motu circulari corporum caelestium*. Posidonius also wrote a separate tract on the size of the sun.

The two things which are sufficiently important to deserve mention here are (1) Posidonius's measurement of the circumference of the earth, (2) his hypothesis as to the distance and size of the sun.

(1) He estimated the circumference of the earth in this way. He assumed (according to Cleomedes[1]) that, whereas the star Canopus, invisible in Greece, was just seen to graze the horizon at Rhodes, rising and setting again immediately, the meridian altitude of the same star at Alexandria was 'a fourth part of a sign, that is, one forty-eighth part of the zodiac circle' ($= 7\frac{1}{2}°$); and he observed that the distance between the two places (supposed to lie on the same meridian) 'was considered to be 5,000 stades'. The circumference of the earth was thus made out to be 240,000 stades. Unfortunately the estimate of the difference of latitude, $7\frac{1}{2}°$, was very far from correct, the true difference being $5\frac{1}{4}°$ only; moreover the estimate of 5,000 stades for the distance was incorrect, being only the maximum estimate put upon it by mariners, while some put it at 4,000 and Eratosthenes, by observations of the shadows of gnomons, found it to be 3,750 stades only. Strabo, on the other hand, says that Posidonius favoured 'the latest of the measurements which gave the smallest dimensions to the earth, namely about 180,000 stades'.[2] This is evidently 48 times 3,750, so that Posidonius combined Eratosthenes's figure of 3,750 stades with the incorrect estimate of $7\frac{1}{2}°$ for the difference of latitude, although Eratosthenes presumably obtained the figure of 3,750 stades from his own estimate (250,000 or 252,000) of the circumference of the earth combined with an estimate of the difference of latitude which was about $5\frac{2}{5}°$ and therefore near the truth.

(2) Cleomedes[3] tells us that Posidonius supposed the circle in which the sun apparently moves round the earth to be 10,000 times the size of a circular section of the earth through its centre, and that with this assumption he combined the

[1] Cleomedes, *De motu circulari*, i. 10, pp. 92–4.
[2] Strabo, ii. c. 95.
[3] Cleomedes, *op. cit.* ii. 1, pp. 144–6, p. 98. 1–5.

statement of Eratosthenes (based apparently upon hearsay) that at Syene, which is under the summer tropic, and throughout a circle round it of 300 stades in diameter, the upright gnomon throws no shadow at noon. It follows from this that the diameter of the sun occupies a portion of the sun's circle 3,000,000 stades in length; in other words, the diameter of the sun is 3,000,000 stades. The assumption that the sun's circle is 10,000 times as large as a great circle of the earth was presumably taken from Archimedes, who had proved in the *Sand-reckoner* that the diameter of the sun's orbit is *less* than 10,000 times that of the earth: Posidonius in fact took the maximum value to be the true value; but his estimate of the sun's size is far nearer the truth than the estimates of Aristarchus, Hipparchus, and Ptolemy. Expressed in terms of the mean diameter of the earth, the estimates of these astronomers give for the diameter of the sun the figures $6\frac{3}{4}$, $12\frac{1}{3}$, and $5\frac{1}{2}$ respectively; Posidonius's estimate gives $39\frac{1}{4}$, the true figure being 108·9.

In elementary geometry Posidonius is credited by Proclus with certain definitions. He defined 'figure' as 'confining limit' (πέρας συγκλεῖον)[1] and 'parallels' as 'those lines which, being in one plane, neither converge nor diverge, but have all the perpendiculars equal which are drawn from the points of one line to the other'.[2] (Both these definitions are included in the *Definitions* of Heron.) He also distinguished seven species of quadrilaterals, and had views on the distinction between *theorem* and *problem*. Another indication of his interest in the fundamentals of elementary geometry is the fact[3] that he wrote a separate work in refutation of the Epicurean Zeno of Sidon, who had objected to the very beginnings of the *Elements* on the ground that they contained unproved assumptions. Thus, said Zeno, even Eucl. I. 1 requires it to be admitted that 'two straight lines cannot have a common segment'; and, as regards the 'proof' of this fact deduced from the bisection of a circle by its diameter, he would object that it has to be assumed that two arcs of circles cannot have a common part. Zeno argued generally that, even if we admit the fundamental principles of geometry, the deductions

[1] Proclus on Eucl. I, p. 143. 8.
[2] *Ib.*, p. 176. 6–10.
[3] *Ib.*, pp. 199. 14–200. 3.

from them cannot be proved without the admission of something else as well which has not been included in the said principles, and he intended by means of these criticisms to destroy the whole of geometry.[1] We can understand, therefore, that the tract of Posidonius was a serious work.

A definition of the centre of gravity by one 'Posidonius a Stoic' is quoted in Heron's *Mechanics*, but, as the writer goes on to say that Archimedes introduced a further distinction, we may fairly assume that the Posidonius in question is not Posidonius of Rhodes, but another, perhaps Posidonius of Alexandria, a pupil of Zeno of Cittium in the third century B.C.

We now come to GEMINUS, a very important authority on many questions belonging to the history of mathematics, as is shown by the numerous quotations from him in Proclus's *Commentary on Euclid, Book I.* His date and birthplace are uncertain, and the discussions on the subject now form a whole literature for which reference must be made to Manitius's edition of the so-called *Gemini elementa astronomiae* (Teubner, 1898) and the article 'Geminus' in Pauly-Wissowa's *Real-Encyclopädie.* The doubts begin with his name. Petau, who included the treatise mentioned in his *Uranologion* (Paris, 1630), took it to be the Latin Gemĭnus. Manitius, the latest editor, satisfied himself that it was Gemīnus, a Greek name, judging from the fact that it consistently appears with the properispomenon accent in Greek (*Γεμῖνος*), while it is also found in inscriptions with the spelling *Γεμεῖνος*; Manitius suggests the derivation from *γεμ*, as *Ἐργῖνος* from *ἐργ*, and *Ἀλεξῖνος* from *ἀλεξ*; he compares also the unmistakably Greek names *Ἰκτῖνος, Κρατῖνος.* Now, however, we are told (by Tittel) that the name is, after all, the Latin Gémĭnus, that *Γεμῖνος* came to be so written through false analogy with *Ἀλεξῖνος*, &c., and that *Γε[μ]εῖνος*, if the reading is correct, is also wrongly formed on the model of *Ἀντωνεῖνος, Ἀγριππεῖνα.* The occurrence of a Latin name in a centre of Greek culture need not surprise us, since Romans settled in such centres in large numbers during the last century B.C. Geminus, however, in spite of his name, was thoroughly Greek.

[1] Proclus on Eucl. I, pp. 214. 18-215. 13, p. 216. 10-19, p. 217. 10-23.

An upper limit for his date is furnished by the fact that he wrote a commentary on or exposition of Posidonius's work περὶ μετεώρων; on the other hand, Alexander Aphrodisiensis (about A.D. 210) quotes an important passage from an 'epitome' of this ἐξήγησις by Geminus. The view most generally accepted is that he was a Stoic philosopher, born probably in the island of Rhodes, and a pupil of Posidonius, and that he wrote about 73–67 B.C.

Of Geminus's works that which has most interest for us is a comprehensive work on mathematics. Proclus, though he makes great use of it, does not mention its title, unless indeed, in the passage where, after quoting from Geminus a classification of lines which never meet, he says 'these remarks I have selected from the φιλοκαλία of Geminus',[1] the word φιλοκαλία is a title or an alternative title. Pappus, however, quotes a work of Geminus 'on the classification of the mathematics' (ἐν τῷ περὶ τῆς τῶν μαθημάτων τάξεως), while Eutocius quotes from 'the sixth book of the doctrine of the mathematics' (ἐν τῷ ἕκτῳ τῆς τῶν μαθημάτων θεωρίας). The former title corresponds well enough to the long extract on the division of the mathematical sciences into arithmetic, geometry, mechanics, astronomy, optics, geodesy, canonic (musical harmony) and logistic which Proclus gives in his first prologue, and also to the fragments contained in the *Anonymi variae collectiones* published by Hultsch in his edition of Heron; but it does not suit most of the other passages borrowed by Proclus. The correct title was most probably that given by Eutocius, *The Doctrine*, or *Theory*, *of the Mathematics*; and Pappus probably refers to one particular section of the work, say the first Book. If the sixth Book treated of conics, as we may conclude from Eutocius's reference, there must have been more Books to follow; for Proclus has preserved us details about higher curves, which must have come later. If again Geminus finished his work and wrote with the same fullness about the other branches of mathematics as he did about geometry, there must have been a considerable number of Books altogether. It seems to have been designed to give a complete view of the whole science of mathematics, and in fact

[1] Proclus on Eucl. I, p. 177. 24.

to have been a sort of encyclopaedia of the subject. The quotations of Proclus from Geminus's work do not stand alone; we have other collections of extracts, some more and some less extensive, and showing varieties of tradition according to the channel through which they came down. The scholia to Euclid's *Elements*, Book I, contain a considerable part of the commentary on the Definitions of Book I, and are valuable in that they give Geminus pure and simple, whereas Proclus includes extracts from other authors. Extracts from Geminus of considerable length are included in the Arabic commentary by an-Nairīzī (about A.D. 900) who got them through the medium of Greek commentaries on Euclid, especially that of Simplicius. It does not appear to be doubted any longer that 'Aganis' in an-Nairīzī is really Geminus; this is inferred from the close agreement between an-Nairīzī's quotations from 'Aganis' and the corresponding passages in Proclus; the difficulty caused by the fact that Simplicius calls Aganis 'socius noster' is met by the suggestion that the particular word *socius* is either the result of the double translation from the Greek or means nothing more, in the mouth of Simplicius, than 'colleague' in the sense of a worker in the same field, or 'authority'. A few extracts again are included in the *Anonymi variae collectiones* in Hultsch's *Heron*. Nos. 5–14 give definitions of geometry, logistic, geodesy and their subject-matter, remarks on bodies as continuous magnitudes, the three dimensions as 'principles' of geometry, the purpose of geometry, and lastly on optics, with its subdivisions, optics proper, *Catoptrica* and *σκηνογραφική*, scene-painting (a sort of perspective), with some fundamental principles of optics, e.g. that all light travels along straight lines (which are broken in the cases of reflection and refraction), and the division between optics and natural philosophy (the theory of light), it being the province of the latter to investigate (what is a matter of indifference to optics) whether (1) visual rays issue from the eye, (2) images proceed from the object and impinge on the eye, or (3) the intervening air is aligned or compacted with the beam-like breath or emanation from the eye.

Nos. 80–6 again in the same collection give the Peripatetic explanation of the name mathematics, adding that the term

was applied by the early Pythagoreans more particularly to geometry and arithmetic, sciences which deal with the pure, the eternal and the unchangeable, but was extended by later writers to cover what we call 'mixed' or applied mathematics, which, though theoretical, has to do with sensible objects, e.g. astronomy and optics. Other extracts from Geminus are found in extant manuscripts in connexion with Damianus's treatise on optics (published by R. Schöne, Berlin, 1897). The definitions of logistic and geometry also appear, but with decided differences, in the scholia to Plato's *Charmides* 165 E. Lastly, isolated extracts appear in Eutocius, (1) a remark reproduced in the commentary on Archimedes's *Plane Equilibriums* to the effect that Archimedes in that work gave the name of postulates to what are really axioms, (2) the statement that before Apollonius's time the conics were produced by cutting different cones (right-angled, acute-angled, and obtuse-angled) by sections perpendicular in each case to a generator.[1]

The object of Geminus's work was evidently the examination of the first principles, the logical building up of mathematics on the basis of those admitted principles, and the defence of the whole structure against the criticisms of the enemies of the science, the Epicureans and Sceptics, some of whom questioned the unproved principles, and others the logical validity of the deductions from them. Thus in geometry Geminus dealt first with the principles or hypotheses (ἀρχαί, ὑποθέσεις) and then with the logical deductions, the theorems and problems (τὰ μετὰ τὰς ἀρχάς). The distinction is between the things which must be taken for granted but are incapable of proof and the things which must not be assumed but are matter for demonstration. The principles consisting of definitions, postulates, and axioms, Geminus subjected them severally to a critical examination from this point of view, distinguishing carefully between postulates and axioms, and discussing the legitimacy or otherwise of those formulated by Euclid in each class. In his notes on the definitions Geminus treated them historically, giving the various alternative definitions which had been suggested for each fundamental concept such as 'line', 'surface', 'figure', 'body', 'angle', &c., and frequently adding instructive classifications

[1] Eutocius, *Comm. on Apollonius's Conics, ad init.*

of the different species of the thing defined. Thus in the case of 'lines' (which include curves) he distinguishes, first, the composite (e.g. a broken line forming an angle) and the incomposite. The incomposite are subdivided into those 'forming a figure' (σχηματοποιοῦσαι) or determinate (e.g. circle, ellipse, cissoid) and those not forming a figure, indeterminate and extending without limit (e.g. straight line, parabola, hyperbola, conchoid). In a second classification incomposite lines are divided into (1) 'simple', namely the circle and straight line, the one 'making a figure', the other extending without limit, and (2) 'mixed'. 'Mixed' lines again are divided into (*a*) 'lines in planes', one kind being a line meeting itself (e.g. the cissoid) and another a line extending without limit, and (*b*) 'lines on solids', subdivided into lines formed by *sections* (e.g. conic sections, *spiric* curves) and 'lines *round* solids' (e.g. a helix round a cylinder, sphere, or cone, the first of which is uniform, homoeomeric, alike in all its parts, while the others are non-uniform). Geminus gave a corresponding division of surfaces into simple and mixed, the former being plane surfaces and spheres, while examples of the latter are the tore or anchor-ring (though formed by the revolution of a circle about an axis) and the conicoids of revolution (the right-angled conoid, the obtuse-angled conoid, and the two spheroids, formed by the revolution of a parabola, a hyperbola, and an ellipse respectively about their axes). He observes that, while there are three *homoeomeric* or uniform 'lines' (the straight line, the circle, and the cylindrical helix), there are only two homoeomeric surfaces, the plane and the sphere. Other classifications are those of 'angles' (according to the nature of the two lines or curves which form them) and of figures and plane figures.

When Proclus gives definitions, &c., by Posidonius, it is evident that he obtained them from Geminus's work. Such are Posidonius's definitions of 'figure' and 'parallels', and his division of quadrilaterals into seven kinds. We may assume further that, even where Geminus did not mention the name of Posidonius, he was, at all events so far as the philosophy of mathematics was concerned, expressing views which were mainly those of his master.

Attempt to prove the Parallel-Postulate.

Geminus devoted much attention to the distinction between postulates and axioms, giving the views of earlier philosophers and mathematicians (Aristotle, Archimedes, Euclid, Apollonius, the Stoics) on the subject as well as his own. It was important in view of the attacks of the Epicureans and Sceptics on mathematics, for (as Geminus says) it is as futile to attempt to prove the indemonstrable (as Apollonius did when he tried to prove the axioms) as it is incorrect to assume what really requires proof, 'as Euclid did in the fourth postulate [that all right angles are equal] and in the fifth postulate [the parallel-postulate]'.[1]

The fifth postulate was the special stumbling-block. Geminus observed that the converse is actually proved by Euclid in I. 17; also that it is conclusively proved that an angle equal to a right angle is not necessarily itself a right angle (e.g. the 'angle' between the circumferences of two semicircles on two equal straight lines with a common extremity and at right angles to one another); we cannot therefore admit that the converses are incapable of demonstration.[2] And

'we have learned from the very pioneers of this science not to have regard to mere plausible imaginings when it is a question of the reasonings to be included in our geometrical doctrine. As Aristotle says, it is as justifiable to ask scientific proofs from a rhetorician as to accept mere plausibilities from a geometer... So in this case (that of the parallel-postulate) the fact that, when the right angles are lessened, the straight lines converge is true and necessary; but the statement that, since they converge more and more as they are produced, they will sometime meet is plausible but not necessary, in the absence of some argument showing that this is true in the case of straight lines. For the fact that some lines exist which approach indefinitely but yet remain non-secant (ἀσύμπτωτοι), although it seems improbable and paradoxical, is nevertheless true and fully ascertained with reference to other species of lines [the hyperbola and its asymptote and the conchoid and its asymptote, as Geminus says elsewhere]. May not then the same thing be possible in the case of

[1] Proclus on Eucl. I, pp. 178-82. 4; 183. 14-184. 10.
[2] *Ib.*, pp. 183. 26-184. 5.

straight lines which happens in the case of the lines referred to? Indeed, until the statement in the postulate is clinched by proof, the facts shown in the case of the other lines may direct our imagination the opposite way. And, though the controversial arguments against the meeting of the straight lines should contain much that is surprising, is there not all the more reason why we should expel from our body of doctrine this merely plausible and unreasoned (hypothesis)? It is clear from this that we must seek a proof of the present theorem, and that it is alien to the special character of postulates.'[1]

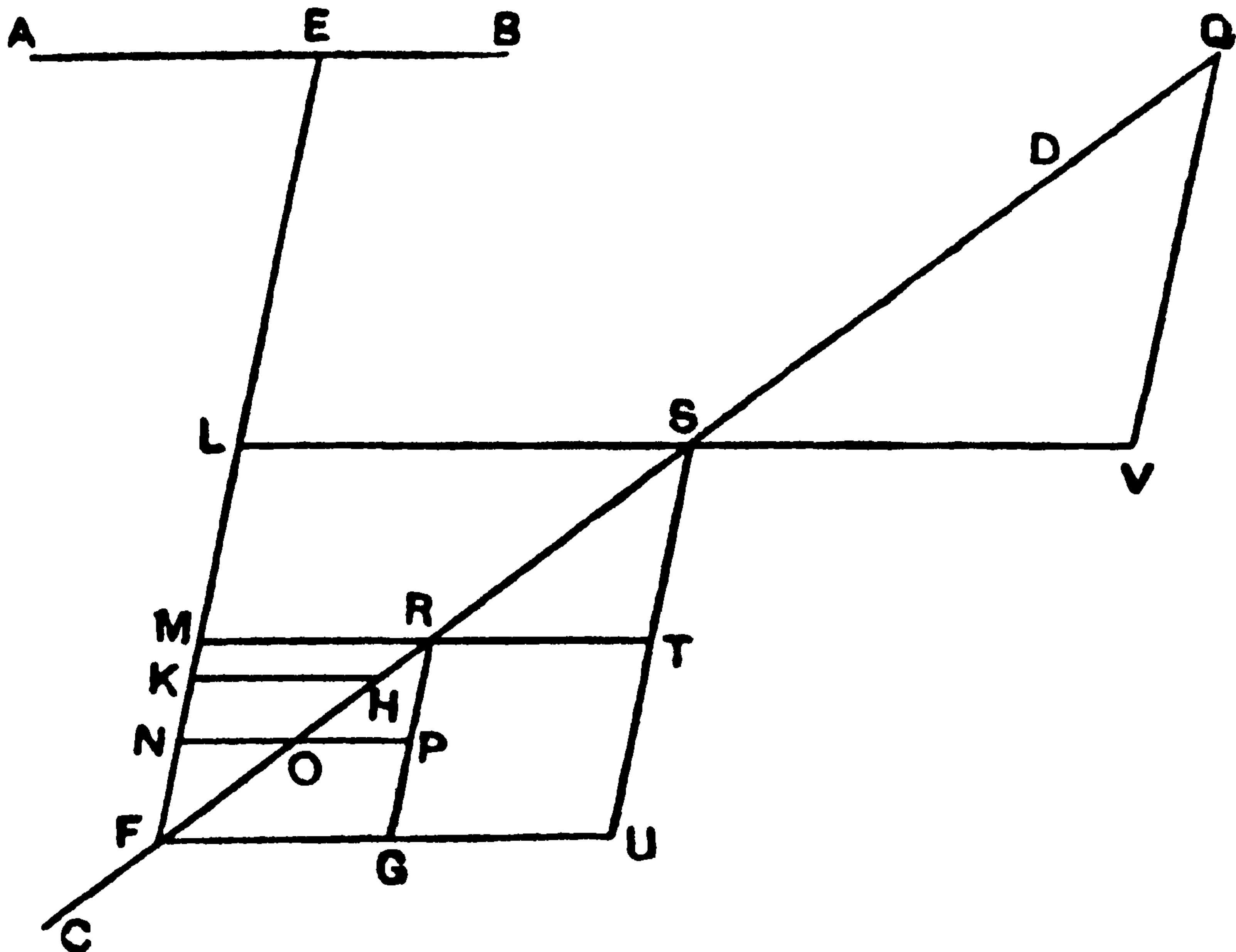
Much of this might have been written by a modern geometer. Geminus's attempted remedy was to substitute a definition of parallels like that of Posidonius, based on the notion of *equidistance.* An-Nairīzī gives the definition as follows: 'Parallel straight lines are straight lines situated in the same plane and such that the distance between them, if they are produced without limit in both directions at the same time, is everywhere the same', to which Geminus adds the statement that the said distance is the shortest straight line that can be drawn between them. Starting from this, Geminus proved to his own satisfaction the propositions of Euclid regarding parallels and finally the parallel-postulate. He first gave the propositions (1) that the 'distance' between the two lines as defined is perpendicular to both, and (2) that, if a straight line is perpendicular to each of two straight lines and meets both, the two straight lines are parallel, and the 'distance' is the intercept on the perpendicular (proved by *reductio ad absurdum*). Next come (3) Euclid's propositions I. 27, 28 that, if two lines are parallel, the alternate angles made by any transversal are equal, &c. (easily proved by drawing the two equal 'distances' through the points of intersection with the transversal), and (4) Eucl. I. 29, the converse of I. 28, which is proved by *reductio ad absurdum*, by means of (2) and (3). Geminus still needs Eucl. I. 30, 31 (about parallels) and I. 33, 34 (the first two propositions relating to parallelograms) for his final proof of the postulate, which is to the following effect.

Let AB, CD be two straight lines met by the straight line

[1] Proclus on Eucl. I, pp. 192. 5-193. 3.

EF, and let the interior angles BEF, EFD be together less than two right angles.

Take any point H on FD and draw HK parallel to AB meeting EF in K. Then, if we bisect EF at L, LF at M, MF at N, and so on, we shall at last have a length, as FN, less



than FK. Draw FG, NOP parallel to AB. Produce FO to Q, and let FQ be the same multiple of FO that FE is of FN; then shall AB, CD meet in Q.

Let S be the middle point of FQ and R the middle point of FS. Draw through R, S, Q respectively the straight lines RPG, STU, QV parallel to EF. Join MR, LS and produce them to T, V. Produce FG to U.

Then, in the triangles FON, ROP, two angles are equal respectively, the vertically opposite angles FON, ROP and the alternate angles NFO, PRO; and $FO = OR$; therefore $RP = FN$.

And FN, PG in the parallelogram $FNPG$ are equal; therefore $RG = 2FN = FM$ (whence MR is parallel to FG or AB)

Similarly we prove that $SU = 2FM = FL$, and LS is parallel to FG or AB.

Lastly, by the triangles FLS, QVS, in which the sides FS, SQ are equal and two angles are respectively equal, $QV = FL$.

Therefore $QV = LE$.

Since then EL, QV are equal and parallel, so are EQ, LV, and (says Geminus) it follows that AB passes through Q.

What follows is actually that both *EQ* and *AB* (or *EB*) are parallel to *LV*, and Geminus assumes that *EQ*, *AB* are coincident (in other words, that through a given point only one parallel can be drawn to a given straight line, an assumption known as Playfair's Axiom, though it is actually stated in Proclus on Eucl. I. 31).

The proof therefore, apparently ingenious as it is, breaks down. Indeed the method is unsound from the beginning, since (as Saccheri pointed out), before even the definition of parallels by Geminus can be used, it has to be *proved* that 'the geometrical locus of points equidistant from a straight line is a straight line', and this cannot be proved without a postulate. But the attempt is interesting as the first which has come down to us, although there must have been many others by geometers earlier than Geminus.

Coming now to the things which follow from the principles (τὰ μετὰ τὰς ἀρχάς), we gather from Proclus that Geminus carefully discussed such generalities as the nature of *elements*, the different views which had been held of the distinction between theorems and problems, the nature and significance of διορισμοί (conditions and limits of possibility), the meaning of 'porism' in the sense in which Euclid used the word in his *Porisms* as distinct from its other meaning of 'corollary', the different sorts of problems and theorems, the two varieties of converses (complete and partial), *topical* or *locus*-theorems, with the classification of loci. He discussed also philosophical questions, e.g. the question whether a line is made up of indivisible parts (ἐξ ἀμερῶν), which came up in connexion with Eucl. I. 10 (the bisection of a straight line).

The book was evidently not less exhaustive as regards higher geometry. Not only did Geminus mention the *spiric* curves, conchoids and cissoids in his classification of curves; he showed how they were obtained, and gave proofs, presumably of their principal properties. Similarly he gave the proof that there are three homoeomeric or uniform lines or curves, the straight line, the circle and the cylindrical helix. The proof of 'uniformity' (the property that any portion of the line or curve will coincide with any other portion of the same length) was preceded by a proof that, if two straight lines be drawn from any point to meet a uniform line or curve

and make equal angles with it, the straight lines are equal.[1] As Apollonius wrote on the cylindrical helix and proved the fact of its uniformity, we may fairly assume that Geminus was here drawing upon Apollonius.

Enough has been said to show how invaluable a source of information Geminus's work furnished to Proclus and all writers on the history of mathematics who had access to it.

In astronomy we know that Geminus wrote an ἐξήγησις of Posidonius's work, the *Meteorologica* or περὶ μετεώρων. This is the source of the famous extract made from Geminus by Alexander Aphrodisiensis, and reproduced by Simplicius in his commentary on the *Physics* of Aristotle,[2] on which Schiaparelli relied in his attempt to show that it was Heraclides of Pontus, not Aristarchus of Samos, who first put forward the heliocentric hypothesis. The extract is on the distinction between physical and astronomical inquiry as applied to the heavens. It is the business of the physicist to consider the substance of the heaven and stars, their force and quality, their coming into being and decay, and he is in a position to prove the facts about their size, shape, and arrangement; astronomy, on the other hand, ignores the physical side, proving the arrangement of the heavenly bodies by considerations based on the view that the heaven is a real κόσμος, and, when it tells us of the shapes, sizes and distances of the earth, sun and moon, of eclipses and conjunctions, and of the quality and extent of the movements of the heavenly bodies, it is connected with the mathematical investigation of quantity, size, form, or shape, and uses arithmetic and geometry to prove its conclusions. Astronomy deals, not with causes, but with facts; hence it often proceeds by hypotheses, stating certain expedients by which the phenomena may be saved. For example, why do the sun, the moon and the planets appear to move irregularly? To explain the observed facts we may assume, for instance, that the orbits are eccentric circles or that the stars describe epicycles on a carrying circle; and then we have to go farther and examine other ways in which it is possible for the phenomena to be brought about. '*Hence we actually find a certain person* [Heraclides

[1] Proclus on Eucl. I, pp. 112. 22–113. 3, p. 251. 3–11.
[2] Simpl. *in Phys.*, pp. 291–2, ed. Diels.

of Pontus] *coming forward and saying that, even on the assumption that the earth moves in a certain way, while the sun is in a certain way at rest, the apparent irregularity with reference to the sun may be saved.*' Philological considerations as well as the other notices which we possess about Heraclides make it practically certain that 'Heraclides of Pontus' is an interpolation and that Geminus said τις simply, 'a certain person', without any name, though he doubtless meant Aristarchus of Samos.[1]

Simplicius says that Alexander quoted this extract from the *epitome* of the ἐξήγησις by Geminus. As the original work was apparently made the subject of an abridgement, we gather that it must have been of considerable scope. It is a question whether ἐξήγησις means 'commentary' or 'exposition'; I am inclined to think that the latter interpretation is the correct one, and that Geminus reproduced Posidonius's work in its entirety with elucidations and comments; this seems to me to be suggested by the words added by Simplicius immediately after the extract 'this is the account given by Geminus, *or Posidonius in Geminus*, of the difference between physics and astronomy', which seems to imply that Geminus in our passage reproduced Posidonius textually.

'Introduction to the Phaenomena' attributed to Geminus.

There remains the treatise, purporting to be by Geminus, which has come down to us under the title Γεμίνου εἰσαγωγὴ εἰς τὰ Φαινόμενα.[2] What, if any, is the relation of this work to the exposition of Posidonius's *Meteorologica* or the epitome of it just mentioned? One view is that the original *Isagoge* of Geminus and the ἐξήγησις of Posidonius were one and the same work, though the *Isagoge* as we have it is not by Geminus, but by an unknown compiler. The objections to this are, first, that it does not contain the extract given by Simplicius, which would have come in usefully at the beginning of an Introduction to Astronomy, nor the other extract given by Alexander from Geminus and relating to the rainbow which seems likewise to have come from the ἐξήγησις[3];

[1] Cf. *Aristarchus of Samos*, pp. 275–83.
[2] Edited by Manitius (Teubner, 1898).
[3] Alex. Aphr. on Aristotle's *Meteorologica*, iii. 4, 9 (Ideler, ii, p. 128; p. 152. 10, Hayduck).

secondly, that it does not anywhere mention the name of Posidonius (not, perhaps, an insuperable objection); and, thirdly, that there are views expressed in it which are not those held by Posidonius but contrary to them. Again, the writer knows how to give a sound judgement as between divergent views, writes in good style on the whole, and can hardly have been the mere compiler of extracts from Posidonius which the view in question assumes him to be. It seems in any case safer to assume that the *Isagoge* and the ἐξήγησις were separate works. At the same time, the *Isagoge*, as we have it, contains errors which we cannot attribute to Geminus. The choice, therefore, seems to lie between two alternatives: either the book is by Geminus in the main, but has in the course of centuries suffered deterioration by interpolations, mistakes of copyists, and so on, or it is a compilation of extracts from an original *Isagoge* by Geminus with foreign and inferior elements introduced either by the compiler himself or by other prentice hands. The result is a tolerable elementary treatise suitable for teaching purposes and containing the most important doctrines of Greek astronomy represented from the standpoint of Hipparchus. Chapter 1 treats of the zodiac, the solar year, the irregularity of the sun's motion, which is explained by the eccentric position of the sun's orbit relatively to the zodiac, the order and the periods of revolution of the planets and the moon. In § 23 we are told that all the fixed stars do not lie on one spherical surface, but some are farther away than others—a doctrine due to the Stoics. Chapter 2, again, treats of the twelve signs of the zodiac, chapter 3 of the constellations, chapter 4 of the axis of the universe and the poles, chapter 5 of the circles on the sphere (the equator and the parallel circles, arctic, summer-tropical, winter-tropical, antarctic, the colure-circles, the zodiac or ecliptic, the horizon, the meridian, and the Milky Way), chapter 6 of Day and Night, their relative lengths in different latitudes, their lengthening and shortening, chapter 7 of the times which the twelve signs take to rise. Chapter 8 is a clear, interesting and valuable chapter on the calendar, the length of months and years and the various cycles, the octaëteris, the 16-years and 160-years cycles, the 19-years cycle of Euctemon (and Meton), and the cycle of Callippus

(76 years). Chapter 9 deals with the moon's phases, chapters 10, 11 with eclipses of the sun and moon, chapter 12 with the problem of accounting for the motions of the sun, moon and planets, chapter 13 with Risings and Settings and the various technical terms connected therewith, chapter 14 with the circles described by the fixed stars, chapters 15 and 16 with mathematical and physical geography, the zones, &c. (Geminus follows Eratosthenes's evaluation of the circumference of the earth, not that of Posidonius). Chapter 17, on weather indications, denies the popular theory that changes of atmospheric conditions depend on the rising and setting of certain stars, and states that the predictions of weather (*ἐπισημασίαι*) in calendars (*παραπήγματα*) are only derived from experience and observation, and have no scientific value. Chapter 18 is on the *ἐξελιγμός*, the shortest period which contains an integral number of synodic months, of days, and of anomalistic revolutions of the moon; this period is three times the Chaldaean period of 223 lunations used for predicting eclipses. The end of the chapter deals with the maximum, mean, and minimum daily motion of the moon. The chapter as a whole does not correspond to the rest of the book; it deals with more difficult matters, and is thought by Manitius to be a fragment only of a discussion to which the compiler did not feel himself equal. At the end of the work is a calendar (*Parapegma*) giving the number of days taken by the sun to traverse each sign of the zodiac, the risings and settings of various stars and the weather indications noted by various astronomers, Democritus, Eudoxus, Dositheus, Euctemon, Meton, Callippus; this calendar is unconnected with the rest of the book and the contents are in several respects inconsistent with it, especially the division of the year into quarters which follows Callippus rather than Hipparchus. Hence it has been, since Boeckh's time, generally considered not to be the work of Geminus. Tittel, however, suggests that it is not impossible that Geminus may have reproduced an older *Parapegma* of Callippus.

XVI

SOME HANDBOOKS

THE description of the handbook on the elements of astronomy entitled the *Introduction to the Phaenomena* and attributed to Geminus might properly have been reserved for this chapter. It was, however, convenient to deal with Geminus in close connexion with Posidonius; for Geminus wrote an exposition of Posidonius's *Meteorologica* related to the original work in such a way that Simplicius, in quoting a long passage from an epitome of this work, could attribute the passage to either Geminus or 'Posidonius in Geminus'; and it is evident that, in other subjects too, Geminus drew from, and was influenced by, Posidonius.

The small work *De motu circulari corporum caelestium* by CLEOMEDES (Κλεομήδους κυκλικὴ θεωρία) in two Books is the production of a much less competent person, but is much more largely based on Posidonius. This is proved by several references to Posidonius by name, but it is specially true of the very long first chapter of Book II (nearly half of the Book) which seems for the most part to be copied bodily from Posidonius, in accordance with the author's remark at the end of Book I that, in giving the refutation of the Epicurean assertion that the sun is just as large as it looks, namely one foot in diameter, he will give so much as suffices for such an introduction of the particular arguments used by 'certain authors who have written whole treatises on this one topic (i.e. the size of the sun), among whom is Posidonius'. The interest of the book then lies mainly in what is quoted from Posidonius; its mathematical interest is almost *nil*.

The date of Cleomedes is not certainly ascertained, but, as he mentions no author later than Posidonius, it is permissible to suppose, with Hultsch, that he wrote about the middle of

the first century B.C. As he seems to know nothing of the works of Ptolemy, he can hardly, in any case, have lived later than the beginning of the second century A.D.

Book I begins with a chapter the object of which is to prove that the universe, which has the shape of a sphere, is limited and surrounded by void extending without limit in all directions, and to refute objections to this view. Then follow chapters on the five parallel circles in the heaven and the zones, habitable and uninhabitable (chap. 2); on the motion of the fixed stars and the independent (προαιρετικαί) movements of the planets including the sun and moon (chap. 3); on the zodiac and the effect of the sun's motion in it (chap. 4); on the inclination of the axis of the universe and its effects on the lengths of days and nights at different places (chap. 5); on the inequality in the rate of increase in the lengths of the days and nights according to the time of year, the different lengths of the seasons due to the motion of the sun in an eccentric circle, the difference between a day-and-night and an exact revolution of the universe owing to the separate motion of the sun (chap. 6); on the habitable regions of the globe including Britain and the 'island of Thule', said to have been visited by Pytheas, where, when the sun is in Cancer and visible, the day is a month long; and so on (chap. 7). Chap. 8 purports to prove that the universe is a sphere by proving first that the earth is a sphere, and then that the air about it, and the ether about that, must necessarily make up larger spheres. The earth is proved to be a sphere by the method of exclusion; it is assumed that the only possibilities are that it is (*a*) flat and plane, or (*b*) hollow and deep, or (*c*) square, or (*d*) pyramidal, or (*e*) spherical, and, the first four hypotheses being successively disposed of, only the fifth remains. Chap. 9 maintains that the earth is in the centre of the universe; chap. 10, on the size of the earth, contains the interesting reproduction of the details of the measurements of the earth by Posidonius and Eratosthenes respectively which have been given above in their proper places (p. 220, pp. 106–7); chap. 11 argues that the earth is in the relation of a point to, i.e. is negligible in size in comparison with, the universe or even the sun's circle, but not the moon's circle (cf. p. 3 above).

Book II, chap. 1, is evidently the *pièce de résistance*, con-

sisting of an elaborate refutation of Epicurus and his followers, who held that the sun is just as large as it *looks*, and further asserted (according to Cleomedes) that the stars are lit up as they rise and extinguished as they set. The chapter seems to be almost wholly taken from Posidonius; it ends with some pages of merely vulgar abuse, comparing Epicurus with Thersites, with more of the same sort. The value of the chapter lies in certain historical traditions mentioned in it, and in the account of Posidonius's speculation as to the size and distance of the sun, which does, as a matter of fact, give results much nearer the truth than those obtained by Aristarchus, Hipparchus, and Ptolemy. Cleomedes observes (1) that by means of water-clocks it is found that the apparent diameter of the sun is 1/750th of the sun's circle, and that this method of measuring it is said to have been first invented by the Egyptians; (2) that Hipparchus is said to have found that the sun is 1,050 times the size of the earth, though, as regards this, we have the better authority of Adrastus (in Theon of Smyrna) and of Chalcidius, according to whom Hipparchus made the sun nearly 1,880 times the size of the earth (both figures refer of course to the solid content). We have already described Posidonius's method of arriving at the size and distance of the sun (pp. 220–1). After he has given this, Cleomedes, apparently deserting his guide, adds a calculation of his own relating to the sizes and distances of the moon and the sun which shows how little he was capable of any scientific inquiry.[1] Chap. 2 purports to prove that the sun is

[1] He says (pp. 146. 17–148. 27) that in an eclipse the breadth of the earth's shadow is stated to be two moon-breadths; hence, he says, it seems credible (πιθανόν) that the earth is twice the size of the moon (this practically assumes that the breadth of the earth's shadow is equal to the diameter of the earth, or that the cone of the earth's shadow is a cylinder!). Since then the circumference of the earth, according to Eratosthenes, is 250,000 stades, and its diameter therefore 'more than 80,000' (he evidently takes $\pi = 3$), the diameter of the moon will be 40,000 stades. Now, the moon's circle being 750 times the moon's diameter, the radius of the moon's circle, i.e. the distance of the moon from the earth, will be $\frac{1}{6}$th of this (i.e. $\pi = 3$) or 125 moon-diameters; therefore the moon's distance is 5,000,000 stades (which is much too great). Again, since the moon traverses its orbit 13 times to the sun's once, he assumes that the sun's orbit is 13 times as large as the moon's, and consequently that the diameter of the sun is 13 times that of the moon, or 520,000 stades and its distance 13 times 5,000,000 or 65,000,000 stades!

larger than the earth; and the remaining chapters deal with the size of the moon and the stars (chap. 3), the illumination of the moon by the sun (chap. 4), the phases of the moon and its conjunctions with the sun (chap. 5), the eclipses of the moon (chap. 6), the maximum deviation in latitude of the five planets (given as 5° for Venus, 4° for Mercury, $2\frac{1}{2}$° for Mars and Jupiter, 1° for Saturn), the maximum elongations of Mercury and Venus from the sun (20° and 50° respectively), and the synodic periods of the planets (Mercury 116 days, Venus 584 days, Mars 780 days, Jupiter 398 days, Saturn 378 days) (chap. 7).

There is only one other item of sufficient interest to be mentioned here. In Book II, chap. 6, Cleomedes mentions that there were stories of extraordinary eclipses which 'the more ancient of the mathematicians had vainly tried to explain'; the supposed 'paradoxical' case was that in which, while the sun seems to be still above the horizon, the *eclipsed* moon rises in the east. The passage has been cited above (vol. i, pp. 6–7), where I have also shown that Cleomedes himself gives the true explanation of the phenomenon, namely that it is due to atmospheric refraction.

The first and second centuries of the Christian era saw a continuation of the work of writing manuals or introductions to the different mathematical subjects. About A.D. 100 came NICOMACHUS, who wrote an *Introduction to Arithmetic* and an *Introduction to Harmony*; if we may judge by a remark of his own,[1] he would appear to have written an introduction to geometry also. The *Arithmetical Introduction* has been sufficiently described above (vol. i, pp. 97–112).

There is yet another handbook which needs to be mentioned separately, although we have had occasion to quote from it several times already. This is the book by THEON OF SMYRNA which goes by the title *Expositio rerum mathematicarum ad legendum Platonem utilium*. There are two main divisions of this work, contained in two Venice manuscripts respectively. The first was edited by Bullialdus (Paris, 1644), the second by T. H. Martin (Paris, 1849); the whole has been

[1] Nicom. *Arith.* ii. 6. 1.

edited by E. Hiller (Teubner, 1878) and finally, with a French translation, by J. Dupuis (Paris, 1892).

Theon's date is approximately fixed by two considerations. He is clearly the person whom Theon of Alexandria called 'the old Theon', τὸν παλαιὸν Θέωνα,[1] and there is no reason to doubt that he is the 'Theon the mathematician' (ὁ μαθηματικός) who is credited by Ptolemy with four observations of the planets Mercury and Venus made in A.D. 127, 129, 130 and 132.[2] The latest writers whom Theon himself mentions are Thrasyllus, who lived in the reign of Tiberius, and Adrastus the Peripatetic, who belongs to the middle of the second century A.D. Theon's work itself is a curious medley, valuable, not intrinsically, but for the numerous historical notices which it contains. The title, which claims that the book contains things useful for the study of Plato, must not be taken too seriously. It was no doubt an elementary *introduction* or vade-mecum for students of philosophy, but there is little in it which has special reference to the mathematical questions raised in Plato. The connexion consists mostly in the long proem quoting the views of Plato on the paramount importance of mathematics in the training of the philosopher, and the mutual relation of the five different branches, arithmetic, geometry, stereometry, astronomy and music. The want of care shown by Theon in the quotations from particular dialogues of Plato prepares us for the patchwork character of the whole book.

In the first chapter he promises to give the mathematical theorems most necessary for the student of Plato to know, in arithmetic, music, and geometry, with its application to stereometry and astronomy.[3] But the promise is by no means kept as regards geometry and stereometry: indeed, in a later passage Theon seems to excuse himself from including theoretical geometry in his plan, on the ground that all those who are likely to read his work or the writings of Plato may be assumed to have gone through an elementary course of theoretical geometry.[4] But he writes at length on figured

[1] Theon of Alexandria, *Comm. on Ptolemy's Syntaxis*, Basel edition, pp. 390, 395, 396.
[2] Ptolemy, *Syntaxis*, ix. 9, x. 1, 2.
[3] Theon of Smyrna, ed. Hiller, p. 1. 10–17.
[4] *Ib.*, p. 16. 17–20.

numbers, plane and solid, which are of course analogous to the corresponding geometrical figures, and he may have considered that he was in this way sufficiently fulfilling his promise with regard to geometry and stereometry. Certain geometrical definitions, of point, line, straight line, the three dimensions, rectilinear plane and solid figures, especially parallelograms and parallelepipedal figures including cubes, *plinthides* (square bricks) and δοκίδες (beams), and *scalene* figures with sides unequal every way (= βωμίσκοι in the classification of solid numbers), are dragged in later (chaps. 53–5 of the section on music)[1] in the middle of the discussion of proportions and means; if this passage is not an interpolation, it confirms the supposition that Theon included in his work only this limited amount of geometry and stereometry.

Section I is on Arithmetic in the same sense as Nicomachus's *Introduction*. At the beginning Theon observes that arithmetic will be followed by music. Of music in its three aspects, music in instruments (ἐν ὀργάνοις), music in numbers, i.e. musical intervals expressed in numbers or pure theoretical music, and the music or harmony in the universe, the first kind (instrumental music) is not exactly essential, but the other two must be discussed immediately after arithmetic.[2] The contents of the arithmetical section have been sufficiently indicated in the chapter on Pythagorean arithmetic (vol. i, pp. 112–13); it deals with the classification of numbers, odd, even, and their subdivisions, prime numbers, composite numbers with equal or unequal factors, plane numbers subdivided into square, oblong, triangular and polygonal numbers, with their respective 'gnomons' and their properties as the sum of successive terms of arithmetical progressions beginning with 1 as the first term, circular and spherical numbers, solid numbers with three factors, pyramidal numbers and truncated pyramidal numbers, perfect numbers with their correlatives, the over-perfect and the deficient; this is practically what we find in Nicomachus. But the special value of Theon's exposition lies in the fact that it contains an account of the famous 'side-' and 'diameter-' numbers of the Pythagoreans.[3]

[1] Theon of Smyrna, ed. Hiller, pp. 111–13. [2] *Ib.*, pp. 16. 24–17. 11.
[3] *Ib.*, pp. 42. 10–45. 9. Cf. vol. i, pp. 91–3.

In the Section on Music Theon says he will first speak of the two kinds of music, the audible or instrumental, and the intelligible or theoretical subsisting in numbers, after which he promises to deal lastly with ratio as predicable of mathematical entities in general and the ratio constituting the harmony in the universe, 'not scrupling to set out once again the things discovered by our predecessors, just as we have given the things handed down in former times by the Pythagoreans, with a view to making them better known, without ourselves claiming to have discovered any of them'.[1] Then follows a discussion of audible music, the intervals which give harmonies, &c., including substantial quotations from Thrasyllus and Adrastus, and references to views of Aristoxenus, Hippasus, Archytas, Eudoxus and Plato. With chap. 17 (p. 72) begins the account of the 'harmony in numbers', which turns into a general discussion of ratios, proportions and means, with more quotations from Plato, Eratosthenes and Thrasyllus, followed by Thrasyllus's *divisio canonis*, chaps. 35, 36 (pp. 87–93). After a promise to apply the latter division to the sphere of the universe, Theon purports to return to the subject of proportion and means. This, however, does not occur till chap. 50 (p. 106), the intervening chapters being taken up with a discussion of the δεκάς and τετρακτύς (with eleven applications of the latter) and the mystic or curious properties of the numbers from 2 to 10; here we have a part of the *theologumena* of arithmetic. The discussion of proportions and the different kinds of means after Eratosthenes and Adrastus is again interrupted by the insertion of the geometrical definitions already referred to (chaps. 53–5, pp. 111–13), after which Theon resumes the question of means for 'more precise' treatment.

The Section on Astronomy begins on p. 120 of Hiller's edition. Here again Theon is mainly dependent upon Adrastus, from whom he makes long quotations. Thus, on the sphericity of the earth, he says that for the necessary conspectus of the arguments it will be sufficient to refer to the grounds stated summarily by Adrastus. In explaining (p. 124) that the unevennesses in the surface of

[1] Theon of Smyrna, ed. Hiller, pp. 46. 20–47. 14.

the earth, represented e.g. by mountains, are negligible in comparison with the size of the whole, he quotes Eratosthenes and Dicaearchus as claiming to have discovered that the perpendicular height of the highest mountain above the normal level of the land is no more than 10 stades; and to obtain the diameter of the earth he uses Eratosthenes's figure of approximately 252,000 stades for the circumference of the earth, which, with the Archimedean value of $\frac{22}{7}$ for π, gives a diameter of about 80,182 stades. The principal astronomical circles in the heaven are next described (chaps. 5–12, pp. 129–35); then (chap. 12) the assumed maximum deviations in latitude are given, that of the sun being put at 1°, that of the moon and Venus at 12°, and those of the planets Mercury, Mars, Jupiter and Saturn at 8°, 5°, 5° and 3° respectively; the obliquity of the ecliptic is given as the side of a regular polygon of 15 sides described in a circle, i.e. as 24° (chap. 23, p. 151). Next the order of the orbits of the sun, moon and planets is explained (the system is of course geocentric); we are told (p. 138) that 'some of the Pythagoreans' made the order (reckoning outwards from the earth) to be moon, Mercury, Venus, sun, Mars, Jupiter, Saturn, whereas (p. 142) Eratosthenes put the sun next to the moon, and the mathematicians, agreeing with Eratosthenes in this, differed only in the order in which they placed Venus and Mercury after the sun, some putting Mercury next and some Venus (p. 143). The order adopted by 'some of the Pythagoreans' is the Chaldaean order, which was not followed by any Greek before Diogenes of Babylon (second century B.C.); 'some of the Pythagoreans' are therefore the later Pythagoreans (of whom Nicomachus was one); the other order, moon, sun, Venus, Mercury, Mars, Jupiter, Saturn, was that of Plato and the early Pythagoreans. In chap. 15 (p. 138 sq.) Theon quotes verses of Alexander 'the Aetolian' (not really the 'Aetolian', but Alexander of Ephesus, a contemporary of Cicero, or possibly Alexander of Miletus, as Chalcidius calls him) assigning to each of the planets (including the earth, though stationary) with the sun and moon and the sphere of the fixed stars one note, the intervals between the notes being so arranged as to bring the nine into an octave, whereas with Eratosthenes and Plato the earth was excluded, and the eight notes of the octachord were assigned

to the seven heavenly bodies and the sphere of the fixed stars. The whole of this passage (chaps. 15 to 16, pp. 138–47) is no doubt intended as the promised account of the 'harmony in the universe', although at the very end of the work Theon implies that this has still to be explained on the basis of Thrasyllus's exposition combined with what he has already given himself.

The next chapters deal with the forward movements, the stationary points, and the retrogradations, as they respectively appear to us, of the five planets, and the 'saving of the phenomena' by the alternative hypotheses of eccentric circles and epicycles (chaps. 17–30, pp. 147–78). These hypotheses are explained, and the identity of the motion produced by the two is shown by Adrastus in the case of the sun (chaps. 26, 27, pp. 166–72). The proof is introduced with the interesting remark that 'Hipparchus says it is worthy of investigation by mathematicians why, on two hypotheses so different from one another, that of eccentric circles and that of concentric circles with epicycles, the same results appear to follow'. It is not to be supposed that the proof of the identity could be other than easy to a mathematician like Hipparchus; the remark perhaps merely suggests that the two hypotheses were discovered quite independently, and it was not till later that the effect was discovered to be the same, when of course the fact would seem to be curious and a mathematical proof would immediately be sought. Another passage (p. 188) says that Hipparchus preferred the hypothesis of the epicycle, as being his own. If this means that Hipparchus claimed to have discovered the epicycle-hypothesis, it must be a misapprehension; for Apollonius already understood the theory of epicycles in all its generality. According to Theon, the epicycle-hypothesis is more 'according to nature'; but it was presumably preferred because it was applicable to all the planets, whereas the eccentric-hypothesis, when originally suggested, applied only to the three superior planets; in order to make it apply to the inferior planets it is necessary to suppose the circle described by the centre of the eccentric to be greater than the eccentric circle itself, which extension of the hypothesis, though known to Hipparchus, does not seem to have occurred to Apollonius.

We next have (chap. 31, p. 178) an allusion to the systems of Eudoxus, Callippus and Aristotle, and a description (p. 180 sq.) of a system in which the 'carrying' spheres (called 'hollow') have between them 'solid spheres which by their own motion will roll (ἀνελίξουσι) the carrying spheres in the opposite direction, being in contact with them'. These 'solid' spheres (which carry the planet fixed at a point on their surface) act in practically the same way as epicycles. In connexion with this description Theon (i.e. Adrastus) speaks (chap. 33, pp. 186–7) of two alternative hypotheses in which, by comparison with Chalcidius,[1] we recognize (after eliminating epicycles erroneously imported into both systems) the hypotheses of Plato and Heraclides respectively. It is this passage which enables us to conclude for certain that Heraclides made Venus and Mercury revolve in circles about the sun, like satellites, while the sun in its turn revolves in a circle about the earth as centre. Theon (p. 187) gives the maximum arcs separating Mercury and Venus respectively from the sun as 20° and 50°, these figures being the same as those given by Cleomedes.

The last chapters (chaps. 37–40), quoted from Adrastus, deal with conjunctions, transits, occultations and eclipses. The book concludes with a considerable extract from Dercyllides, a Platonist with Pythagorean leanings, who wrote (before the time of Tiberius and perhaps even before Varro) a book on Plato's philosophy. It is here (p. 198. 14) that we have the passage so often quoted from Eudemus:

'Eudemus relates in his Astronomy that it was Oenopides who first discovered the girdling of the zodiac and the revolution (or cycle) of the Great Year, that Thales was the first to discover the eclipse of the sun and the fact that the sun's period with respect to the solstices is not always the same, that Anaximander discovered that the earth is (suspended) on high and lies (substituting κεῖται for the reading of the manuscripts, κινεῖται, moves) about the centre of the universe, and that Anaximenes said that the moon has its light from the sun and (explained) how its eclipses come about' (Anaximenes is here apparently a mistake for Anaxagoras).

[1] Chalcidius, *Comm. on Timaeus*, c. 110. Cf. *Aristarchus of Samos*, pp. 256–8.

XVII

TRIGONOMETRY: HIPPARCHUS, MENELAUS, PTOLEMY

WE have seen that *Sphaeric*, the geometry of the sphere, was very early studied, because it was required so soon as astronomy became mathematical; with the Pythagoreans the word *Sphaeric*, applied to one of the subjects of the quadrivium, actually meant astronomy. The subject was so far advanced before Euclid's time that there was in existence a regular textbook containing the principal propositions about great and small circles on the sphere, from which both Autolycus and Euclid quoted the propositions as generally known. These propositions, with others of purely astronomical interest, were collected afterwards in a work entitled *Sphaerica*, in three Books, by THEODOSIUS.

Suidas has a notice, *s. v.* *Θεοδόσιος*, which evidently confuses the author of the *Sphaerica* with another Theodosius, a Sceptic philosopher, since it calls him 'Theodosius, a philosopher', and attributes to him, besides the mathematical works, 'Sceptic chapters' and a commentary on the chapters of Theudas. Now the commentator on Theudas must have belonged, at the earliest, to the second half of the second century A.D., whereas our Theodosius was earlier than Menelaus (*fl.* about A.D. 100), who quotes him by name. The next notice by Suidas is of yet another Theodosius, a poet, who came from Tripolis. Hence it was at one time supposed that our Theodosius was of Tripolis. But Vitruvius[1] mentions a Theodosius who invented a sundial 'for any climate'; and Strabo, in speaking of certain Bithynians distinguished in their particular sciences, refers to 'Hipparchus, Theodosius and his sons, mathematicians'[2]. We conclude that our Theo-

[1] *De architectura* ix. 9. [2] Strabo, xii. 4, 9, p. 566.

dosius was of Bithynia and not later in date than Vitruvius (say 20 B.C.); but the order in which Strabo gives the names makes it not unlikely that he was contemporary with Hipparchus, while the character of his *Sphaerica* suggests a date even earlier rather than later.

Works by Theodosius.

Two other works of Theodosius besides the *Sphaerica*, namely *On habitations* and *On Days and Nights*, seem to have been included in the 'Little Astronomy' (μικρὸς ἀστρονομούμενος, sc. τόπος). These two treatises need not detain us long. They are extant in Greek (in the great MS. Vaticanus Graecus 204 and others), but the Greek text has not apparently yet been published. In the first, *On habitations*, in 12 propositions, Theodosius explains the different phenomena due to the daily rotation of the earth, and the particular portions of the whole system which are visible to inhabitants of the different zones. In the second, *On Days and Nights*, containing 13 and 19 propositions in the two Books respectively, Theodosius considers the arc of the ecliptic described by the sun each day, with a view to determining the conditions to be satisfied in order that the solstice may occur in the meridian at a given place, and in order that the day and the night may really be equal at the equinoxes; he shows also that the variations in the day and night must recur exactly after a certain time, if the length of the solar year is commensurable with that of the day, while on the contrary assumption they will not recur so exactly.

In addition to the works bearing on astronomy, Theodosius is said[1] to have written a commentary, now lost, on the ἐφόδιον or *Method* of Archimedes (see above, pp. 27–34).

Contents of the *Sphaerica*.

We come now to the *Sphaerica*, which deserves a short description from the point of view of this chapter. A text-book on the geometry of the sphere was wanted as a supplement to the *Elements* of Euclid. In the *Elements* themselves

[1] Suidas, *loc. cit.*

(Books XII and XIII) Euclid included no general properties of the sphere except the theorem proved in XII. 16–18, that the volumes of two spheres are in the triplicate ratio of their diameters; apart from this, the sphere is only introduced in the propositions about the regular solids, where it is proved that they are severally inscribable in a sphere, and it was doubtless with a view to his proofs of this property in each case that he gave a new definition of a sphere as the figure described by the revolution of a semicircle about its diameter, instead of the more usual definition (after the manner of the definition of a circle) as the locus of all points (in space instead of in a plane) which are equidistant from a fixed point (the centre). No doubt the exclusion of the geometry of the sphere from the *Elements* was due to the fact that it was regarded as belonging to astronomy rather than pure geometry.

Theodosius defines the sphere as 'a solid figure contained by one surface such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another', which is exactly Euclid's definition of a circle with 'solid' inserted before 'figure' and 'surface' substituted for 'line'. The early part of the work is then generally developed on the lines of Euclid's Book III on the circle. Any plane section of a sphere is a circle (Prop. 1). The straight line from the centre of the sphere to the centre of a circular section is perpendicular to the plane of that section (1, Por. 2; cf. 7, 23); thus a plane section serves for finding the centre of the sphere just as a chord does for finding that of a circle (Prop. 2). The propositions about tangent planes (3–5) and the relation between the sizes of circular sections and their distances from the centre (5, 6) correspond to Euclid III. 16–19 and 15; as the small circle corresponds to any chord, the great circle ('greatest circle' in Greek) corresponds to the diameter. The poles of a circular section correspond to the extremities of the diameter bisecting a chord of a circle at right angles (Props. 8–10). Great circles bisecting one another (Props. 11–12) correspond to chords which bisect one another (diameters), and great circles bisecting small circles at right angles and passing through their poles (Props. 13–15) correspond to diameters bisecting chords at right angles. The distance of any point of a great

circle from its pole is equal to the side of a square inscribed in the great circle and conversely (Props. 16, 17). Next come certain problems: To find a straight line equal to the diameter of any circular section or of the sphere itself (Props. 18, 19); to draw the great circle through any two given points on the surface (Prop. 20); to find the pole of any given circular section (Prop. 21). Prop. 22 applies Eucl. III. 3 to the sphere.

Book II begins with a definition of circles on a sphere which touch one another; this happens 'when the common section of the planes (of the circles) touches both circles'. Another series of propositions follows, corresponding again to propositions in Eucl., Book III, for the circle. Parallel circular sections have the same poles, and conversely (Props. 1, 2). Props. 3–5 relate to circles on the sphere touching one another and therefore having their poles on a great circle which also passes through the point of contact (cf. Eucl. III. 11, [12] about circles touching one another). If a great circle touches a small circle, it also touches another small circle equal and parallel to it (Props. 6, 7), and if a great circle be obliquely inclined to another circular section, it touches each of two equal circles parallel to that section (Prop. 8). If two circles on a sphere cut one another, the great circle drawn through their poles bisects the intercepted segments of the circles (Prop. 9). If there are any number of parallel circles on a sphere, and any number of great circles drawn through their poles, the arcs of the parallel circles intercepted between any two of the great circles are similar, and the arcs of the great circles intercepted between any two of the parallel circles are equal (Prop. 10).

The last proposition forms a sort of transition to the portion of the treatise (II. 11–23 and Book III) which contains propositions of purely astronomical interest, though expressed as propositions in pure geometry without any specific reference to the various circles in the heavenly sphere. The propositions are long and complicated, and it would neither be easy nor worth while to attempt an enumeration. They deal with circles or parts of circles (arcs intercepted on one circle by series of other circles and the like). We have no difficulty in recognizing particular circles which come into many proposi-

tions. A particular small circle is the circle which is the limit of the stars which do not set, as seen by an observer at a particular place on the earth's surface; the pole of this circle is the pole in the heaven. A great circle which touches this circle and is obliquely inclined to the 'parallel circles' is the circle of the horizon; the parallel circles of course represent the apparent motion of the fixed stars in the diurnal rotation, and have the pole of the heaven as pole. A second great circle obliquely inclined to the parallel circles is of course the circle of the zodiac or ecliptic. The greatest of the 'parallel circles' is naturally the equator. All that need be said of the various propositions (except two which will be mentioned separately) is that the sort of result proved is like that of Props. 12 and 13 of Euclid's *Phaenomena* to the effect that in the half of the zodiac circle beginning with Cancer (or Capricornus) equal arcs set (or rise) in unequal times; those which are nearer the tropic circle take a longer time, those further from it a shorter; those which take the shortest time are those adjacent to the equinoctial points; those which are equidistant from the equator rise and set in equal times. In like manner Theodosius (III. 8) in effect takes equal and contiguous arcs of the ecliptic all on one side of the equator, draws through their extremities great circles touching the circumpolar 'parallel' circle, and proves that the corresponding arcs of the equator intercepted between the latter great circles are unequal and that, of the said arcs, that corresponding to the arc of the ecliptic which is nearer the tropic circle is the greater. The successive great circles touching the circumpolar circle are of course successive positions of the horizon as the earth revolves about its axis, that is to say, the same length of arc on the ecliptic takes a longer or shorter time to rise according as it is nearer to or farther from the tropic, in other words, farther from or nearer to the equinoctial points.

It is, however, obvious that investigations of this kind, which only prove that certain arcs are greater than others, and do not give the actual numerical ratios between them, are useless for any practical purpose such as that of telling the hour of the night by the stars, which was one of the fundamental problems in Greek astronomy; and in order to find

the required numerical ratios a new method had to be invented, namely trigonometry.

No actual trigonometry in Theodosius.

It is perhaps hardly correct to say that spherical triangles are nowhere referred to in Theodosius, for in III. 3 the congruence-theorem for spherical triangles corresponding to Eucl. I. 4 is practically proved; but there is nothing in the book that can be called trigonometrical. The nearest approach is in III. 11, 12, where ratios between certain straight lines are compared with ratios between arcs. ACc (Prop. 11) is a great circle through the poles A, A'; CDc, $C'D$ are two other great circles, both of which are at right angles to the plane of $AC'c$, but CDc is perpendicular to AA', while $C'D$ is inclined to it at an acute angle. Let any other great circle $AB'BA'$ through AA' cut CD in any point B between C and D, and $C'D$ in B'. Let the 'parallel' circle $EB'e$ be drawn through B', and let $C'c'$ be the diameter of the 'parallel' circle touching the great circle $C'D$. Let L, K be the centres of the 'parallel' circles, and let R, ρ be the radii of the 'parallel' circles CDc, $C'c'$ respectively. It is required to prove that

$$2R : 2\rho > (\text{arc } CB) : (\text{arc } C'B').$$

Let $C'O$, Ee meet in N, and join NB'.

Then $B'N$, being the intersection of two planes perpendicular to the plane of $AC'CA'$, is perpendicular to that plane and therefore to both Ee and $C'O$.

Now, the triangle NLO being right-angled at L, $NO > NL$.

Measure NT along NO equal to NL, and join TB'.

Then in the triangles $B'NT$, $B'NL$ two sides $B'N$, NT are equal to two sides $B'N$, NL, and the included angles (both being right) are equal; therefore the triangles are equal in all respects, and $\angle NLB' = \angle NTB'$.

Now
$$\begin{aligned} 2R : 2\rho &= OC' : C'K \\ &= ON : NL \\ &= ON : NT \\ [&= \tan NTB' : \tan NOB'] \\ &> \angle NTB' : \angle NOB' \\ &> \angle NLB' : \angle NOB' \\ &> \angle COB : \angle NOB' \\ &> (\text{arc } BC) : (\text{arc } B'C'). \end{aligned}$$

If a', b', c' are the sides of the spherical triangle $AB'C'$, this result is equivalent (since the angle COB subtended by the arc CB is equal to A) to

$$\begin{aligned} 1 : \sin b' &= \tan A : \tan a' \\ &> a : a', \end{aligned}$$

where $a = BC$, the side opposite A in the triangle ABC.

The proof is based on the fact (proved in Euclid's *Optics* and assumed as known by Aristarchus of Samos and Archimedes) that, if α, β are angles such that $\frac{1}{2}\pi > \alpha > \beta$, $\tan\alpha/\tan\beta > \alpha/\beta$.

While, therefore, Theodosius proves the equivalent of the formula, applicable in the solution of a spherical triangle right-angled at C, that $\tan a = \sin b \tan A$, he is unable, for want of trigonometry, to find the actual value of a/a', and can only find a limit for it. He is exactly in the same position as Aristarchus, who can only approximate to the values of the trigonometrical ratios which he needs, e.g. $\sin 1°$, $\cos 1°$, $\sin 3°$, by bringing them within upper and lower limits with the aid of the inequalities

$$\frac{\tan\alpha}{\tan\beta} > \frac{\alpha}{\beta} > \frac{\sin\alpha}{\sin\beta},$$

where $\frac{1}{2}\pi > \alpha > \beta$.

We may contrast with this proposition of Theodosius the corresponding proposition in Menelaus's *Sphaerica* (III. 15) dealing with the more general case in which C', instead of being the tropical point on the ecliptic, is, like B', any point between the tropical point and D. If R, ρ have the same meaning as above and r_1, r_2 are the radii of the parallel circles through B' and the new C', Menelaus proves that

$$\frac{\sin a}{\sin a'} = \frac{R\rho}{r_1 r_2},$$

which, of course, with the aid of Tables, gives the means of finding the actual values of a or a' when the other elements are given.

The proposition III. 12 of Theodosius proves a result similar to that of III. 11 for the case where the great circles $AB'B$, $AC'C$, instead of being great circles through the poles, are great circles touching 'the circle of the always-visible stars', i.e. different positions of the horizon, and the points C', B' are any points on the arc of the oblique circle between the tropical and the equinoctial points; in this case, with the same notation, $4R : 2\rho >$ (arc BC) : (arc $B'C'$).

It is evident that Theodosius was simply a laborious compiler, and that there was practically nothing original in his work. It has been proved, by means of propositions quoted *verbatim* or assumed as known by Autolycus in his *Moving Sphere* and by Euclid in his *Phaenomena*, that the following propositions in Theodosius are pre-Euclidean, I. **1**, 6 a, **7**, **8**, **11**, **12**, **13**, **15**, 20; II. 1, **2**, 3, **5**, 8, **9**, **10 a**, **13**, **15**, **17**, **18**, **19**, 20, **22**; III. **1 b**, 2, **3**, 7, **8**, those shown in thick type being quoted word for word.

The beginnings of trigonometry.

But this is not all. In Menelaus's *Sphaerica*, III. 15, there is a reference to the proposition (III. 11) of Theodosius proved above, and in Gherard of Cremona's translation from the Arabic, as well as in Halley's translation from the Hebrew of Jacob b. Machir, there is an addition to the effect that this proposition was used by Apollonius in a book the title of which is given in the two translations in the alternative

forms '*liber aggregativus*' and 'liber de principiis universalibus'. Each of these expressions may well mean the work of Apollonius which Marinus refers to as the 'General Treatise' (*ἡ καθόλου πραγματεία*). There is no apparent reason to doubt that the remark in question was really contained in Menelaus's original work; and, even if it is an Arabian interpolation, it is not likely to have been made without some definite authority. If then Apollonius was the discoverer of the proposition, the fact affords some ground for thinking that the beginnings of trigonometry go as far back, at least, as Apollonius. Tannery[1] indeed suggested that not only Apollonius but Archimedes before him may have compiled a 'table of chords', or at least shown the way to such a compilation, Archimedes in the work of which we possess only a fragment in the *Measurement of a Circle*, and Apollonius in the *ὠκυτόκιον*, where he gave an approximation to the value of π closer than that obtained by Archimedes; Tannery compares the Indian Table of Sines in the *Sūrya-Siddhānta*, where the angles go by 24ths of a right angle ($1/24\text{th}=3°\ 45'$, $2/24\text{ths}=7°\ 30'$, &c.), as possibly showing Greek influence. This is, however, in the region of conjecture; the first person to make systematic use of trigonometry is, so far as we know, Hipparchus.

Hipparchus, the greatest astronomer of antiquity, was born at Nicaea in Bithynia. The period of his activity is indicated by references in Ptolemy to observations made by him the limits of which are from 161 B.C. to 126 B.C. Ptolemy further says that from Hipparchus's time to the beginning of the reign of Antoninus Pius (A.D. 138) was 265 years.[2] The best and most important observations made by Hipparchus were made at Rhodes, though an observation of the vernal equinox at Alexandria on March 24, 146 B.C., recorded by him may have been his own. His main contributions to theoretical and practical astronomy can here only be indicated in the briefest manner.

[1] Tannery, *Recherches sur l'hist. de l'astronomie ancienne*, p. 64.
[2] Ptolemy, *Syntaxis*, vii. 2 (vol. ii, p. 15).

The work of Hipparchus.

Discovery of precession.

1. The greatest is perhaps his discovery of the precession of the equinoxes. Hipparchus found that the bright star Spica was, at the time of his observation of it, 6° distant from the autumnal equinoctial point, whereas he deduced from observations recorded by Timocharis that Timocharis had made the distance 8°. Consequently the motion had amounted to 2° in the period between Timocharis's observations, made in 283 or 295 B.C., and 129/8 B.C., a period, that is, of 154 or 166 years; this gives about 46·8″ or 43·4″ a year, as compared with the true value of 50·3757″.

Calculation of mean lunar month.

2. The same discovery is presupposed in his work *On the length of the Year*, in which, by comparing an observation of the summer solstice by Aristarchus in 281/0 B.C. with his own in 136/5 B.C., he found that after 145 years (the interval between the two dates) the summer solstice occurred half a day-and-night earlier than it should on the assumption of exactly 365¼ days to the year; hence he concluded that the *tropical* year contained about $\frac{1}{300}$th of a day-and-night less than 365¼ days. This agrees very nearly with Censorinus's statement that Hipparchus's cycle was 304 years, four times the 76 years of Callippus, but with 111,035 days in it instead of 111,036 ($=27{,}759 \times 4$). Counting in the 304 years $12 \times 304 + 112$ (intercalary) months, or 3,760 months in all, Hipparchus made the mean lunar month 29 days 12 hrs. 44 min. 2½ sec., which is less than a second out in comparison with the present accepted figure of 29·53059 days!

3. Hipparchus attempted a new determination of the sun's motion by means of exact equinoctial and solstitial observations; he reckoned the eccentricity of the sun's course and fixed the apogee at the point 5° 30′ of *Gemini*. More remarkable still was his investigation of the moon's course. He determined the eccentricity and the inclination of the orbit to the ecliptic, and by means of records of observations of eclipses determined the moon's period with extraordinary accuracy (as remarked above). We now learn

that the lengths of the mean synodic, the sidereal, the anomalistic and the draconitic month obtained by Hipparchus agree exactly with Babylonian cuneiform tables of date not later than Hipparchus, and it is clear that Hipparchus was in full possession of all the results established by Babylonian astronomy.

Improved estimates of sizes and distances of sun and moon.

4. Hipparchus improved on Aristarchus's calculations of the sizes and distances of the sun and moon, determining the apparent diameters more exactly and noting the changes in them; he made the mean distance of the sun 1,245 D, the mean distance of the moon $33\frac{2}{3}$ D, the diameters of the sun and moon $12\frac{1}{3}$ D and $\frac{1}{3}$ D respectively, where D is the mean diameter of the earth.

Epicycles and eccentrics.

5. Hipparchus, in investigating the motions of the sun, moon and planets, proceeded on the alternative hypotheses of epicycles and eccentrics; he did not invent these hypotheses, which were already fully understood and discussed by Apollonius. While the motions of the sun and moon could with difficulty be accounted for by the simple epicycle and eccentric hypotheses, Hipparchus found that for the planets it was necessary to combine the two, i.e. to superadd epicycles to motion in eccentric circles.

Catalogue of stars.

6. He compiled a catalogue of fixed stars including 850 or more such stars; apparently he was the first to state their positions in terms of coordinates in relation to the ecliptic (latitude and longitude), and his table distinguished the apparent sizes of the stars. His work was continued by Ptolemy, who produced a catalogue of 1,022 stars which, owing to an error in his solar tables affecting all his longitudes, has by many erroneously been supposed to be a mere reproduction of Hipparchus's catalogue. That Ptolemy took many observations himself seems certain.[1]

[1] See two papers by Dr. J. L. E. Dreyer in the *Monthly Notices of the Royal Astronomical Society*, 1917, pp. 528-39, and 1918, pp. 343-9.

Improved Instruments.

7. He made great improvements in the instruments used for observations. Among those which he used were an improved dioptra, a 'meridian-instrument' designed for observations in the meridian only, and a universal instrument (ἀστρολάβον ὄργανον) for more general use. He also made a globe on which he showed the positions of the fixed stars as determined by him; it appears that he showed a larger number of stars on his globe than in his catalogue.

Geography.

In geography Hipparchus wrote a criticism of Eratosthenes, in great part unfair. He checked Eratosthenes's data by means of a sort of triangulation; he insisted on the necessity of applying astronomy to geography, of fixing the position of places by latitude and longitude, and of determining longitudes by observations of lunar eclipses.

Outside the domain of astronomy and geography, Hipparchus wrote a book *On things borne down by their weight* from which Simplicius (on Aristotle's *De caelo*, p. 264 sq.) quotes two propositions. It is possible, however, that even in this work Hipparchus may have applied his doctrine to the case of the heavenly bodies.

In pure mathematics he is said to have considered a problem in permutations and combinations, the problem of finding the number of different possible combinations of 10 axioms or assumptions, which he made to be 103,049 (*v.l.* 101,049) or 310,952 according as the axioms were affirmed or denied[1]: it seems impossible to make anything of these figures. When the *Fihrist* attributes to him works 'On the art of algebra, known by the title of the Rules' and 'On the division of numbers', we have no confirmation: Suter suspects some confusion, in view of the fact that the article immediately following in the *Fihrist* is on Diophantus, who also 'wrote on the art of algebra'.

[1] Plutarch, *Quaest. Conviv.* viii. 9. 3, 732 F, *De Stoicorum repugn.* 29. 1047 D.

First systematic use of Trigonometry.

We come now to what is the most important from the point of view of this work, Hipparchus's share in the development of trigonometry. Even if he did not invent it, Hipparchus is the first person of whose systematic use of trigonometry we have documentary evidence. (1) Theon of Alexandria says on the *Syntaxis* of Ptolemy, à propos of Ptolemy's Table of Chords in a circle (equivalent to sines), that Hipparchus, too, wrote a treatise in twelve books on straight lines (i.e. chords) in a circle, while another in six books was written by Menelaus.[1] In the *Syntaxis* I. 10 Ptolemy gives the necessary explanations as to the notation used in his Table. The circumference of the circle is divided into 360 parts or degrees; the diameter is also divided into 120 parts, and one of such parts is the unit of length in terms of which the length of each chord is expressed; each part, whether of the circumference or diameter, is divided into 60 parts, each of these again into 60, and so on, according to the system of sexagesimal fractions. Ptolemy then sets out the minimum number of propositions in plane geometry upon which the calculation of the chords in the Table is based (*διὰ τῆς ἐκ τῶν γραμμῶν μεθοδικῆς αὐτῶν συστάσεως*). The propositions are famous, and it cannot be doubted that Hipparchus used a set of propositions of the same kind, though his exposition probably ran to much greater length. As Ptolemy definitely set himself to give the necessary propositions in the shortest form possible, it will be better to give them under Ptolemy rather than here. (2) Pappus, in speaking of Euclid's propositions about the inequality of the times which equal arcs of the zodiac take to rise, observes that 'Hipparchus in his book *On the rising of the twelve signs of the zodiac* shows *by means of numerical calculations* (*δι' ἀριθμῶν*) that equal arcs of the semicircle beginning with Cancer which set in times having a certain relation to one another do not everywhere show the same relation between the times in which they rise',[2] and so on. We have seen that Euclid, Autolycus, and even Theodosius could only prove that the said times are greater or less

[1] Theon, *Comm. on Syntaxis*, p. 110, ed. Halma.
[2] Pappus, vi, p. 600. 9-13.

in relation to one another; they could not calculate the actual times. As Hipparchus proved corresponding propositions by means of *numbers*, we can only conclude that he used propositions in spherical trigonometry, calculating arcs from others which are given, by means of tables. (3) In the only work of his which survives, the *Commentary on the Phaenomena of Eudoxus and Aratus* (an early work anterior to the discovery of the precession of the equinoxes), Hipparchus states that (presumably in the latitude of Rhodes) a star which lies $27\frac{1}{3}°$ north of the equator describes above the horizon an arc containing 3 minutes less than 15/24ths of the whole circle[1]; then, after some more inferences, he says, 'For each of the aforesaid facts is proved *by means of lines* (*διὰ τῶν γραμμῶν*) in the general treatises on these matters compiled by me'. In other places[2] of the *Commentary* he alludes to a work *On simultaneous risings* (*τὰ περὶ τῶν συνανατολῶν*), and in II. 4. 2 he says he will state summarily, about each of the fixed stars, along with what sign of the zodiac it rises and sets and from which degree to which degree of each sign it rises or sets in the regions about Greece or wherever the longest day is $14\frac{1}{2}$ equinoctial hours, adding that he has given special proofs in another work designed so that it is possible in practically every place in the inhabited earth to follow the differences between the concurrent risings and settings.[3] Where Hipparchus speaks of proofs 'by means of lines', he does not mean a merely graphical method, by construction only, but theoretical determination by geometry, followed by calculation, just as Ptolemy uses the expression *ἐκ τῶν γραμμῶν* of his calculation of chords and the expressions *σφαιρικαὶ δείξεις* and *γραμμικαὶ δείξεις* of the fundamental proposition in spherical trigonometry (Menelaus's theorem applied to the sphere) and its various applications to particular cases. It is significant that in the *Syntaxis* VIII. 5, where Ptolemy applies the proposition to the very problem of finding the times of concurrent rising, culmination and setting of the fixed stars, he says that the times can be obtained 'by lines only' (*διὰ μόνων τῶν γραμμῶν*).[4] Hence we may be certain that, in the other books of his own to which Hipparchus refers

[1] Ed. Manitius, pp. 148–50.
[2] *Ib.*, pp. 128. 5, 148. 20.
[3] *Ib.*, pp. 182. 19–184. 5.
[4] *Syntaxis*, vol. ii, p. 193.

in his *Commentary*, he used the formulae of spherical trigonometry to get his results. In the particular case where it is required to find the time in which a star of $27\frac{1}{3}°$ northern declination describes, in the latitude of Rhodes, the portion of its arc above the horizon, Hipparchus must have used the equivalent of the formula in the solution of a right-angled spherical triangle, $\tan b = \cos A \tan c$, where C is the right angle. Whether, like Ptolemy, Hipparchus obtained the formulae, such as this one, which he used from different applications of the one general theorem (Menelaus's theorem) it is not possible to say. There was of course no difficulty in calculating the tangent or other trigonometrical function of an angle if only a table of sines was given; for Hipparchus and Ptolemy were both aware of the fact expressed by $\sin^2 \alpha + \cos^2 \alpha = 1$ or, as they would have written it,

$$(\text{crd. } 2\alpha)^2 + \{\text{crd. } (180° - 2\alpha)\}^2 = 4r^2,$$

where (crd. 2α) means the chord subtending an arc 2α, and r is the radius, of the circle of reference.

Table of Chords.

We have no details of Hipparchus's Table of Chords sufficient to enable us to compare it with Ptolemy's, which goes by half-degrees, beginning with angles of $\frac{1}{2}°$, $1°$, $1\frac{1}{2}°$, and so on. But Heron[1] in his *Metrica* says that 'it is proved in the books about chords in a circle' that, if a_9 and a_{11} are the sides of a regular enneagon (9-sided figure) and hendecagon (11-sided figure) inscribed in a circle of diameter d, then (1) $a_9 = \frac{1}{3}d$, (2) $a_{11} = \frac{7}{25}d$ very nearly, which means that sin 20° was taken as equal to 0·3333 ... (Ptolemy's table makes it $\frac{1}{60}\left(20 + \frac{31}{60} + \frac{16\frac{1}{2}}{60^2}\right)$, so that the first approximation is $\frac{1}{3}$), and $\sin \frac{1}{11} . 180°$ or sin 16° 21′ 49″ was made equal to 0·28 (this corresponds to the chord subtending an angle of 32° 43′ 38″, nearly half-way between $32\frac{1}{2}°$ and 33°, and the mean between the two chords subtending the latter angles gives $\frac{1}{60}\left(16 + \frac{54}{60} + \frac{55}{60^2}\right)$ as the required sine, while $\frac{1}{60}(16\frac{9}{10}) = \frac{169}{600}$, which only differs

[1] Heron, *Metrica*, i. 22, 24, pp. 58. 19 and 62. 17.

by $\frac{1}{600}$ from $\frac{188}{600}$ or $\frac{7}{25}$, Heron's figure). There is little doubt that it is to Hipparchus's work that Heron refers, though the author is not mentioned.

While for our knowledge of Hipparchus's trigonometry we have to rely for the most part upon what we can infer from Ptolemy, we fortunately possess an original source of information about Greek trigonometry in its highest development in the *Sphaerica* of Menelaus.

The date of MENELAUS of Alexandria is roughly indicated by the fact that Ptolemy quotes an observation of his made in the first year of Trajan's reign (A.D. 98). He was therefore a contemporary of Plutarch, who in fact represents him as being present at the dialogue *De facie in orbe lunae*, where (chap. 17) Lucius apologizes to Menelaus 'the mathematician' for questioning the fundamental proposition in optics that the angles of incidence and reflection are equal.

He wrote a variety of treatises other than the *Sphaerica*. We have seen that Theon mentions his work on *Chords in a Circle* in six Books. Pappus says that he wrote a treatise (πραγματεία) on the setting (or perhaps only rising) of different arcs of the zodiac.[1] Proclus quotes an alternative proof by him of Eucl. I. 25, which is direct instead of by *reductio ad absurdum*,[2] and he would seem to have avoided the latter kind of proof throughout. Again, Pappus, speaking of the many complicated curves 'discovered by Demetrius of Alexandria (in his "Linear considerations") and by Philon of Tyana as the result of interweaving plectoids and other surfaces of all kinds', says that one curve in particular was investigated by Menelaus and called by him 'paradoxical' (παράδοξος)[3]; the nature of this curve can only be conjectured (see below).

But Arabian tradition refers to other works by Menelaus, (1) *Elements of Geometry*, edited by Thābit b. Qurra, in three Books, (2) a Book on triangles, and (3) a work the title of which is translated by Wenrich *de cognitione quantitatis discretae corporum permixtorum*. Light is thrown on this last title by one al-Chāzinī who (about A.D. 1121) wrote a

[1] Pappus, vi, pp. 600–2.
[2] Proclus on Eucl. I, pp. 345. 14–346. 11.
[3] Pappus, iv, p. 270. 25.

treatise about the hydrostatic balance, i.e. about the determination of the specific gravity of homogeneous or mixed bodies, in the course of which he mentions Archimedes and Menelaus (among others) as authorities on the subject; hence the treatise (3) must have been a book on hydrostatics discussing such problems as that of the crown solved by Archimedes. The alternative proof of Eucl. I. 25 quoted by Proclus might have come either from the *Elements of Geometry* or the Book on triangles. With regard to the geometry, the 'liber trium fratrum' (written by three sons of Mūsā b. Shākir in the ninth century) says that it contained a solution of the duplication of the cube, which is none other than that of Archytas. The solution of Archytas having employed the intersection of a tore and a cylinder (with a cone as well), there would, on the assumption that Menelaus reproduced the solution, be a certain appropriateness in the suggestion of Tannery[1] that the curve which Menelaus called the παράδοξος γραμμή was in reality the curve of double curvature, known by the name of Viviani, which is the intersection of a sphere with a cylinder touching it internally and having for its diameter the radius of the sphere. This curve is a particular case of Eudoxus's *hippopede*, and it has the property that the portion left outside the curve of the surface of the hemisphere on which it lies is equal to the square on the diameter of the sphere; the fact of the said area being squareable would justify the application of the word παράδοξος to the curve, and the quadrature itself would not probably be beyond the powers of the Greek mathematicians, as witness Pappus's determination of the area cut off between a complete turn of a certain spiral on a sphere and the great circle touching it at the origin.[2]

The *Sphaerica* of Menelaus.

This treatise in three Books is fortunately preserved in the Arabic, and although the extant versions differ considerably in form, the substance is beyond doubt genuine; the original translator was apparently Isḥāq b. Ḥunain (died A.D. 910). There have been two editions, (1) a Latin

[1] Tannery, *Mémoires scientifiques*, ii, p. 17. [2] Pappus, iv, pp. 264–8.

translation by Maurolycus (Messina, 1558) and (2) Halley's edition (Oxford, 1758). The former is unserviceable because Maurolycus's manuscript was very imperfect, and, besides trying to correct and restore the propositions, he added several of his own. Halley seems to have made a free translation of the Hebrew version of the work by Jacob b. Machir (about 1273), although he consulted Arabic manuscripts to some extent, following them, e.g., in dividing the work into three Books instead of two. But an earlier version direct from the Arabic is available in manuscripts of the thirteenth to fifteenth centuries at Paris and elsewhere; this version is without doubt that made by the famous translator Gherard of Cremona (1114–87). With the help of Halley's edition, Gherard's translation, and a Leyden manuscript (930) of the redaction of the work by Abū-Nasr-Manṣūr made in A.D. 1007–8, Björnbo has succeeded in presenting an adequate reproduction of the contents of the *Sphaerica*.[1]

Book I.

In this Book for the first time we have the conception and definition of a *spherical triangle*. Menelaus does not trouble to give the usual definitions of points and circles related to the sphere, e.g. pole, great circle, small circle, but begins with that of a spherical triangle as 'the area included by arcs of great circles on the surface of a sphere', subject to the restriction (Def. 2) that each of the sides or legs of the triangle is an arc less than a semicircle. The angles of the triangle are the angles contained by the arcs of great circles on the sphere (Def. 3), and one such angle is equal to or greater than another according as the planes containing the arcs forming the first angle are inclined at the same angle as, or a greater angle than, the planes of the arcs forming the other (Defs. 4, 5). The angle is a right angle if the planes of the arcs are at right angles (Def. 6). Pappus tells us that Menelaus in his *Sphaerica* calls the figure in question (the spherical triangle) a 'three-side' (τρίπλευρον)[2]; the word *triangle* (τρίγωνον) was of course

[1] Björnbo, *Studien über Menelaos' Sphärik* (Abhandlungen zur Gesch. d. math. Wissenschaften, Heft xiv. 1902).
[2] Pappus, vi, p. 476. 16.

already appropriated for the plane triangle. We should gather from this, as well as from the restriction of the definitions to the spherical triangle and its parts, that the discussion of the spherical triangle as such was probably new; and if the preface in the Arabic version addressed to a prince and beginning with the words, 'O prince! I have discovered an excellent method of proof...' is genuine, we have confirmatory evidence in the writer's own claim.

Menelaus's object, so far as Book I is concerned, seems to have been to give the main propositions about spherical triangles corresponding to Euclid's propositions about plane triangles. At the same time he does not restrict himself to Euclid's methods of proof even where they could be adapted to the case of the sphere; he avoids the form of proof by *reductio ad absurdum*, but, subject to this, he prefers the easiest proofs. In some respects his treatment is more complete than Euclid's treatment of the analogous plane cases. In the congruence-theorems, for example, we have I. 4a corresponding to Eucl. I. 4, I. 4b to Eucl. I. 8, I. 14, 16 to Eucl. I. 26a, b; but Menelaus includes (I. 13) what we know as the 'ambiguous case', which is enunciated on the lines of Eucl. VI. 7. I. 12 is a particular case of I. 16. Menelaus includes also the further case which has no analogue in plane triangles, that in which the three angles of one triangle are severally equal to the three angles of the other (I. 17). He makes, moreover, no distinction between the congruent and the symmetrical, regarding both as covered by congruent. I. 1 is a problem, to construct a spherical angle equal to a given spherical angle, introduced only as a lemma because required in later propositions. I. 2, 3 are the propositions about isosceles triangles corresponding to Eucl. I. 5, 6; Eucl. I. 18, 19 (greater side opposite greater angle and vice versa) have their analogues in I. 7, 9, and Eucl. I. 24, 25 (two sides respectively equal and included angle, or third side, in one triangle greater than included angle, or third side, in the other) in I. 8. I. 5 (two sides of a triangle together greater than the third) corresponds to Eucl. I. 20. There is yet a further group of propositions comparing parts of spherical triangles, I. 6, 18, 19, where I. 6 (corresponding to Eucl. I. 21) is deduced from I. 5, just as the first part of Eucl. I. 21 is deduced from Eucl. I. 20.

Eucl. I. 16, 32 are not true of spherical triangles, and Menelaus has therefore the corresponding but different propositions. I. 10 proves that, with the usual notation a, b, c, A, B, C, for the sides and opposite angles of a spherical triangle, the exterior angle at C, or $180°-C$, $<=$ or $>A$ according as $c+a >=$ or $< 180°$, and vice versa. The proof of this and the next proposition shall be given as specimens.

In the triangle ABC suppose that $c+a >=$ or $< 180°$; let D be the pole opposite to A.

Then, according as $c+a >=$ or $< 180°$, $BC >=$ or $< BD$ (since $AD = 180°$),

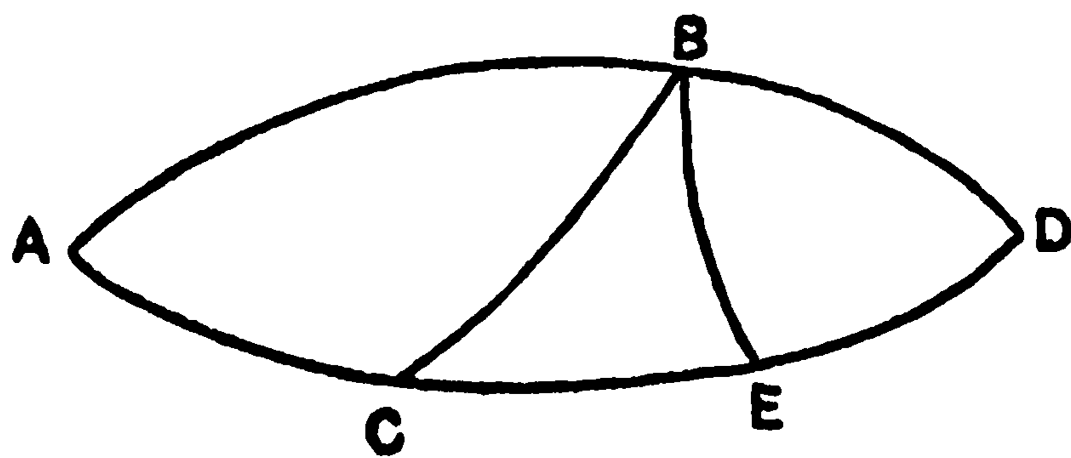
and therefore $\angle D >=$ or $< \angle BCD\,(= 180°-C)$, [I. 9]

i.e. (since $\angle D = \angle A$) $180°-C <=$ or $> A$.

Menelaus takes the converse for granted.

As a consequence of this, I. 11 proves that $A+B+C > 180°$.

Take the same triangle ABC, with the pole D opposite



to A, and from B draw the great circle BE such that $\angle DBE = \angle BDE$.

Then $CE+EB = CD < 180°$, so that, by the preceding proposition, the exterior angle ACB to the triangle BCE is greater than $\angle CBE$,

i.e. $C > \angle CBE$.

Add A or D $(= \angle EBD)$ to the unequals;

therefore $C+A > \angle CBD$,

whence $A+B+C > \angle CBD+B$ or $180°$.

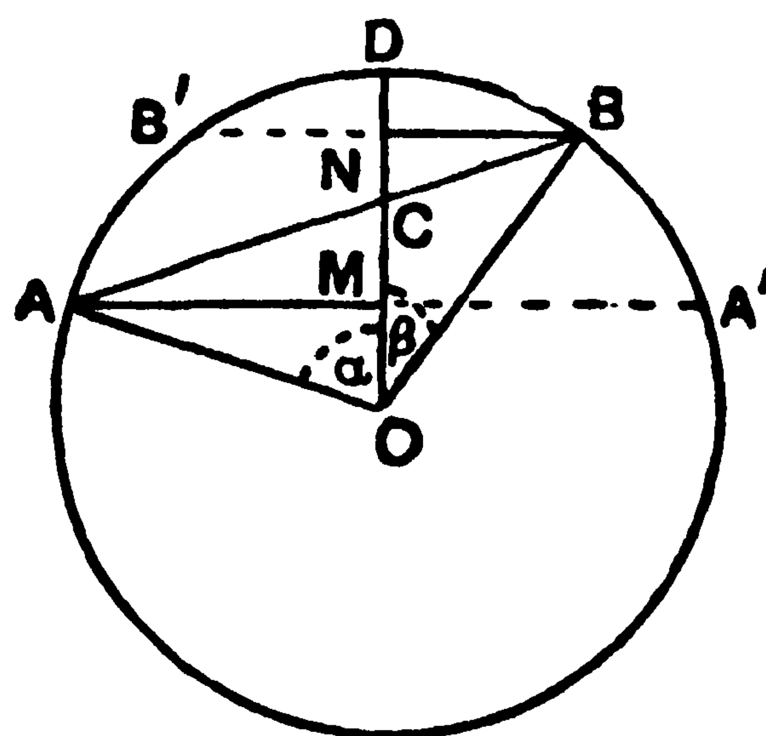
After two lemmas I. 21, 22 we have some propositions introducing M, N, P the middle points of a, b, c respectively. I. 23 proves, e.g., that the arc MN of a great circle $> \frac{1}{2}c$, and I. 20 that $AM <=$ or $> \frac{1}{2}a$ according as $A >=$ or $< (B+C)$. The last group of propositions, 26–35, relate to the figure formed

by the triangle ABC with great circles drawn through B to meet AC (between A and C) in D, E respectively, and the case where D and E coincide, and they prove different results arising from different relations between a and c ($a > c$), combined with the equality of AD and EC (or DC), of the angles ABD and EBC (or DBC), or of $a+c$ and $BD+BE$ (or $2BD$) respectively, according as $a+c <= $ or $> 180°$.

Book II has practically no interest for us. The object of it is to establish certain propositions, of astronomical interest only, which are nothing more than generalizations or extensions of propositions in Theodosius's *Sphaerica*, Book III. Thus Theodosius III. 5, 6, 9 are included in Menelaus II. 10, Theodosius III. 7–8 in Menelaus II. 12, while Menelaus II. 11 is an extension of Theodosius III. 13. The proofs are quite different from those of Theodosius, which are generally very long-winded.

Book III. Trigonometry.

It will have been noticed that, while Book I of Menelaus gives the geometry of the spherical triangle, neither Book I nor Book II contains any trigonometry. This is reserved for Book III. As I shall throughout express the various results obtained in terms of the trigonometrical ratios, sine, cosine, tangent, it is necessary to explain once for all that the Greeks did not use this terminology, but, instead of sines, they used the chords subtended by arcs of a circle. In the accompanying figure let the arc AD of a circle subtend an angle α at the centre O. Draw AM perpendicular to OD, and produce it to meet the circle again in A'. Then $\sin\alpha = AM/AO$, and AM is $\frac{1}{2}AA'$ or half the chord subtended by an angle 2α at the centre, which may shortly be denoted by $\frac{1}{2}$(crd. 2α).



Since Ptolemy expresses the chords as so many 120th parts of the diameter of the circle, while $AM/AO = AA'/2AO$, it follows that $\sin\alpha$ and $\frac{1}{2}$(crd. 2α) are equivalent. Cos α is of course $\sin(90°-\alpha)$ and is therefore equivalent to $\frac{1}{2}$ crd. $(180°-2\alpha)$.

(*a*) *'Menelaus's theorem' for the sphere.*

The first proposition of Book III is the famous 'Menelaus's theorem' with reference to a spherical triangle and any transversal (great circle) cutting the sides of a triangle, produced if necessary. Menelaus does not, however, use a spherical triangle in his enunciation, but enunciates the proposition in terms of intersecting great circles. 'Between two arcs ADB, AEC of great circles are two other arcs of great circles DFC and BFE which intersect them and also intersect each other in F. All the arcs are less than a semicircle. It is required to prove that

$$\frac{\sin CE}{\sin EA} = \frac{\sin CF}{\sin FD} \cdot \frac{\sin DB}{\sin BA}\text{'}.$$

It appears that Menelaus gave three or four cases, sufficient to prove the theorem completely. The proof depends on two simple propositions which Menelaus assumes without proof; the proof of them is given by Ptolemy.

(1) In the figure on the last page, if OD be a radius cutting a chord AB in C, then

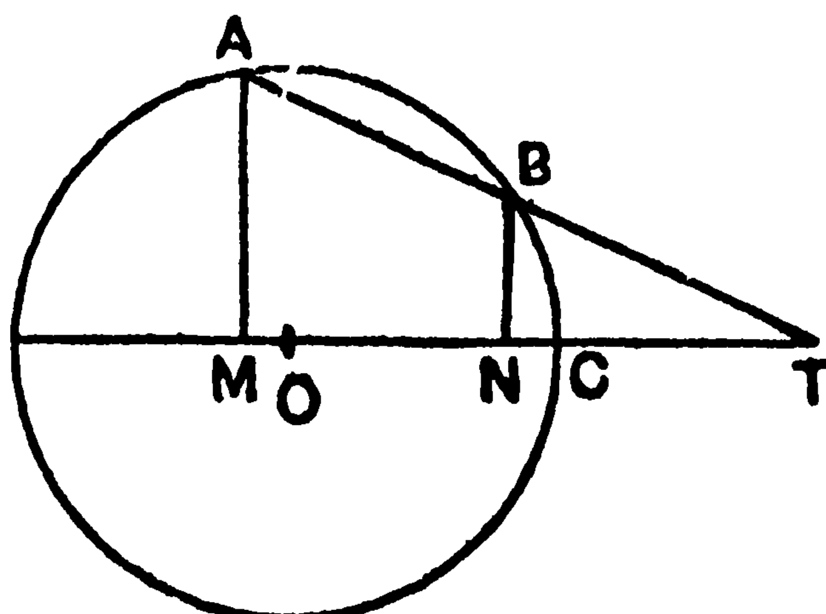
$$AC : CB = \sin AD : \sin DB.$$

For draw AM, BN perpendicular to OD. Then

$$\begin{aligned} AC : CB &= AM : BN \\ &= \tfrac{1}{2}(\text{crd. } 2AD) : \tfrac{1}{2}(\text{crd. } 2DB) \\ &= \sin AD : \sin DB. \end{aligned}$$

(2) If AB meet the radius OC produced in T, then

$$AT : BT = \sin AC : \sin BC.$$



For, if AM, BN are perpendicular to OC, we have, as before,

$$AT : TB = AM : BN$$
$$= \tfrac{1}{2}(\text{crd. } 2AC) : \tfrac{1}{2}(\text{crd. } 2BC)$$
$$= \sin AC : \sin BC.$$

Now let the arcs of great circles ADB, AEC be cut by the arcs of great circles DFC, BFE which themselves meet in F.

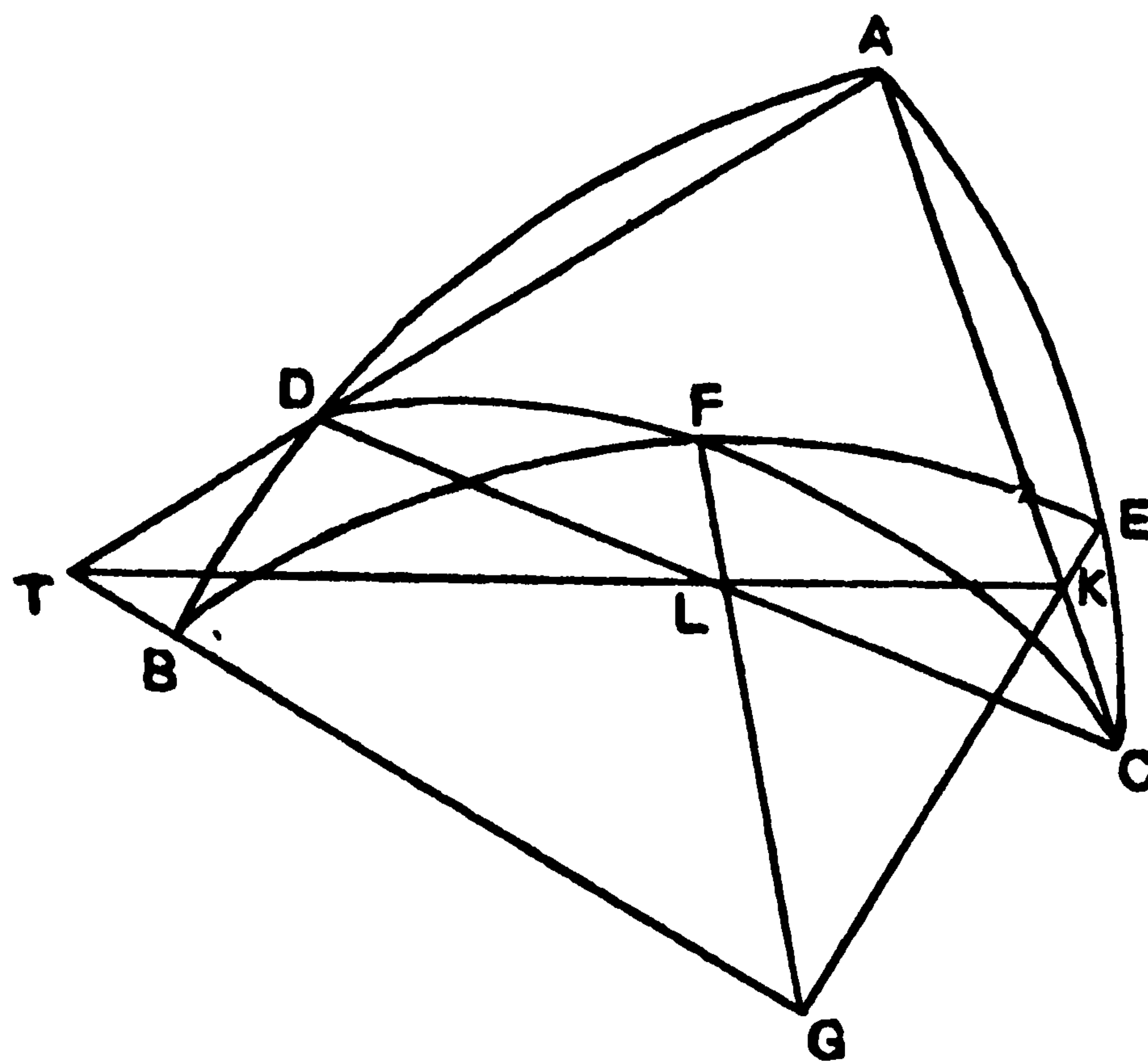
Let G be the centre of the sphere and join GB, GF, GE, AD.

Then the straight lines AD, GB, being in one plane, are either parallel or not parallel. If they are not parallel, they will meet either in the direction of D, B or of A, G.

Let AD, GB meet in T.

Draw the straight lines AKC, DLC meeting GE, GF in K, L respectively.

Then K, L, T must lie on a straight line, namely the straight line which is the section of the planes determined by the arc EFB and by the triangle ACD.[1]



Thus we have two straight lines AC, AT cut by the two straight lines CD, TK which themselves intersect in L.

Therefore, by Menelaus's proposition in plane geometry,

$$\frac{CK}{KA} = \frac{CL}{LD} \cdot \frac{DT}{TA}.$$

[1] So Ptolemy. In other words, since the straight lines GB, GE, GF, which are in one plane, respectively intersect the straight lines AD, AC, CD which are also in one plane, the points of intersection T, K, L are in both planes, and therefore lie on the straight line in which the planes intersect.

But, by the propositions proved above,

$$\frac{CK}{KA} = \frac{\sin CE}{\sin EA}, \quad \frac{CL}{LD} = \frac{\sin CF}{\sin FD}, \quad \text{and} \quad \frac{DT}{TA} = \frac{\sin DB}{\sin BA};$$

therefore, by substitution, we have

$$\frac{\sin CE}{\sin EA} = \frac{\sin CF}{\sin FD} \cdot \frac{\sin DB}{\sin BA}.$$

Menelaus apparently also gave the proof for the cases in which AD, GB meet towards A, G, and in which AD, GB are parallel respectively, and also proved that in like manner, in the above figure,

$$\frac{\sin CA}{\sin AE} = \frac{\sin CD}{\sin DF} \cdot \frac{\sin FB}{\sin BE}$$

(the triangle cut by the transversal being here CFE instead of ADC). Ptolemy[1] gives the proof of the above case only, and dismisses the last-mentioned result with a 'similarly'.

(β) *Deductions from Menelaus's Theorem.*

III. 2 proves, by means of I. 14, 10 and III. 1, that, if ABC, $A'B'C'$ be two spherical triangles in which $A = A'$, and C, C' are either equal or supplementary, $\sin c/\sin a = \sin c'/\sin a'$ and conversely. The particular case in which C, C' are right angles gives what was afterwards known as the 'regula quattuor quantitatum' and was fundamental in Arabian trigonometry.[2] A similar association attaches to the result of III. 3, which is the so-called 'tangent' or 'shadow-rule' of the Arabs. If ABC, $A'B'C'$ be triangles right-angled at A, A', and C, C' are equal and both either > or < 90°, and if P, P' be the poles of AC, $A'C'$, then

$$\frac{\sin AB}{\sin AC} = \frac{\sin A'B'}{\sin A'C'} \cdot \frac{\sin BP}{\sin B'P'}.$$

Apply the triangles so that C' falls on C, $C'B'$ on CB as CE, and $C'A'$ on CA as CD; then the result follows directly from III. 1. Since $\sin BP = \cos AB$, and $\sin B'P' = \cos A'B'$, the result becomes

$$\frac{\sin CA}{\sin C'A'} = \frac{\tan AB}{\tan A'B'},$$

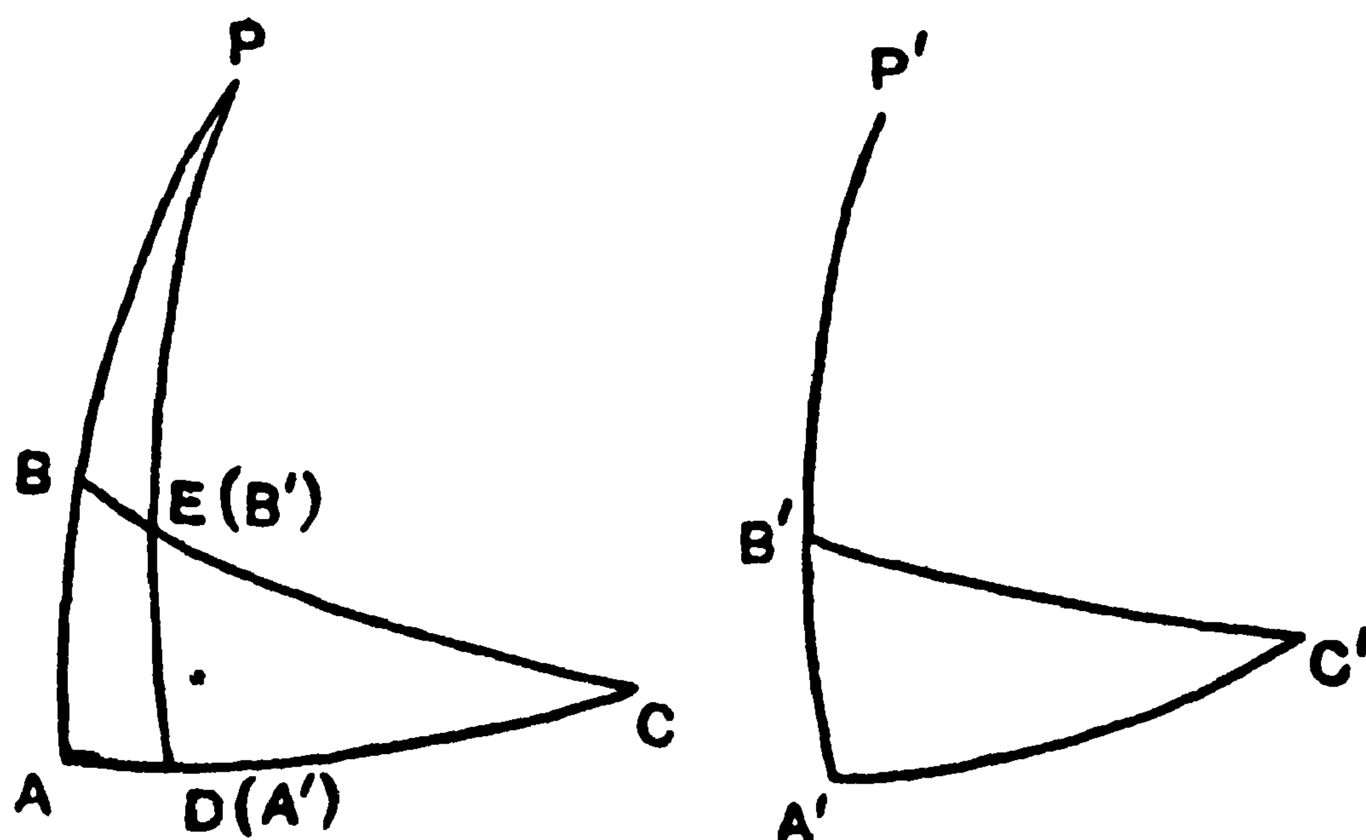
which is the 'tangent-rule' of the Arabs.[3]

[1] Ptolemy, *Syntaxis*, i. 13, vol. i, p. 76.
[2] See Braunmühl, *Gesch. der Trig.* i, pp. 17, 47, 58–60, 127–9.
[3] Cf. Braunmühl, *op. cit.* i, pp. 17–18, 58, 67–9, &c.

It follows at once (Prop. 4) that, if AM, $A'M'$ are great circles drawn perpendicular to the bases BC, $B'C'$ of two spherical triangles ABC, $A'B'C'$ in which $B = B'$, $C = C'$,

$$\frac{\sin BM}{\sin B'M'} = \frac{\sin MC}{\sin M'C'} \left(\text{since both are equal to } \frac{\tan AM}{\tan A'M'}\right).$$

III. 5 proves that, if there are two spherical triangles ABC,



$A'B'C'$ right-angled at A, A' and such that $C = C'$, while b and b' are less than 90°,

$$\frac{\sin (a+b)}{\sin (a-b)} = \frac{\sin (a'+b')}{\sin (a'-b')},$$

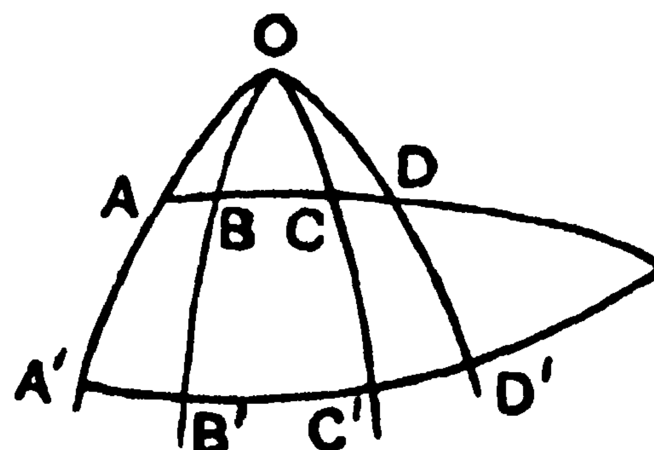
from which we may deduce[1] the formula

$$\frac{\sin (a+b)}{\sin (a-b)} = \frac{1+\cos C}{1-\cos C},$$

which is equivalent to $\tan b = \tan a \cos C$.

(γ) *Anharmonic property of four great circles through one point.*

But more important than the above result is the fact that the proof assumes as known the anharmonic property of four great circles drawn from a point on a sphere in relation to any great circle intersecting them all, viz. that, if $ABCD$, $A'B'C'D'$ be two transversals,



$$\frac{\sin AD}{\sin DC} \cdot \frac{\sin BC}{\sin AB} = \frac{\sin A'D'}{\sin D'C'} \cdot \frac{\sin B'C'}{\sin A'B'}.$$

[1] Braunmühl, *op. cit.* i, p. 18; Björnbo, p. 96.

It follows that this proposition was known before Menelaus's time. It is most easily proved by means of 'Menelaus's Theorem', III. 1, or alternatively it may be deduced for the sphere from the corresponding proposition in plane geometry, just as Menelaus's theorem is transferred by him from the plane to the sphere in III. 1. We may therefore fairly conclude that both the anharmonic property and Menelaus's theorem with reference to the sphere were already included in some earlier text-book; and, as Ptolemy, who built so much upon Hipparchus, deduces many of the trigonometrical formulae which he uses from the one theorem (III. 1) of Menelaus, it seems probable enough that both theorems were known to Hipparchus. The corresponding plane theorems appear in Pappus among his lemmas to Euclid's *Porisms*,[1] and there is therefore every probability that they were assumed by Euclid as known.

(δ) *Propositions analogous to Eucl. VI. 3.*

Two theorems following, III. 6, 8, have their analogy in Eucl. VI. 3. In III. 6 the vertical angle A of a spherical triangle is bisected by an arc of a great circle meeting BC in D, and it is proved that $\sin BD/\sin DC = \sin BA/\sin AC$; in III. 8 we have the vertical angle bisected both internally and externally by arcs of great circles meeting BC in D and E, and the proposition proves the harmonic property

$$\frac{\sin BE}{\sin EC} = \frac{\sin BD}{\sin DC}.$$

III. 7 is to the effect that, if arcs of great circles be drawn through B to meet the opposite side AC of a spherical triangle in D, E so that $\angle ABD = \angle EBC$, then

$$\frac{\sin EA \cdot \sin AD}{\sin DC \cdot \sin CE} = \frac{\sin^2 AB}{\sin^2 BC}.$$

As this is analogous to plane propositions given by Pappus as lemmas to different works included in the *Treasury of Analysis*, it is clear that these works were familiar to Menelaus.

[1] Pappus, vii, pp. 870-2, 874.

III. 9 and III. 10 show, for a spherical triangle, that (1) the great circles bisecting the three angles, (2) the great circles through the angular points meeting the opposite sides at right angles meet in a point.

The remaining propositions, III. 11–15, return to the same sort of astronomical problem as those dealt with in Euclid's *Phaenomena*, Theodosius's *Sphaerica* and Book II of Menelaus's own work. Props. 11–14 amount to theorems in spherical trigonometry such as the following.

Given arcs $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$, such that

$$90^\circ \geqq \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4,$$

$$90^\circ > \beta_1 > \beta_2 > \beta_3 > \beta_4,$$

and also $\alpha_1 > \beta_1,\ \alpha_2 > \beta_2,\ \alpha_3 > \beta_3,\ \alpha_4 > \beta_4,$

(1) If $\sin\alpha_1 : \sin\alpha_2 : \sin\alpha_3 : \sin\alpha_4 = \sin\beta_1 : \sin\beta_2 : \sin\beta_3 : \sin\beta_4$,

then
$$\frac{\alpha_1-\alpha_2}{\alpha_3-\alpha_4} > \frac{\beta_1-\beta_2}{\beta_3-\beta_4}.$$

(2) If
$$\frac{\sin(\alpha_1+\beta_1)}{\sin(\alpha_1-\beta_1)} = \frac{\sin(\alpha_2+\beta_2)}{\sin(\alpha_2-\beta_2)} = \frac{\sin(\alpha_3+\beta_3)}{\sin(\alpha_3-\beta_3)} = \frac{\sin(\alpha_4+\beta_4)}{\sin(\alpha_4-\beta_4)},$$

then
$$\frac{\alpha_1-\alpha_2}{\alpha_3-\alpha_4} < \frac{\beta_1-\beta_2}{\beta_3-\beta_4}.$$

(3) If
$$\frac{\sin(\alpha_1-\alpha_2)}{\sin(\alpha_3-\alpha_4)} < \frac{\sin(\beta_1-\beta_2)}{\sin(\beta_3-\beta_4)}$$

then
$$\frac{\alpha_1-\alpha_2}{\alpha_3-\alpha_4} < \frac{\beta_1-\beta_2}{\beta_3-\beta_4}.$$

Again, given three series of three arcs such that

$$\alpha_1 > \alpha_2 > \alpha_3,\quad \beta_1 > \beta_2 > \beta_3,\quad 90^\circ > \gamma_1 > \gamma_2 > \gamma_3,$$

and
$$\sin(\alpha_1-\gamma_1) : \sin(\alpha_2-\gamma_2) : \sin(\alpha_3-\gamma_3)$$
$$= \sin(\beta_1-\gamma_1) : \sin(\beta_2-\gamma_2) : \sin(\beta_3-\gamma_3)$$
$$= \sin\gamma_1 : \sin\gamma_2 : \sin\gamma_3$$

(1) If $\alpha_1 > \beta_1 > 2\gamma_1$, $\alpha_2 > \beta_2 > 2\gamma_2$, $\alpha_3 > \beta_3 > 2\gamma_3$,

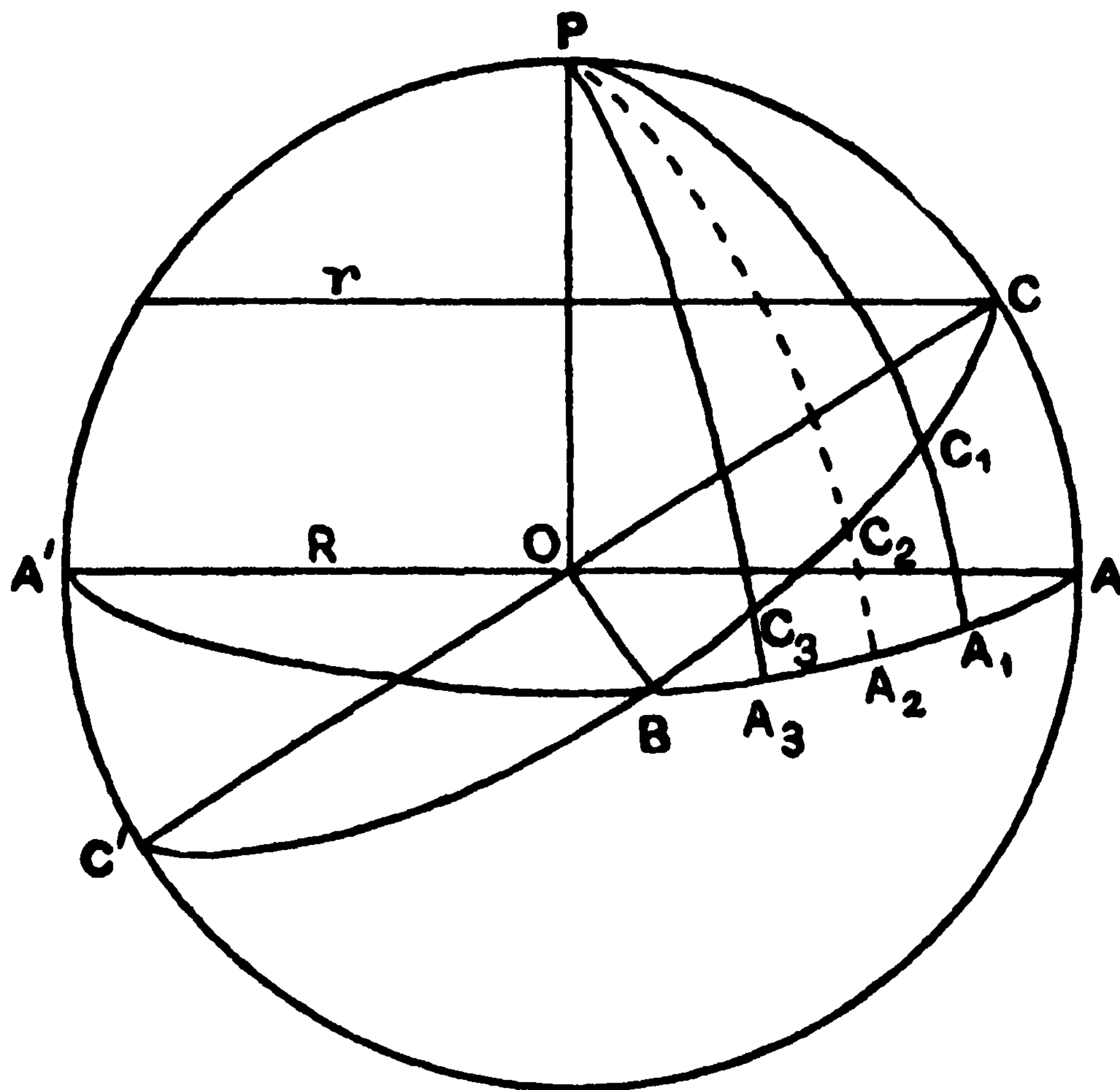
then
$$\frac{\alpha_1 - \alpha_2}{\alpha_2 - \alpha_3} > \frac{\beta_1 - \beta_2}{\beta_2 - \beta_3}; \text{ and}$$

(2) If $\beta_1 < \alpha_1 < \gamma_1$, $\beta_2 < \alpha_2 < \gamma_2$, $\beta_3 < \alpha_3 < \gamma_3$,

then
$$\frac{\alpha_1 - \alpha_2}{\alpha_2 - \alpha_3} < \frac{\beta_1 - \beta_2}{\beta_2 - \beta_3}.$$

III. 15, the last proposition, is in four parts. The first part is the proposition corresponding to Theodosius III. 11 above alluded to. Let BA, BC be two quadrants of great circles (in which we easily recognize the equator and the ecliptic), P the pole of the former, PA_1, PA_3 quadrants of great circles meeting the other quadrants in A_1, A_3 and C_1, C_3 respectively. Let R be the radius of the sphere, r, r_1, r_3 the radii of the 'parallel circles' (with pole P) through C, C_1, C_3 respectively.

Then shall
$$\frac{\sin A_1A_3}{\sin C_1C_3} = \frac{Rr}{r_1 r_3}.$$



In the triangles PCC_3, BA_3C_3 the angles at C, A_3 are right, and the angles at C_3 equal; therefore (III. 2)

$$\frac{\sin PC}{\sin PC_3} = \frac{\sin BA_3}{\sin BC_3}.$$

But, by III. 1 applied to the triangle BC_1A_1 cut by the transversal PC_3A_3,

$$\frac{\sin A_1A_3}{\sin BA_3} = \frac{\sin C_1C_3}{\sin BC_3} \cdot \frac{\sin PA_1}{\sin PC_1},$$

or $$\frac{\sin A_1A_3}{\sin C_1C_3} = \frac{\sin PA_1}{\sin PC_1} \cdot \frac{\sin BA_3}{\sin BC_3} = \frac{\sin PA_1}{\sin PC_1} \cdot \frac{\sin PC}{\sin PC_3},$$

from above,

$$= \frac{Rr}{r_1 r_3}.$$

Part 2 of the proposition proves that, if PC_2A_2 be drawn such that $\sin^2 PC_2 = \sin PA_2 . \sin PC$, or $r_2{}^2 = Rr$ (where r_2 is the radius of the parallel circle through C_2), $BC_2 - BA_2$ is a maximum, while Parts 3, 4 discuss the limits to the value of the ratio between the arcs A_1A_3 and C_1C_3.

Nothing is known of the life of CLAUDIUS PTOLEMY except that he was of Alexandria, made observations between the years A.D. 125 and 141 or perhaps 151, and therefore presumably wrote his great work about the middle of the reign of Antoninus Pius (A.D. 138–61). A tradition handed down by the Byzantine scholar Theodorus Meliteniota (about 1361) states that he was born, not at Alexandria, but at Ptolemais ἡ Ἑρμείου. Arabian traditions, going back probably to Ḥunain b. Isḥāq, say that he lived to the age of 78, and give a number of personal details to which too much weight must not be attached.

The *Μαθηματικὴ σύνταξις* (Arab. *Almagest*).

Ptolemy's great work, the definitive achievement of Greek astronomy, bore the title *Μαθηματικῆς Συντάξεως βιβλία ιγ*, the *Mathematical Collection* in thirteen Books. By the time of the commentators who distinguished the lesser treatises on astronomy forming an introduction to Ptolemy's work as *μικρὸς ἀστρονομούμενος (τόπος)*, the 'Little Astronomy', the book came to be called the 'Great Collection', *μεγάλη σύνταξις*. Later still the Arabs, combining the article Al with

the superlative μέγιστος, made up a word Al-majisṭi, which became *Almagest*; and it has been known by this name ever since. The complicated character of the system expounded by Ptolemy is no doubt responsible for the fact that it speedily became the subject of elaborate commentaries.

Commentaries on the *Syntaxis*.

Pappus[1] cites a passage from his own commentary on Book I of the *Mathematica*, which evidently means Ptolemy's work. Part of Pappus's commentary on Book V, as well as his commentary on Book VI, are actually extant in the original. Theon of Alexandria, who wrote a commentary on the *Syntaxis* in eleven Books, incorporated as much as was available of Pappus's commentary on Book V with full acknowledgement, though not in Pappus's exact words. In his commentary on Book VI Theon made much more partial quotations from Pappus; indeed the greater part of the commentary on this Book is Theon's own or taken from other sources. Pappus's commentaries are called *scholia*, Theon's ὑπομνήματα. Passages in Pappus's commentary on Book V allude to 'the scholia preceding this one' (in the plural), and in particular to the scholium on Book IV. It is therefore all but certain that he wrote on all the Books from I to VI at least. The text of the eleven Books of Theon's commentary was published at Basel by Joachim Camerarius in 1538, but it is rare and, owing to the way in which it is printed, with insufficient punctuation marks, gaps in places, and any number of misprints, almost unusable; accordingly little attention has so far been paid to it except as regards the first two Books, which were included, in a more readable form and with a Latin translation, by Halma in his edition of Ptolemy.

Translations and editions.

The *Syntaxis* was translated into Arabic, first (we are told) by translators unnamed at the instance of Yaḥyā b. Khālid b. Barmak, then by al-Ḥajjāj, the translator of Euclid (about 786–835), and again by the famous translator Isḥāq b. Ḥunain (d. 910), whose translation, as improved by Thābit b. Qurra

[1] Pappus, viii, p. 1106. 13.

(died 901), is extant in part, as well as the version by Naṣīrad-dīn aṭ-Ṭūsī (1201–74).

The first edition to be published was the Latin translation made by Gherard of Cremona from the Arabic, which was finished in 1175 but was not published till 1515, when it was brought out, without the author's name, by Peter Liechtenstein at Venice. A translation from the Greek had been made about 1160 by an unknown writer for a certain Henricus Aristippus, Archdeacon of Catania, who, having been sent by William I, King of Sicily, on a mission to the Byzantine Emperor Manuel I. Comnenus in 1158, brought back with him a Greek manuscript of the *Syntaxis* as a present; this translation, however, exists only in manuscripts in the Vatican and at Florence. The first Latin translation from the Greek to be published was that made by Georgius 'of Trebizond' for Pope Nicolas V in 1451; this was revised and published by Lucas Gauricus at Venice in 1528. The *editio princeps* of the Greek text was brought out by Grynaeus at Basel in 1538. The next complete edition was that of Halma published 1813–16, which is now rare. All the more welcome, therefore, is the definitive Greek text of the astronomical works of Ptolemy edited by Heiberg (1899–1907), to which is now added, so far as the *Syntaxis* is concerned, a most valuable supplement in the German translation (with notes) by Manitius (Teubner, 1912–13).

Summary of Contents.

The *Syntaxis* is most valuable for the reason that it contains very full particulars of observations and investigations by Hipparchus, as well as of the earlier observations recorded by him, e.g. that of a lunar eclipse in 721 B.C. Ptolemy based himself very largely upon Hipparchus, e.g. in the preparation of a Table of Chords (equivalent to sines), the theory of eccentrics and epicycles, &c.; and it is questionable whether he himself contributed anything of great value except a definite theory of the motion of the five planets, for which Hipparchus had only collected material in the shape of observations made by his predecessors and himself. A very short indication of the subjects of the different Books is all that can

be given here. Book I: Indispensable preliminaries to the study of the Ptolemaic system, general explanations of the different motions of the heavenly bodies in relation to the earth as centre, propositions required for the preparation of Tables of Chords, the Table itself, some propositions in spherical geometry leading to trigonometrical calculations of the relations of arcs of the equator, ecliptic, horizon and meridian, a 'Table of Obliquity', for calculating declinations for each degree-point on the ecliptic, and finally a method of finding the right ascensions for arcs of the ecliptic equal to one-third of a sign or 10°. Book II: The same subject continued, i.e. problems on the sphere, with special reference to the differences between various latitudes, the length of the longest day at any degree of latitude, and the like. Book III: On the length of the year and the motion of the sun on the eccentric and epicycle hypotheses. Book IV: The length of the months and the theory of the moon. Book V: The construction of the astrolabe, and the theory of the moon continued, the diameters of the sun, the moon and the earth's shadow, the distance of the sun and the dimensions of the sun, moon and earth. Book VI: Conjunctions and oppositions of sun and moon, solar and lunar eclipses and their periods. Books VII and VIII are about the fixed stars and the precession of the equinoxes, and Books IX–XIII are devoted to the movements of the planets.

Trigonometry in Ptolemy.

What interests the historian of mathematics is the trigonometry in Ptolemy. It is evident that no part of the trigonometry, or of the matter preliminary to it, in Ptolemy was new. What he did was to abstract from earlier treatises, and to condense into the smallest possible space, the minimum of propositions necessary to establish the methods and formulae used. Thus at the beginning of the preliminaries to the Table of Chords in Book I he says:

'We will first show how we can establish a systematic and speedy method of obtaining the lengths of the chords based on the uniform use of the smallest possible number of propositions, so that we may not only have the lengths of the chords

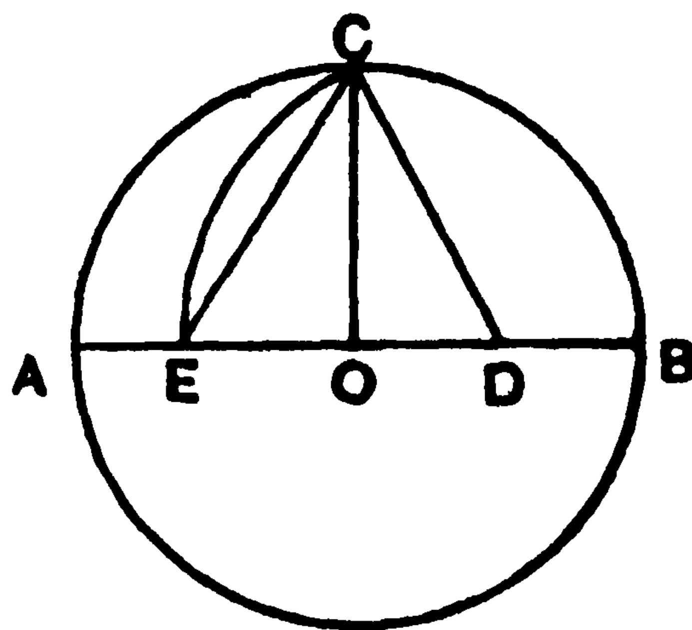
set out correctly, but may be in possession of a ready proof of our method of obtaining them based on geometrical considerations.'[1]

He explains that he will use the division (1) of the circle into 360 equal parts or degrees and (2) of the diameter into 120 equal parts, and will express fractions of these parts on the sexagesimal system. Then come the geometrical propositions, as follows.

(*a*) *Lemma for finding* sin 18° *and* sin 36°.

To find the side of a pentagon and decagon inscribed in a circle or, in other words, the chords subtending arcs of 72° and 36° respectively.

Let AB be the diameter of a circle, O the centre, OC the radius perpendicular to AB.



Bisect OB at D, join DC, and measure DE along DA equal to DC. Join EC.

Then shall OE be the side of the inscribed regular decagon, and EC the side of the inscribed regular pentagon.

For, since OB is bisected at D,

$$BE \cdot EO + OD^2 = DE^2$$
$$= DC^2 = DO^2 + OC^2.$$

Therefore $$BE \cdot EO = OC^2 = OB^2,$$

and BE is divided in extreme and mean ratio.

But (Eucl. XIII. 9) the sides of the regular hexagon and the regular decagon inscribed in a circle when placed in a straight line with one another form a straight line divided in extreme and mean ratio at the point of division.

Therefore, BO being the side of the hexagon, EO is the side of the decagon.

Also (by Eucl. XIII. 10)

$$(\text{side of pentagon})^2 = (\text{side of hexagon})^2 + (\text{side of decagon})^2$$
$$= CO^2 + OE^2 = EC^2;$$

therefore EC is the side of the regular pentagon inscribed in the circle.

[1] Ptolemy, *Syntaxis*, i. 10, pp. 31-2.

The construction in fact easily leads to the results

$$EO = \tfrac{1}{2}a(\sqrt{5}-1), \quad EC = \tfrac{1}{2}a\sqrt{(10-2\sqrt{5})},$$

where a is the radius of the circle.

Ptolemy does not however use these radicals, but calculates the lengths in terms of 'parts' of the diameter thus.

$$DO = 30, \text{ and } DO^2 = 900; \quad OC = 60 \text{ and } OC^2 = 3600;$$

therefore $DE^2 = DC^2 = 4500$, and $DE = 67^p\,4'\,55''$ nearly;

therefore side of decagon or (crd. 36°) $= DE - DO = 37^p\,4'\,55''$.

Again $OE^2 = (37^p\,4'\,55'')^2 = 1375\,.\,4'\,15''$, and $OC^2 = 3600$;

therefore $CE^2 = 4975\,.\,4'\,15''$, and $CE = 70^p\,32'\,3''$ nearly,

i.e. side of pentagon or (crd. 72°) $= 70^p\,32'\,3''$.

The method of extracting the square root is explained by Theon in connexion with the first of these cases, $\sqrt{4500}$ (see above, vol. i, pp. 61–3).

The chords which are the sides of other regular inscribed figures, the hexagon, the square and the equilateral triangle, are next given, namely,

$$\text{crd. } 60° = 60^p,$$

$$\text{crd. } 90° = \sqrt{(2\,.\,60^2)} = \sqrt{(7200)} = 84^p\,51'\,10'',$$

$$\text{crd. } 120° = \sqrt{(3\,.\,60^2)} = \sqrt{(10800)} = 103^p\,55'\,23''.$$

(β) *Equivalent of* $\sin^2\theta + \cos^2\theta = 1$.

It is next observed that, if x be any arc,

$$(\text{crd. } x)^2 + \{\text{crd. } (180° - x)\}^2 = (\text{diam.})^2 = 120^2,$$

a formula which is of course equivalent to $\sin^2\theta + \cos^2\theta = 1$.

We can therefore, from crd. 72°, derive crd. 108°, from crd. 36°, crd. 144°, and so on.

(γ) *'Ptolemy's theorem', giving the equivalent of* $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$.

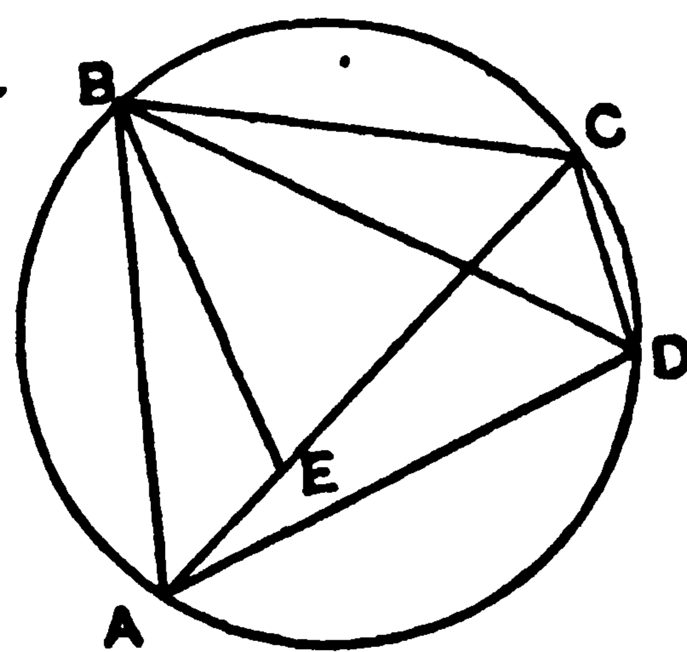
The next step is to find a formula which will give us crd. $(\alpha - \beta)$ when crd. α and crd. β are given. (This for instance enables us to find crd. 12° from crd. 72° and crd. 60°.)

The proposition giving the required formula depends upon a lemma, which is the famous 'Ptolemy's theorem'.

Given a quadrilateral $ABCD$ inscribed in a circle, the diagonals being AC, BD, to prove that

$$AC.BD = AB.DC + AD.BC.$$

The proof is well known. Draw BE so that the angle ABE is equal to the angle DBC, and let BE meet AC in E.



Then the triangles ABE, DBC are equiangular, and therefore

$$AB : AE = BD : DC,$$

or $$AB.DC = AE.BD. \qquad (1)$$

Again, to each of the equal angles ABE, DBC add the angle EBD;

then the angle ABD is equal to the angle EBC, and the triangles ABD, EBC are equiangular;

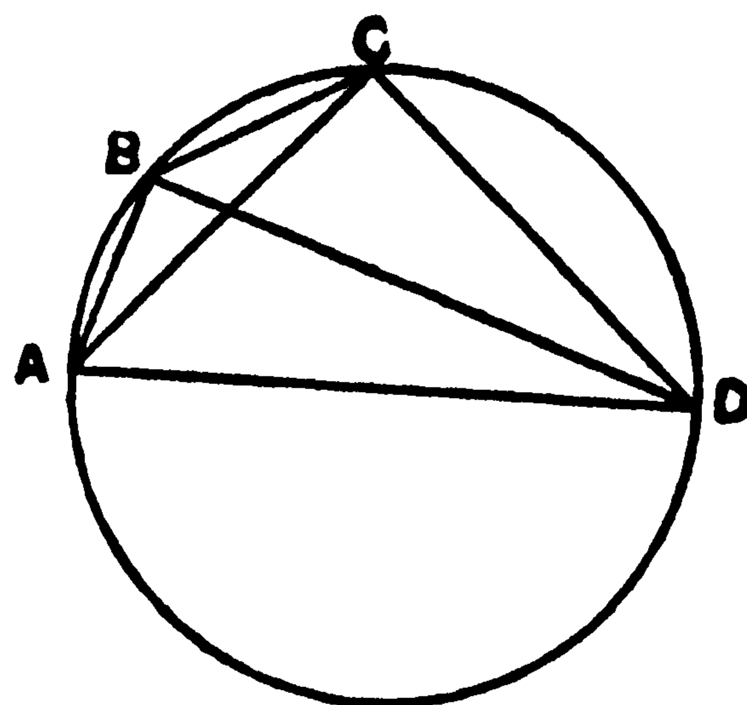
therefore $$BC : CE = BD : DA,$$

or $$AD.BC = CE.BD. \qquad (2)$$

By adding (1) and (2), we obtain

$$AB.DC + AD.BC = AC.BD.$$

Now let AB, AC be two arcs terminating at A, the extremity of the diameter AD of a circle, and let AC $(= \alpha)$ be greater than AB $(= \beta)$. Suppose that (crd. AC) and (crd. AB) are given: it is required to find (crd. BC).



Join BD, CD.

Then, by the above theorem,

$$AC.BD = BC.AD + AB.CD.$$

Now AB, AC are given; therefore $BD =$ crd. $(180° - AB)$ and $CD =$ crd. $(180° - AC)$ are known. And AD is known. Hence the remaining chord BC (crd. BC) is known.

The equation in fact gives the formula,

$$\{\text{crd.}\ (\alpha-\beta)\}\,.\,(\text{crd.}\ 180^\circ) = (\text{crd.}\ \alpha)\,.\,\{\text{crd.}\ (180^\circ-\beta)\} \\ -(\text{crd.}\ \beta)\,.\,\{\text{crd.}\ (180^\circ-\alpha)\},$$

which is, of course, equivalent to

$$\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi,\ \text{where}\ \alpha = 2\theta, \beta = 2\phi.$$

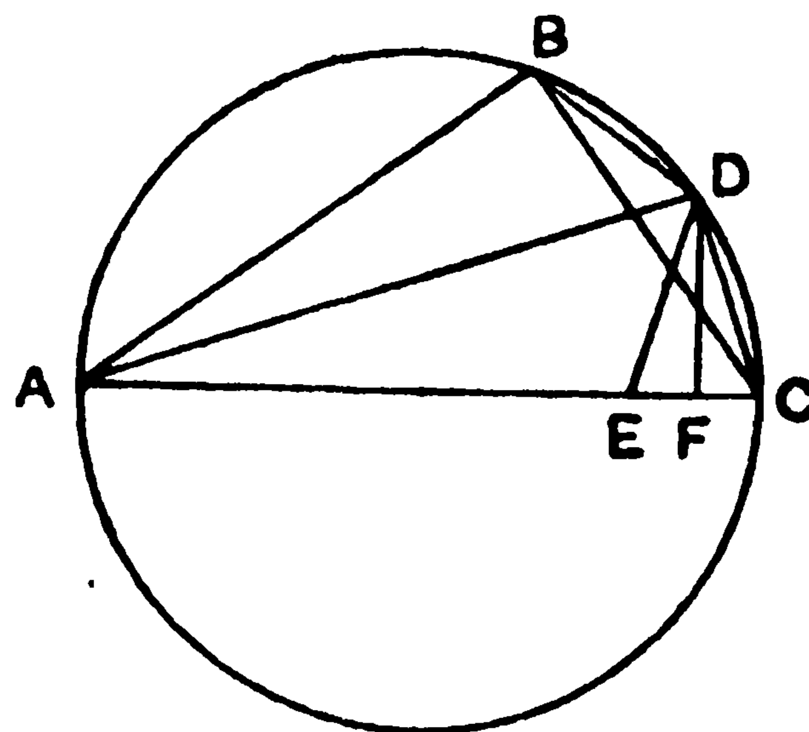
By means of this formula Ptolemy obtained

$$\text{crd.}\ 12^\circ = \text{crd.}\ (72^\circ-60^\circ) = 12^{p}\ 32'\ 36''.$$

(δ) *Equivalent of* $\sin^2\frac{1}{2}\theta = \frac{1}{2}(1-\cos\theta)$.

But, in order to get the chords of smaller angles still, we want a formula for finding the chord of half an arc when the chord of the arc is given. This is the subject of Ptolemy's next proposition.

Let BC be an arc of a circle with diameter AC, and let the arc BC be bisected at D. Given (crd. BC), it is required to find (crd. DC).



Draw DF perpendicular to AC, and join AB, AD, BD, DC. Measure AE along AC equal to AB, and join DE.

Then shall FC be equal to EF, or FC shall be half the difference between AC and AB.

For the triangles ABD, AED are equal in all respects, since two sides of the one are equal to two sides of the other and the included angles BAD, EAD, standing on equal arcs, are equal.

Therefore $$ED = BD = DC,$$

and the right-angled triangles DEF, DCF are equal in all respects, whence $EF = FC$, or $CF = \frac{1}{2}(AC-AB)$.

Now $$AC\,.\,CF = CD^2,$$

whence $$(\text{crd.}\ CD)^2 = \tfrac{1}{2}AC\,(AC-AB)$$
$$= \tfrac{1}{2}(\text{crd.}\ 180^\circ)\,.\,\{(\text{crd.}\ 180^\circ) - (\text{crd.}\ \overline{180^\circ - BC})\}.$$

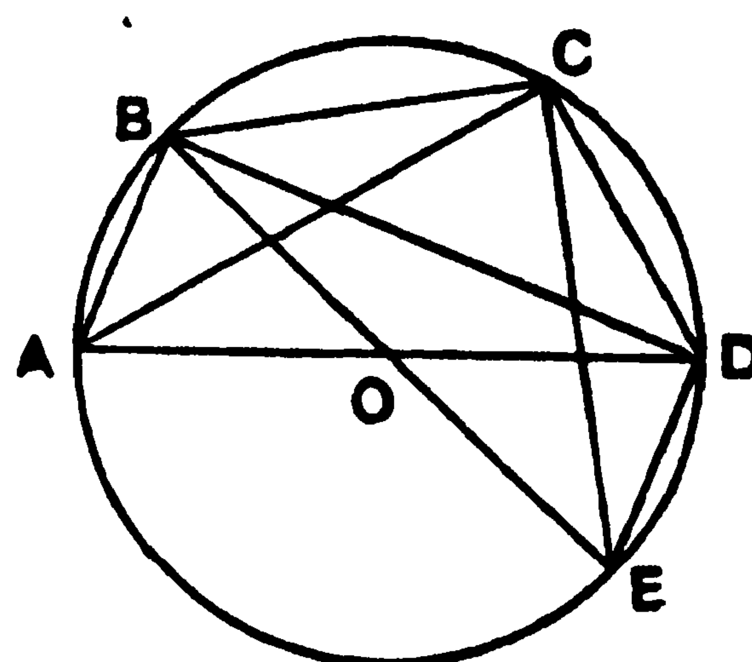
This is, of course, equivalent to the formula

$$\sin^2\tfrac{1}{2}\theta = \tfrac{1}{2}(1-\cos\theta).$$

By successively applying this formula, Ptolemy obtained (crd. 6°), (crd. 3°) and finally (crd. $1\frac{1}{2}°$) = 1^p 34′ 15″ and (crd. $\frac{3}{4}°$) = 0^p 47′ 8″. But we want a table going by half-degrees, and hence two more things are necessary; we have to get a value for (crd. 1°) lying between (crd. $1\frac{1}{2}°$) and (crd. $\frac{3}{4}°$), and we have to obtain an *addition* formula enabling us when (crd. α) is given to find {crd. $(\alpha+\frac{1}{2}°)$}, and so on.

(ε) *Equivalent of* $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$.

To find the addition formula. Suppose *AD* is the diameter of a circle, and *AB*, *BC* two arcs. Given (crd. *AB*) and (crd. *BC*), to find (crd. *AC*). Draw the diameter *BOE*, and join *CE*, *CD*, *DE*, *BD*.



Now, (crd. *AB*) being known, (crd. *BD*) is known, and therefore also (crd. *DE*), which is equal to (crd. *AB*); and, (crd. *BC*) being known, (crd. *CE*) is known.

And, by Ptolemy's theorem,

$$BD.CE = BC.DE + BE.CD.$$

The diameter *BE* and all the chords in this equation except *CD* being given, we can find *CD* or crd. $(180° - AC)$. We have in fact

$$(\text{crd. } 180°).\{\text{crd. }(180°-AC)\}$$
$$=\{\text{crd. }(180°-AB)\}.\{\text{crd. }(180°-BC)\} - (\text{crd.} AB).(\text{crd.} BC);$$

thus crd. $(180°-AC)$ and therefore (crd. *AC*) is known.

If $AB = 2\theta$, $BC = 2\phi$, the result is equivalent to

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

(ζ) *Method of interpolation based on formula*
$\sin\alpha/\sin\beta < \alpha/\beta$ (*where* $\frac{1}{2}\pi > \alpha > \beta$).

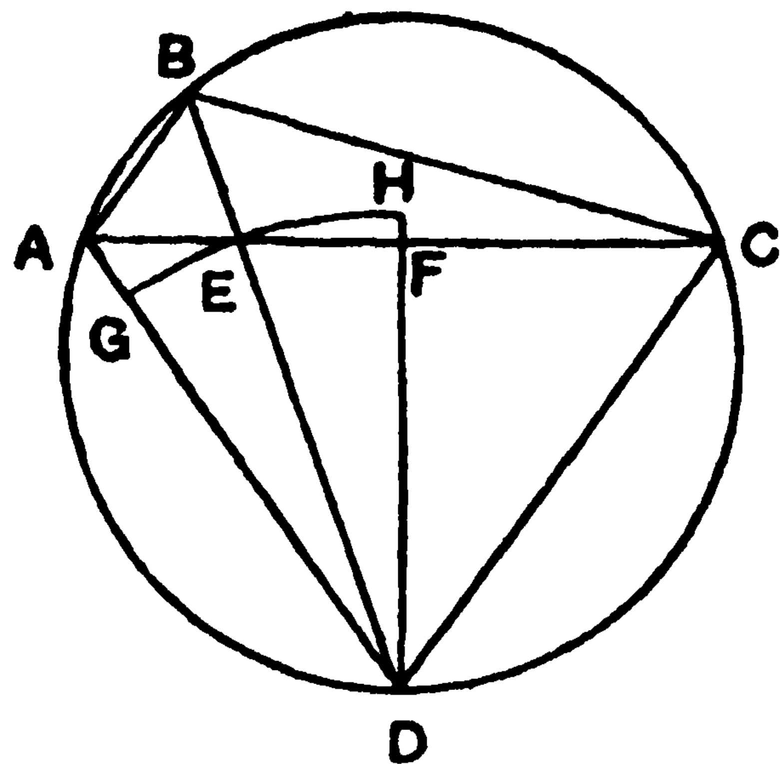
Lastly we have to find (crd. 1°), having given (crd. $1\frac{1}{2}°$) and (crd. $\frac{3}{4}°$).

Ptolemy uses an ingenious method of *interpolation* based on a proposition already assumed as known by Aristarchus.

If *AB*, *BC* be unequal chords in a circle, *BC* being the

greater, then shall the ratio of CB to BA be less than the ratio of the arc CB to the arc BA.

Let BD bisect the angle ABC, meeting AC in E and the circumference in D. The arcs AD, DC are then equal, and so are the chords AD, DC. Also $CE > EA$ (since $CB:BA = CE:EA$).



Draw DF perpendicular to AC; then $AD > DE > DF$, so that the circle with centre D and radius DE will meet DA in G and DF produced in H.

Now $\quad FE:EA = \triangle FED : \triangle AED$

$$< (\text{sector } HED) : (\text{sector } GED)$$

$$< \angle FDE : \angle EDA.$$

Componendo, $\quad FA : AE < \angle FDA : \angle ADE.$

Doubling the antecedents, we have

$$CA : AE < \angle CDA : \angle ADE,$$

and, *separando*, $\quad CE : EA < \angle CDE : \angle EDA$;

therefore (since $CB : BA = CE : EA$)

$$CB : BA < \angle CDB : \angle BDA$$

$$< (\text{arc } CB) : (\text{arc } BA),$$

i.e. $\quad (\text{crd. } CB) : (\text{crd. } BA) < (\text{arc } CB) : (\text{arc } BA).$

[This is of course equivalent to $\sin\alpha : \sin\beta < \alpha : \beta$, where $\frac{1}{2}\pi > \alpha > \beta$.]

It follows (1) that $\quad (\text{crd. } 1^\circ) : (\text{crd. } \frac{3}{4}^\circ) < 1 : \frac{3}{4}$,

and (2) $\quad$ that $\quad (\text{crd. } 1\frac{1}{2}^\circ) : (\text{crd. } 1^\circ) < 1\frac{1}{2} : 1.$

That is, $\quad \frac{4}{3} . (\text{crd. } \frac{3}{4}^\circ) > (\text{crd. } 1^\circ) > \frac{2}{3} . (\text{crd. } 1\frac{1}{2}^\circ).$

But $(\text{crd. } \frac{3}{4}^\circ) = 0^p\ 47'\ 8''$, so that $\frac{4}{3}(\text{crd. } \frac{3}{4}^\circ) = 1^p\ 2'\ 50''$ nearly (actually $1^p\ 2'\ 50\frac{2}{3}''$);

and $(\text{crd. } 1\frac{1}{2}^\circ) = 1^p\ 34'\ 15''$, so that $\frac{2}{3}(\text{crd. } 1\frac{1}{2}^\circ) = 1^p\ 2'\ 50''$.

Since, then, (crd. 1°) is both less and greater than a length which only differs inappreciably from $1^p\ 2'\ 50''$, we may say that $(\text{crd. } 1^\circ) = 1^p\ 2'\ 50''$ as nearly as possible.

(η) *Table of Chords.*

From this Ptolemy deduces that (crd. $\frac{1}{2}°$) is very nearly $0^p\ 31'\ 25''$, and by the aid of the above propositions he is in a position to complete his Table of Chords for arcs subtending angles increasing from $\frac{1}{2}°$ to $180°$ by steps of $\frac{1}{2}°$; in other words, a Table of Sines for angles from $\frac{1}{4}°$ to $90°$ by steps of $\frac{1}{4}°$.

(θ) *Further use of proportional increase.*

Ptolemy carries further the principle of proportional increase as a method of finding approximately the chords of arcs containing an odd number of minutes between $0'$ and $30'$. Opposite each chord in the Table he enters in a third column $\frac{1}{30}$th of the excess of that chord over the one before, i.e. the chord of the arc containing $30'$ less than the chord in question. For example (crd. $2\frac{1}{2}°$) is stated in the second column of the Table as $2^p\ 37'\ 4''$. The excess of (crd. $2\frac{1}{2}°$) over (crd. $2°$) in the Table is $0^p\ 31'\ 24''$: $\frac{1}{30}$th of this is $0^p\ 1'\ 2''\ 48'''$, which is therefore the amount entered in the third column opposite (crd. $2\frac{1}{2}°$). Accordingly, if we want (crd. $2°\ 25'$), we take (crd. $2°$) or $2^p\ 5'\ 40''$ and add 25 times $0^p\ 1'\ 2''\ 48'''$; or we take (crd. $2\frac{1}{2}°$) or $2^p\ 37'\ 4''$ and subtract 5 times $0^p\ 1'\ 2''\ 48'''$. Ptolemy adds that if, by using the approximation for $1°$ and $\frac{1}{2}°$, we gradually accumulate an error, we can check the calculation by comparing the chord with that of other related arcs, e.g. the double, or the supplement (the difference between the arc and the semicircle).

Some particular results obtained from the Table may be mentioned. Since (crd. $1°$) $= 1^p\ 2'\ 50''$, the whole circumference $= 360\ (1^p\ 2'\ 50'')$, nearly, and, the length of the diameter being 120^p, the value of π is $3\ (1 + \frac{2}{60} + \frac{50}{3600}) = 3 + \frac{8}{60} + \frac{30}{3600}$, which is the value used later by Ptolemy and is equivalent to $3 \cdot 14166\ldots$ Again, $\sqrt{3} = 2 \sin 60°$ and, 2 (crd. $120°$) being equal to $2\ (103^p\ 55'\ 23'')$, we have $\sqrt{3} = \frac{2}{120}\ (103 + \frac{55}{60} + \frac{23}{3600})$

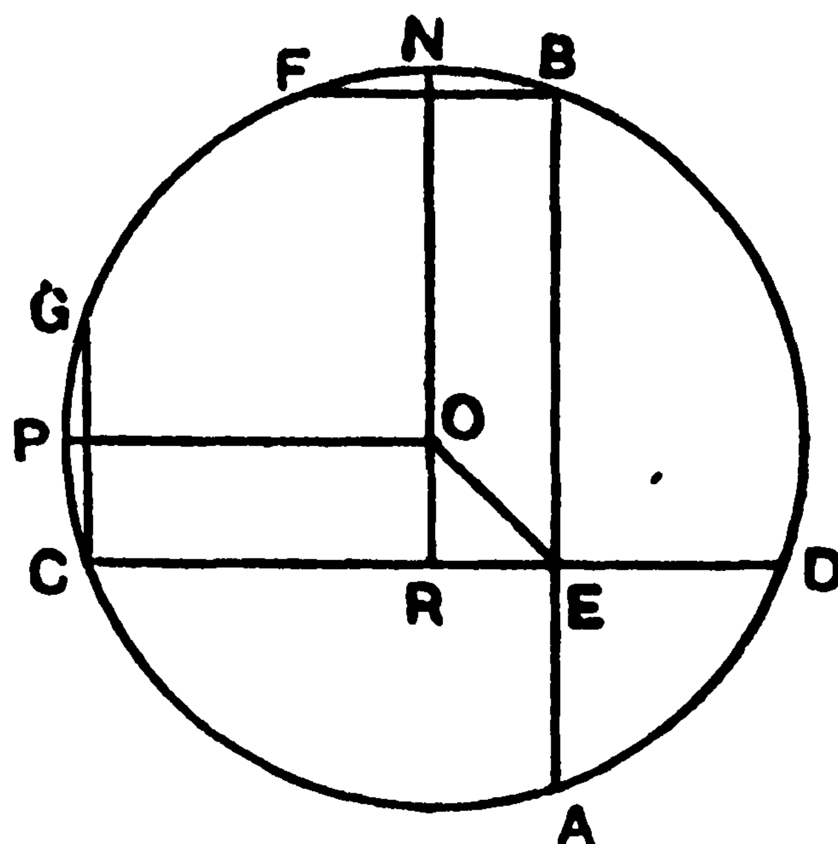
$$= 1 + \frac{43}{60} + \frac{55}{60^2} + \frac{23}{60^3} = 1 \cdot 7320509,$$

which is correct to 6 places of decimals. Speaking generally,

the sines obtained from Ptolemy's Table are correct to 5 places.

(ι) *Plane trigonometry in effect used.*

There are other cases in Ptolemy in which plane trigonometry is in effect used, e.g. in the determination of the eccentricity of the sun's orbit.[1] Suppose that $ACBD$ is the eccentric circle with centre O, and AB, CD are chords at right angles through E, the centre of the earth. To find OE. The arc BC is known ($=\alpha$, say) as also the arc CA ($=\beta$). If BF be the chord parallel to CD, and CG the chord parallel to AB, and if N, P be the middle points of the arcs BF, GC, Ptolemy finds (1) the arc BF ($=\alpha+\beta-180°$), then the chord BF, crd. $(\alpha+\beta-180°)$, then the half of it, (2) the arc GC $=$ arc $(\alpha+\beta-2\beta)$ or arc $(\alpha-\beta)$, then the chord GC, and lastly half of it. He then adds the squares on the half-chords, i.e. he obtains



$$OE^2 = \tfrac{1}{4}\{\text{crd.}\,(\alpha+\beta-180)\}^2+\tfrac{1}{4}\{\text{crd.}\,(\alpha-\beta)\}^2,$$

that is, $$OE^2/r^2 = \cos^2\tfrac{1}{2}(\alpha+\beta)+\sin^2\tfrac{1}{2}(\alpha-\beta).$$

He proceeds to obtain the angle OEC from its sine OR/OE, which he expresses as a chord of double the angle in the circle on OE as diameter in relation to that diameter.

Spherical trigonometry: formulae in solution of spherical triangles.

In spherical trigonometry, as already stated, Ptolemy obtains everything that he wants by using the one fundamental proposition known as 'Menelaus's theorem' applied to the sphere (Menelaus III. 1), of which he gives a proof following that given by Menelaus of the first case taken in his proposition. Where Ptolemy has occasion for other propositions of Menelaus's *Sphaerica*, e.g. III. 2 and 3, he does

[1] Ptolemy, *Syntaxis*, iii. 4, vol. i, pp. 234–7.

not quote those propositions, as he might have done, but proves them afresh by means of Menelaus's theorem.[1] The application of the theorem in other cases gives in effect the following different formulae belonging to the solution of a spherical triangle ABC right-angled at C, viz.

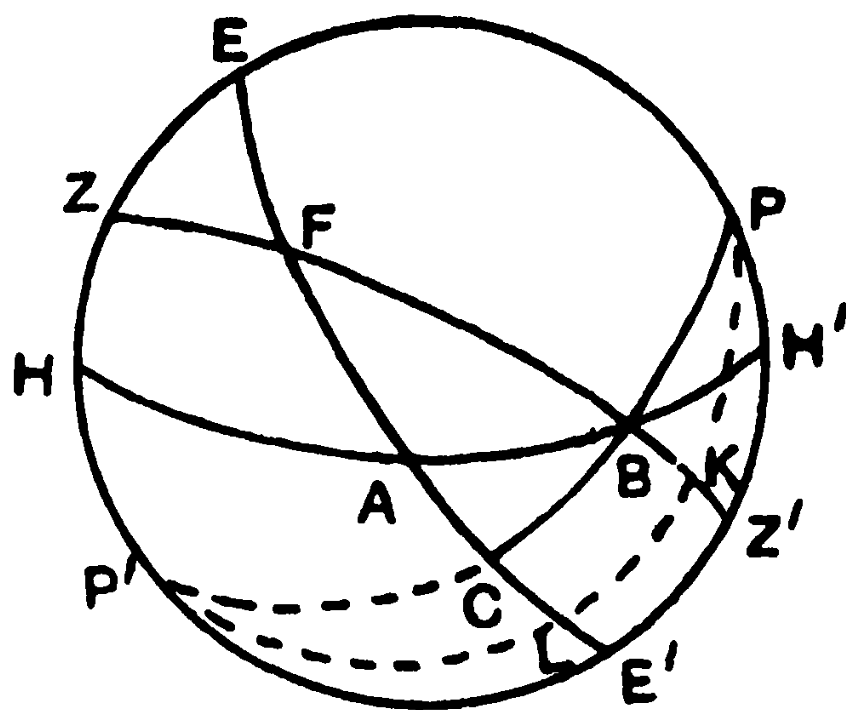
$$\sin a = \sin c \sin A,$$

$$\tan a = \sin b \tan A,$$

$$\cos c = \cos a \cos b,$$

$$\tan b = \tan c \cos A.$$

One illustration of Ptolemy's procedure will be sufficient.[2] Let HAH' be the horizon, $PEZH$ the meridian circle, EE' the equator, ZZ' the ecliptic, F an equinoctial point. Let EE', ZZ' cut the horizon in A, B. Let P be the pole, and let the great circle through P, B cut the equator at C. Now let it be required to find the time which the arc FB of the ecliptic takes to rise: this time will be measured by the arc FA of the equator. (Ptolemy has previously found the length of the arcs BC, the declination, and FC, the right ascension, of B, I. 14, 16.)



By Menelaus's theorem applied to the arcs AE', $E'P$ cut by the arcs AH', PC which also intersect one another in B,

$$\frac{\text{crd. } 2PH'}{\text{crd. } 2H'E'} = \frac{\text{crd. } 2PB}{\text{crd. } 2BC} \cdot \frac{\text{crd. } 2CA}{\text{crd. } 2AE'};$$

that is,
$$\frac{\sin PH'}{\sin H'E'} = \frac{\sin PB}{\sin BC} \cdot \frac{\sin CA}{\sin AE'}.$$

Now $\sin PH' = \cos H'E'$, $\sin PB = \cos BC$, and $\sin AE' = 1$:

therefore
$$\cot H'E' = \cot BC \,.\, \sin CA,$$

in other words, in the triangle ABC right-angled at C,

$$\cot A = \cot a \sin b,$$

or
$$\tan a = \sin b \tan A.$$

[1] *Syntaxis*, vol. i, p. 169 and pp. 126–7 respectively.
[2] *Ib.*, vol. i, pp. 121–2.

Thus AC is found, and therefore $FC-AC$ or FA.

The lengths of BC, FC are found in I. 14, 16 by the same method, the four intersecting great circles used in the figure being in that case the equator EE', the ecliptic ZZ', the great circle $PBCP'$ through the poles, and the great circle $PKLP'$ passing through the poles of both the ecliptic and the equator. In this case the two arcs PL, AE' are cut by the intersecting great circles PC, FK, and Menelaus's theorem gives (1)

$$\frac{\sin PL}{\sin KL} = \frac{\sin CP}{\sin BC} \cdot \frac{\sin BF}{\sin FK}.$$

But $\sin PL = 1$, $\sin KL = \sin BFC$, $\sin CP = 1$, $\sin FK = 1$, and it follows that

$$\sin BC = \sin BF \sin BFC,$$

corresponding to the formula for a triangle right-angled at C,

$$\sin a = \sin c \sin A.$$

(2) We have

$$\frac{\sin PK}{\sin KL} = \frac{\sin PB}{\sin BC} \cdot \frac{\sin CF}{\sin FL},$$

and $\sin PK = \cos KL = \cos BFC$, $\sin PB = \cos BC$, $\sin FL = 1$,

so that $$\tan BC = \sin CF \tan BFC,$$

corresponding to the formula

$$\tan a = \sin b \tan A.$$

While, therefore, Ptolemy's method implicitly gives the formulae for the solution of right-angled triangles above quoted, he does not speak of right-angled triangles at all, but only of arcs of intersecting great circles. The advantage from his point of view is that he works in sines and cosines only, avoiding tangents as such, and therefore he requires tables of only one trigonometrical ratio, namely the sine (or, as he has it, the chord of the double arc).

The *Analemma*.

Two other works of Ptolemy should be mentioned here. The first is the *Analemma*. The object of this is to explain a method of representing on one plane the different points

and arcs of the heavenly sphere by means of *orthogonal projection* upon three planes mutually at right angles, the meridian, the horizon, and the 'prime vertical'. The definite problem attacked is that of showing the position of the sun at any given time of the day, and the use of the method and of the instruments described in the book by Ptolemy was connected with the construction of sundials, as we learn from Vitruvius.[1] There was another *ἀνάλημμα* besides that of Ptolemy; the author of it was Diodorus of Alexandria, a contemporary of Caesar and Cicero ('Diodorus, famed among the makers of gnomons, tell me the time!' says the Anthology [2]), and Pappus wrote a commentary upon it in which, as he tells us,[3] he used the conchoid in order to trisect an angle, a problem evidently required in the *Analemma* in order to divide any arc of a circle into six equal parts (hours). The word *ἀνάλημμα* evidently means 'taking up' ('Aufnahme') in the sense of 'making a graphic representation' of something, in this case the representation on a plane of parts of the heavenly sphere. Only a few fragments remain of the Greek text of the *Analemma* of Ptolemy; these are contained in a palimpsest (Ambros. Gr. L. 99 sup., now 491) attributed to the seventh century but probably earlier. Besides this, we have a translation by William of Moerbeke from an Arabic version. This Latin translation was edited with a valuable commentary by the indefatigable Commandinus (Rome, 1562); but it is now available in William of Moerbeke's own words, Heiberg having edited it from Cod. Vaticanus Ottobon. lat. 1850 of the thirteenth century (written in William's own hand), and included it with the Greek fragments (so far as they exist) in parallel columns in vol. ii of Ptolemy's works (Teubner, 1907).

The figure is referred to three fixed planes (1) the meridian, (2) the horizon, (3) the prime vertical; these planes are the planes of the three circles $APZB$, ACB, ZQC respectively shown in the diagram below. Three other great circles are used, one of which, the equator with pole P, is fixed; the other two are movable and were called by special names; the first is the circle represented by any position of the circle of the horizon as it revolves round COC' as diameter (CSM in

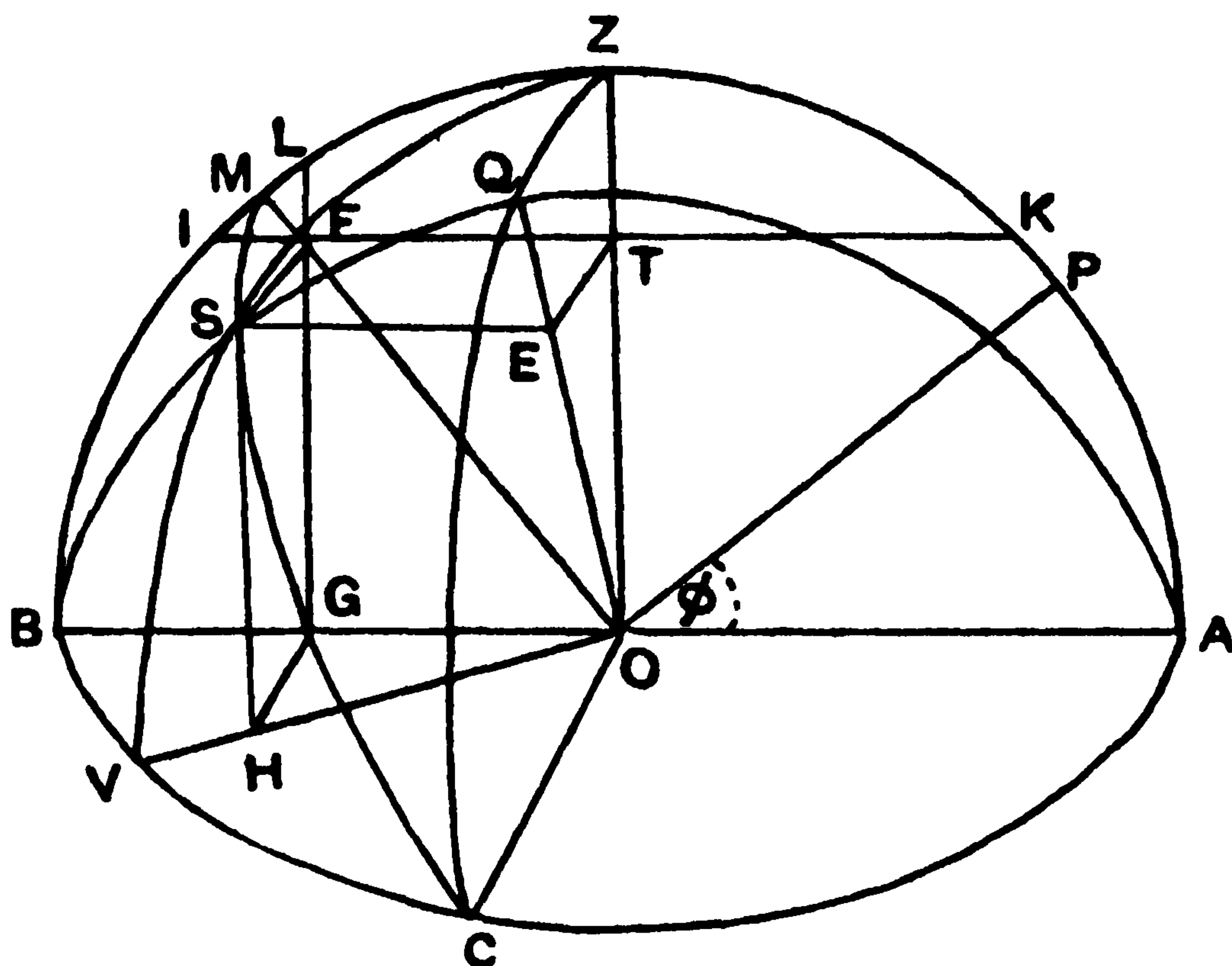
[1] Vitruvius, *De architect.* ix. 4. [2] *Anth. Palat.* xiv. 139.
[3] Pappus, iv, p. 246. 1.

the diagram is one position of it, coinciding with the equator), and it was called ἑκτήμορος κύκλος ('the circle in six parts') because the highest point of it above the horizon corresponds to the lapse of six hours; the second, called the *hour-circle*, is the circle represented by any position, as *BSQA*, of the circle of the horizon as it revolves round *BA* as axis.

The problem is, as above stated, to find the position of the sun at a given hour of the day. In order to illustrate the method, it is sufficient, with A. v. Braunmühl,[1] to take the simplest case where the sun is on the equator, i.e. at one of the equinoctial points, so that the *hectemoron* circle coincides with the equator.

Let *S* be the position of the sun, lying on the equator *MSC*, *P* the pole, *MZA* the meridian, *BCA* the horizon, *BSQA* the *hour-circle*, and let the vertical great circle *ZSV* be drawn through *S*, and the vertical great circle *ZQC* through *Z* the zenith and *C* the east-point.

We are given the arc $SC = 90° - t$, where t is the hour-angle, and the arc $MB = 90° - \phi$, where ϕ is the elevation of the pole; and we have to find the arcs *SV* (the sun's altitude),



VC, the 'ascensional difference', *SQ* and *QC*. Ptolemy, in fact, practically determines the position of *S* in terms of certain spherical coordinates.

Draw the perpendiculars, *SF* to the plane of the meridian, *SH* to that of the horizon, and *SE* to the plane of the prime

[1] Braunmühl, *Gesch. der Trigonometrie*, i, pp. 12, 13.

vertical; and draw FG perpendicular to BA, and ET to OZ. Join HG, and we have $FG = SH$, $GH = FS = ET$.

We now represent SF in a separate figure (for clearness' sake, as Ptolemy uses only one figure), where $B'Z'A'$ corresponds to BZA, P' to P and $O'M'$ to OM. Set off the arc $P'S'$ equal to CS ($= 90° - t$), and draw $S'F'$ perpendicular to $O'M'$. Then $S'M' = SM$, and $S'F' = SF$; it is as if in the original figure we had turned the quadrant MSC round MO till it coincided with the meridian circle.

In the two figures draw IFK, $I'F'K'$ parallel to BA, $B'A'$, and LFG, $L'F'G'$ parallel to OZ, $O'Z'$.

Then (1) arc ZI = arc ZS = arc ($90° - SV$), because if we turn the quadrant ZSV about ZO till it coincides with the meridian, S falls on I, and V on B. It follows that the required arc SV = arc $B'I'$ in the second figure.

(2) To find the arc VC, set off $G'X$ (in the second figure) along $G'F'$ equal to FS or $F'S'$, and draw $O'X$ through to meet the circle in X'. Then arc $Z'X'$ = arc VC; for it is as if we had turned the quadrant BVC about BO till it coincided with the meridian, when (since $G'X = FS = GH$) H would coincide with X and V with X'. Therefore BV is also equal to $B'X'$.

(3) To find QC or ZQ, set off along $T'F'$ in the second figure $T'Y$ equal to $F'S'$, and draw $O'Y$ through to Y' on the circle.

Then arc $B'Y'$ = arc QC; for it is as if we turned the prime vertical ZQC about ZO till it coincided with the meridian, when (since $T'Y = S'F' = TE$) E would fall on Y, the radius OEQ on $O'YY'$ and Q on Y'.

(4) Lastly, arc BS = arc BL = arc $B'L'$, because S, L are

both in the plane $LSHG$ at right angles to the meridian; therefore arc SQ = arc $L'Z'$.

Hence all four arcs SV, VC, QC, QS are represented in the auxiliary figure in one plane.

So far the procedure amounts to a method of *graphically* constructing the arcs required as parts of an auxiliary circle in one plane. But Ptolemy makes it clear that practical calculation followed on the basis of the figure.[1] The lines used in the construction are $SF = \sin t$ (where the radius $= 1$), $FT = OF \sin\phi$, $FG = OF \sin(90° - \phi)$, and this was fully realized by Ptolemy. Thus he shows how to calculate the arc SZ, the zenith distance ($= d$, say) or its complement SV, the height of the sun ($= h$, say), in the following way. He says in effect: Since G is known, and $\angle F'O'G' = 90° - \phi$, the ratios $O'F' : F'T'$ and $O'F' : O'T'$ are known.

[In fact $\dfrac{O'F'}{O'T'} = \dfrac{D}{\text{crd.}\,(180° - 2\phi)}$, where D is the diameter of the sphere.]

Next, since the arc MS or $M'S'$ is known [$= t$], and therefore the arc $P'S'$ [$= 90° - t$], the ratio of $O'F'$ to D is known [in fact $O'F'/D = \{\text{crd.}\ (180 - 2t)\}/2D$.

It follows from these two results that

$$O'T' = \frac{\text{crd.}\,(180° - 2t)}{2D}\,.\,\text{crd.}\ (180° - 2\phi)].$$

Lastly, the arc SV ($= h$) being equal to $B'I'$, the angle h is equal to the angle $O'I'T'$ in the triangle $I'O'T'$. And in this triangle $O'I'$, the radius, is known, while $O'T'$ has been found; and we have therefore

$$\frac{O'T'}{O'I'} = \frac{\text{crd.}\,(2h)}{D} = \frac{\text{crd.}\,(180° - 2t)}{D}\,.\,\frac{\text{crd.}\,(180° - 2\phi)}{D},\ \text{from above.}$$

[In other words, $\sin h = \cos t \cos\phi$; or, if $u = SC = 90° - t$, $\sin h = \sin u \cos\phi$, the formula for finding $\sin h$ in the right-angled spherical triangle SVC.]

For the azimuth ω (arc BV = arc $B'X'$), the figure gives

$$\tan\omega = \frac{XG'}{G'O'} = \frac{S'F'}{F'T'} = \frac{S'F'}{O'F'}\cdot\frac{O'F'}{F'T'} = \tan t\,.\,\frac{1}{\sin\phi},$$

[1] See Zeuthen in *Bibliotheca mathematica*, i_3, 1900, pp. 23–7.

or $\tan VC = \tan SC \cos SCV$ in the right-angled spherical triangle SVC.

Thirdly,

$$\tan QZ = \tan Z'Y' = \frac{S'F'}{O'T'} = \frac{S'F'}{O'F'} \cdot \frac{O'F'}{O'T'} = \tan t \, . \, \frac{1}{\cos\phi};$$

that is, $\dfrac{\tan QZ}{\tan SM} = \dfrac{\sin BZ}{\sin BM}$, which is Menelaus, *Sphaerica*, III. 3, applied to the right-angled spherical triangles ZBQ, MBS with the angle B common.

Zeuthen points out that later in the same treatise Ptolemy finds the arc 2α described above the horizon by a star of given declination δ', by a procedure equivalent to the formula

$$\cos\alpha = \tan\delta' \tan\phi,$$

and this is the same formula which, as we have seen, Hipparchus must in effect have used in his *Commentary on the Phaenomena of Eudoxus and Aratus*.

Lastly, with regard to the calculations of the height h and the azimuth ω in the general case where the sun's declination is δ', Zeuthen has shown that they may be expressed by the formulae

$$\sin h = (\cos\delta' \cos t - \sin\delta' \tan\phi)\cos\phi,$$

and
$$\tan\omega = \frac{\cos\delta' \sin t}{\dfrac{\sin\delta'}{\cos\phi} + (\cos\delta' \cos t - \sin\delta' \tan\phi)\sin\phi},$$

or
$$\frac{\cos\delta' \sin t}{\sin\delta' \cos\phi + \cos\delta' \cos t \sin\phi}.$$

The statement therefore of A. v. Braunmühl[1] that the Indians were the first to utilize the method of projection contained in the *Analemma* for actual trigonometrical calculations with the help of the Table of Chords or Sines requires modification in so far as the Greeks at all events showed the way to such use of the figure. Whether the practical application of the method of the *Analemma* for what is equivalent to the solution of spherical triangles goes back as far as Hipparchus is not certain; but it is quite likely that it does.

[1] Braunmühl, i, pp. 13, 14, 38–41.

seeing that Diodorus wrote his *Analemma* in the next century. The other alternative source for Hipparchus's spherical trigonometry is the Menelaus-theorem applied to the sphere, on which alone Ptolemy, as we have seen, relies in his *Syntaxis*. In any case the Table of Chords or Sines was in full use in Hipparchus's works, for it is presupposed by either method.

The *Planisphaerium*.

With the *Analemma* of Ptolemy is associated another work of somewhat similar content, the *Planisphaerium*. This again has only survived in a Latin translation from an Arabic version made by one Maslama b. Aḥmad al-Majrīṭī, of Cordova (born probably at Madrid, died 1007/8); the translation is now found to be, not by Rudolph of Bruges, but by 'Hermannus Secundus', whose pupil Rudolph was; it was first published at Basel in 1536, and again edited, with commentary, by Commandinus (Venice, 1558). It has been re-edited from the manuscripts by Heiberg in vol. ii. of his text of Ptolemy. The book is an explanation of the system of projection known as *stereographic*, by which points on the heavenly sphere are represented on the plane of the equator by projection from one point, a pole; Ptolemy naturally takes the south pole as centre of projection, as it is the northern hemisphere which he is concerned to represent on a plane. Ptolemy is aware that the projections of all circles on the sphere (great circles—other than those through the poles which project into straight lines—and small circles either parallel or not parallel to the equator) are likewise circles. It is curious, however, that he does not give any general proof of the fact, but is content to prove it of particular circles, such as the ecliptic, the horizon, &c. This is remarkable, because it is easy to show that, if a cone be described with the pole as vertex and passing through any circle on the sphere, i.e. a circular cone, in general oblique, with that circle as base, the section of the cone by the plane of the equator satisfies the criterion found for the 'subcontrary sections' by Apollonius at the beginning of his *Conics*, and is therefore a circle. The fact that the method of stereographic projection is so easily connected with the property of subcontrary sections

of oblique circular cones has led to the conjecture that Apollonius was the discoverer of the method. But Ptolemy makes no mention of Apollonius, and all that we know is that Synesius of Cyrene (a pupil of Hypatia, and born about A.D. 365–370) attributes the discovery of the method and its application to Hipparchus; it is curious that he does not mention Ptolemy's treatise on the subject, but speaks of himself alone as having perfected the theory. While Ptolemy is fully aware that circles on the sphere become circles in the projection, he says nothing about the other characteristic of this method of projection, namely that the angles on the sphere are represented by equal angles on the projection.

We must content ourselves with the shortest allusion to other works of Ptolemy. There are, in the first place, other minor astronomical works as follows:

(1) *Φάσεις ἀπλανῶν ἀστέρων* of which only Book II survives, (2) *Ὑποθέσεις τῶν πλανωμένων* in two Books, the first of which is extant in Greek, the second in Arabic only, (3) the inscription in Canobus, (4) *Προχείρων κανόνων διάτασις καὶ ψηφοφορία.* All these are included in Heiberg's edition, vol. ii.

The *Optics*.

Ptolemy wrote an *Optics* in five Books, which was translated from an Arabic version into Latin in the twelfth century by a certain Admiral Eugenius Siculus[1]; Book I, however, and the end of Book V are wanting. Books I, II were physical, and dealt with generalities; in Book III Ptolemy takes up the theory of mirrors, Book IV deals with concave and composite mirrors, and Book V with refraction. The theoretical portion would suggest that the author was not very proficient in geometry. Many questions are solved incorrectly owing to the assumption of a principle which is clearly false, namely that 'the image of a point on a mirror is at the point of concurrence of two lines, one of which is drawn from the luminous point to the centre of curvature of the mirror, while the other is the line from the eye to the point

[1] See G. Govi, *L'ottica di Claudio Tolomeo di Eugenio Ammiraglio di Sicilia,*... Torino, 1884; and particulars in G. Loria, *Le scienze esatte nell' antica Grecia*, pp. 570, 571.

on the mirror where the reflection takes place'; Ptolemy uses the principle to solve various special cases of the following problem (depending in general on a biquadratic equation and now known as the problem of Alhazen), 'Given a reflecting surface, the position of a luminous point, and the position of a point through which the reflected ray is required to pass, to find the point on the mirror where the reflection will take place.' Book V is the most interesting, because it seems to be the first attempt at a theory of refraction. It contains many details of experiments with different media, air, glass, and water, and gives tables of angles of refraction (r) corresponding to different angles of incidence (i); these are calculated on the supposition that r and i are connected by an equation of the following form,

$$r = ai - bi^2,$$

where a, b are constants, which is worth noting as the first recorded attempt to state a law of refraction.

The discovery of Ptolemy's *Optics* in the Arabic at once made it clear that the work *De speculis* formerly attributed to Ptolemy is not his, and it is now practically certain that it is, at least in substance, by Heron. This is established partly by internal evidence, e.g. the style and certain expressions recalling others which are found in the same author's *Automata* and *Dioptra*, and partly by a quotation by Damianus (*On hypotheses in Optics*, chap. 14) of a proposition proved by 'the mechanician Heron in his own *Catoptrica*', which appears in the work in question, but is not found in Ptolemy's *Optics*, or in Euclid's. The proposition in question is to the effect that of all broken straight lines from the eye to the mirror and from that again to the object, that particular broken line is shortest in which the two parts make equal angles with the surface of the mirror; the inference is that, as nature does nothing in vain, we must assume that, in reflection from a mirror, the ray takes the shortest course, i.e. the angles of incidence and reflection are equal. Except for the notice in Damianus and a fragment in Olympiodorus[1] containing the proof of the proposition, nothing remains of the Greek text;

[1] Olympiodorus on Aristotle, *Meteor.* iii. 2, ed. Ideler, ii, p. 96, ed. Stüve, pp. 212. 5–213. 20.

but the translation into Latin (now included in the Teubner edition of Heron, ii, 1900, pp. 316–64), which was made by William of Moerbeke in 1269, was evidently made from the Greek and not from the Arabic, as is shown by Graecisms in the translation.

A mechanical work, *Περὶ ῥοπῶν*.

There are allusions in Simplicius[1] and elsewhere to a book by Ptolemy of mechanical content, *περὶ ῥοπῶν*, on balancings or turnings of the scale, in which Ptolemy maintained as against Aristotle that air or water (e.g.) in their own 'place' have no weight, and, when they are in their own 'place', either remain at rest or rotate simply, the tendency to go up or to fall down being due to the desire of things which are not in their own places to move to them. Ptolemy went so far as to maintain that a bottle full of air was not only not heavier than the same bottle empty (as Aristotle held), but actually lighter when inflated than when empty. The same work is apparently meant by the 'book on the elements' mentioned by Simplicius.[2] Suidas attributes to Ptolemy three Books of *Mechanica*.

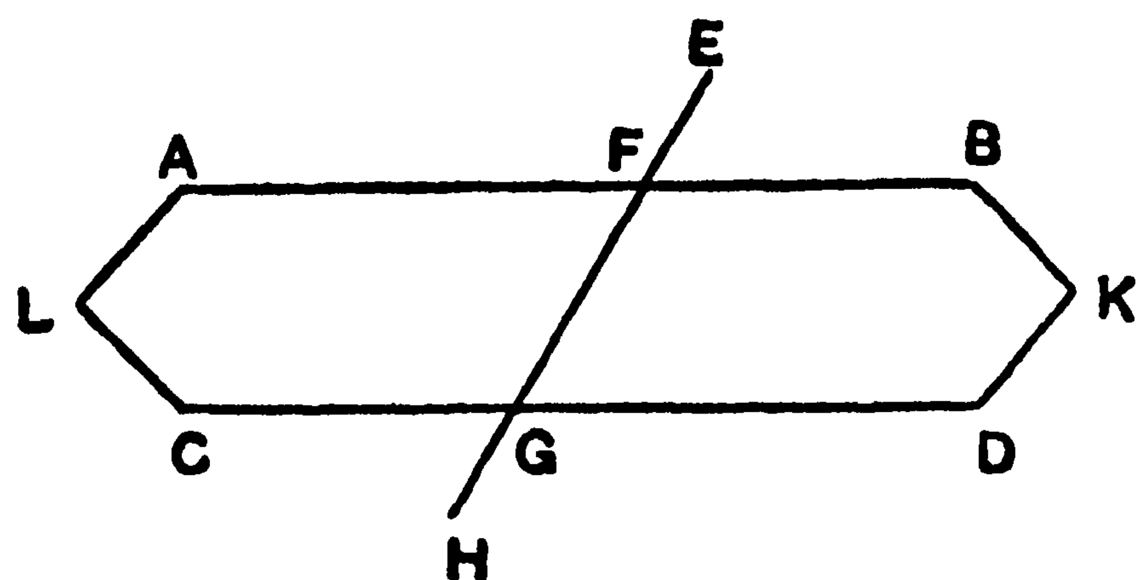
Simplicius[3] also mentions a single book, *περὶ διαστάσεως*, '*On dimension*', i.e. dimensions, in which Ptolemy tried to show that the possible number of dimensions is limited to three.

Attempt to prove the Parallel-Postulate.

Nor should we omit to notice Ptolemy's attempt to prove the Parallel-Postulate. Ptolemy devoted a tract to this subject, and Proclus[4] has given us the essentials of the argument used. Ptolemy gives, first, a proof of Eucl. I. 28, and then an attempted proof of I. 29, from which he deduces Postulate 5.

[1] Simplicius on Arist. *De caelo*, p. 710. 14, Heib. (Ptolemy, ed. Heib., vol. ii, p. 263).
[2] *Ib.*, p. 20. 10 sq.
[3] *Ib.*, p. 9. 21 sq., (Ptolemy, ed. Heib., vol. ii, p. 265).
[4] Proclus on Eucl. I, pp. 362. 14 sq., 365. 7–367. 27 (Ptolemy, ed. Heib., vol. ii, pp. 266–70).

I. To prove I. 28, Ptolemy takes two straight lines *AB*, *CD*, and a transversal *EFGH*. We have to prove that, if the sum



of the angles *BFG*, *FGD* is equal to two right angles, the straight lines *AB*, *CD* are parallel, i.e. non-secant.

Since *AFG* is the supplement of *BFG*, and *FGC* of *FGD*, it follows that the sum of the angles *AFG*, *FGC* is also equal to two right angles.

Now suppose, if possible, that *FB*, *GD*, making the sum of the angles *BFG*, *FGD* equal to two right angles, meet at *K*; then similarly *FA*, *GC* making the sum of the angles *AFG*, *FGC* equal to two right angles must also meet, say at *L*.

[Ptolemy would have done better to point out that not only are the two sums equal but the angles themselves are equal in pairs, i.e. *AFG* to *FGD* and *FGC* to *BFG*, and we can therefore take the triangle *KFG* and apply it to *FG* on the other side so that the sides *FK*, *GK* may lie along *GC*, *FA* respectively, in which case *GC*, *FA* will meet at the point where *K* falls.]

Consequently the straight lines *LABK*, *LCDK* enclose a space: which is impossible.

It follows that *AB*, *CD* cannot meet in either direction; they are therefore parallel.

II. To prove I. 29, Ptolemy takes two parallel lines *AB*, *CD* and the transversal *FG*, and argues thus. It is required to prove that $\angle AFG + \angle CGF =$ two right angles.

For, if the sum is not equal to two right angles, it must be either (1) greater or (2) less.

(1) If it is greater, the sum of the angles on the other side, *BFG*, *FGD*, which are the supplements of the first pair of angles, must be *less* than two right angles.

But *AF*, *CG* are no more parallel than *FB*, *GD*, *so that, if FG makes one pair of angles AFG, FGC together greater than*

two right angles, it must also make the other pair BFG, FGD together greater than two right angles.

But the latter pair of angles were proved less than two right angles: which is impossible.

Therefore the sum of the angles AFG, FGC cannot be *greater* than two right angles.

(2) Similarly we can show that the sum of the two angles AFG, FGC cannot be *less* than two right angles.

Therefore $\angle AFG + \angle CGF =$ two right angles.

[The fallacy here lies in the inference which I have marked by italics. When Ptolemy says that AF, CG are no more parallel than FB, GD, he is in effect assuming that *through any one point only one parallel can be drawn to a given straight line*, which is an equivalent for the very Postulate he is endeavouring to prove. The alternative Postulate is known as 'Playfair's axiom', but it is of ancient origin, since it is distinctly enunciated in Proclus's note on Eucl. I. 31.]

III. Post. 5 is now deduced, thus.

Suppose that the straight lines making with a transversal angles the sum of which is less than two right angles do not meet on the side on which those angles are.

Then, *a fortiori*, they will not meet on the other side on which are the angles the sum of which is *greater* than two right angles. [This is enforced by a supplementary proposition showing that, if the lines met on that side, Eucl. I. 16 would be contradicted.]

Hence the straight lines cannot meet in either direction: they are therefore *parallel*.

But in that case the angles made with the transversal are *equal* to two right angles: which contradicts the assumption.

Therefore the straight lines will meet.

XVIII

MENSURATION: HERON OF ALEXANDRIA

Controversies as to Heron's date.

THE vexed question of Heron's date has perhaps called forth as much discussion as any doubtful point in the history of mathematics. In the early stages of the controversy much was made of the supposed relation of Heron to Ctesibius. The *Belopoeïca* of Heron has, in the best manuscript, the heading Ἥρωνος Κτησιβίου Βελοποιϊκά, and from this, coupled with an expression used by an anonymous Byzantine writer of the tenth century, ὁ Ἀσκρηνὸς Κτησίβιος ὁ τοῦ Ἀλεξανδρέως Ἥρωνος καθηγητής, 'Ctesibius of Ascra, the teacher of Heron of Alexandria', it was inferred that Heron was a pupil of Ctesibius. The question then was, when did Ctesibius live? Martin took him to be a certain barber of that name who lived in the time of Ptolemy Euergetes II, that is, Ptolemy VII, called Physcon (died 117 B.C.), and who is said to have made an improved water-organ [1]; Martin therefore placed Heron at the beginning of the first century (say 126–50) B.C. But Philon of Byzantium, who repeatedly mentions Ctesibius by name, says that the first mechanicians (τεχνῖται) had the great advantage of being under kings who loved fame and supported the arts.[2] This description applies much better to Ptolemy II Philadelphus (285–247) and Ptolemy III Euergetes I (247–222). It is more probable, therefore, that Ctesibius was the mechanician Ctesibius who is mentioned by Athenaeus as having made an elegant drinking-horn in the time of Ptolemy Philadelphus [3]; a pupil then of Ctesibius would probably belong to the end of the third and the beginning of the second century B.C. But in truth we cannot safely conclude that Heron was an immediate pupil of Ctesibius. The Byzantine writer probably only inferred this from the title

[1] Athenaeus, *Deipno-Soph.* iv. c. 75, p. 174 b–e: cf. Vitruvius, x. 9, 13.
[2] Philon, *Mechan Synt.*, p. 50. 38, ed. Schöne.
[3] Athenaeus, xi. c. 97, p. 497 b–e.

above quoted; the title, however, in itself need not imply more than that Heron's work was a new edition of a similar work by Ctesibius, and the *Κτησιβίου* may even have been added by some well-read editor who knew both works and desired to indicate that the greater part of the contents of Heron's work was due to Ctesibius. One manuscript has *Ἥρωνος Ἀλεξανδρέως Βελοποιϊκά*, which corresponds to the titles of the other works of Heron and is therefore more likely to be genuine.

The discovery of the Greek text of the *Metrica* by R. Schöne in 1896 made it possible to fix with certainty an upper limit. In that work there are a number of allusions to Archimedes, three references to the *χωρίου ἀποτομή* of Apollonius, and two to 'the (books) about straight lines (chords) in a circle' (*δέδεικται δὲ ἐν τοῖς περὶ τῶν ἐν κύκλῳ εὐθειῶν*). Now, although the first beginnings of trigonometry may go back as far as Apollonius, we know of no work giving an actual Table of Chords earlier than that of Hipparchus. We get, therefore, at once the date 150 B.C. or thereabouts as the *terminus post quem*. A *terminus ante quem* is furnished by the date of the composition of Pappus's *Collection*; for Pappus alludes to, and draws upon, the works of Heron. As Pappus was writing in the reign of Diocletian (A.D. 284–305), it follows that Heron could not be much later than, say, A.D. 250. In speaking of the solutions by 'the old geometers' (*οἱ παλαιοὶ γεωμέτραι*) of the problem of finding the two mean proportionals, Pappus may seem at first sight to include Heron along with Eratosthenes, Nicomedes and Philon in that designation, and it has been argued, on this basis, that Heron lived long before Pappus. But a close examination of the passage[1] shows that this is by no means necessary. The relevant words are as follows:

'The ancient geometers were not able to solve the problem of the two straight lines [the problem of finding two mean proportionals to them] by ordinary geometrical methods, since the problem is by nature "solid" . . . but by attacking it with mechanical means they managed, in a wonderful way, to reduce the question to a practical and convenient construction, as may be seen in the *Mesolabon* of Eratosthenes and in the mechanics of Philon and Heron . . . Nicomedes also solved it by means of the cochloid curve, with which he also trisected an angle.'

[1] Pappus, iii, pp. 54–6.

Pappus goes on to say that he will give four solutions, one of which is his own; the first, second, and third he describes as those of Eratosthenes, Nicomedes and Heron. But in the earlier sentence he mentions Philon along with Heron, and we know from Eutocius that Heron's solution is practically the same as Philon's. Hence we may conclude that by the third solution Pappus really meant Philon's, and that he only mentioned Heron's *Mechanics* because it was a convenient place in which to find the same solution.

Another argument has been based on the fact that the extracts from Heron's *Mechanics* given at the end of Pappus's Book VIII, as we have it, are introduced by the author with a complaint that the copies of Heron's works in which he found them were in many respects corrupt, having lost both beginning and end.[1] But the extracts appear to have been added, not by Pappus, but by some later writer, and the argument accordingly falls to the ground.

The limits of date being then, say, 150 B.C. to A.D. 250, our only course is to try to define, as well as possible, the relation in time between Heron and the other mathematicians who come, roughly, within the same limits. This method has led one of the most recent writers on the subject (Tittel[2]) to place Heron not much later than 100 B.C., while another,[3] relying almost entirely on a comparison between passages in Ptolemy and Heron, arrives at the very different conclusion that Heron was later than Ptolemy and belonged in fact to the second century A.D.

In view of the difference between these results, it will be convenient to summarize the evidence relied on to establish the earlier date, and to consider how far it is or is not conclusive against the later. We begin with the relation of Heron to Philon. Philon is supposed to come not more than a generation later than Ctesibius, because it would appear that machines for throwing projectiles constructed by Ctesibius and Philon respectively were both available at one time for inspection by experts on the subject[4]; it is inferred that

[1] Pappus, viii, p. 1116. 4–7.
[2] Art. 'Heron von Alexandreia' in Pauly-Wissowa's *Real-Encyclopädie der class. Altertumswissenschaft*, vol. 8. 1, 1912.
[3] I. Hammer-Jensen in *Hermes*, vol. 48, 1913, pp. 224–35.
[4] Philon, *Mech. Synt.* iv, pp. 68. 1, 72. 36.

Philon's date cannot be later than the end of the second century B.C. (If Ctesibius flourished before 247 B.C. the argument would apparently suggest rather the beginning than the end of the second century.) Next, Heron is supposed to have been a younger contemporary of Philon, the grounds being the following. (1) Heron mentions a 'stationary-automaton' representation by Philon of the Nauplius-story,[1] and this is identified by Tittel with a representation of the same story by some contemporary of Heron's (οἱ καθ' ἡμᾶς [2]). But a careful perusal of the whole passage seems to me rather to suggest that the latter representation was not Philon's, and that Philon was included by Heron among the 'ancient' automaton-makers, and not among his contemporaries.[3] (2) Another argument adduced to show that Philon was contemporary

[1] Heron, *Autom.*, pp. 404. 11–408. 9. [2] *Ib.*, p. 412. 13.

[3] The relevant remarks of Heron are as follows. (1) He says that he has found no arrangements of 'stationary automata' better or more instructive than those described by Philon of Byzantium (p. 404. 11). As an instance he mentions Philon's setting of the Nauplius-story, in which he found everything good except two things (*a*) the mechanism for the appearance of Athene, which was too difficult (ἐργωδέστερον), and (*b*) the absence of an incident promised by Philon in his description, namely the falling of a thunderbolt on Ajax with a sound of thunder accompanying it (pp. 404. 15–408. 9). This latter incident Heron could not find anywhere in Philon, though he had consulted a great number of copies of his work. He continues (p. 408. 9–13) that we are not to suppose that he is running down Philon or charging him with not being capable of carrying out what he promised. On the contrary, the omission was probably due to a slip of memory, for it is easy enough to make stage-thunder (he proceeds to show how to do it). But the rest of Philon's arrangements seemed to him satisfactory, and this, he says, is why he has not ignored Philon's work: 'for I think that my readers will get the most benefit if they are shown, first what has been well said by the ancients and then, separately from this, what the ancients overlooked or what in their work needed improvement' (pp. 408. 22–410. 6). (2) The next chapter (pp. 410. 7-412. 2) explains generally the sort of thing the automaton-picture has to show, and Heron says he will give one example which he regards as the best. Then (3), after drawing a contrast between the simpler pictures made by 'the ancients', which involved three different movements only, and the contemporary (οἱ καθ' ἡμᾶς) representations of interesting stories by means of more numerous and varied movements (p. 412. 3-15), he proceeds to describe a setting of the Nauplius-story. This is the representation which Tittel identifies with Philon's. But it is to be observed that the description includes that of the episode of the thunderbolt striking Ajax (c. 30, pp. 448. 1–452. 7) which Heron expressly says that Philon omitted. Further, the mechanism for the appearance of Athene described in c. 29 is clearly not Philon's 'more difficult' arrangement, but the simpler device described (pp. 404. 18–408. 5) as possible and preferable to Philon's (cf. Heron, vol. i, ed. Schmidt, pp. lxviii–lxix).

with Heron is the fact that Philon has some criticisms of details of construction of projectile-throwers which are found in Heron, whence it is inferred that Philon had Heron's work specifically in view. But if Heron's *Βελοποιϊκά* was based on the work of Ctesibius, it is equally possible that Philon may be referring to Ctesibius.

A difficulty in the way of the earlier date is the relation in which Heron stands to Posidonius. In Heron's *Mechanics*, i. 24, there is a definition of 'centre of gravity' which is attributed by Heron to 'Posidonius a Stoic'. But this can hardly be Posidonius of Apamea, Cicero's teacher, because the next sentence in Heron, stating a distinction drawn by Archimedes in connexion with this definition, seems to imply that the Posidonius referred to lived before Archimedes. But the *Definitions* of Heron do contain definitions of geometrical notions which are put down by Proclus to Posidonius of Apamea or Rhodes, and, in particular, definitions of 'figure' and of 'parallels'. Now Posidonius lived from 135 to 51 B.C., and the supporters of the earlier date for Heron can only suggest that either Posidonius was not the first to give these definitions, or alternatively, if he was, and if they were included in Heron's *Definitions* by Heron himself and not by some later editor, all that this obliges us to admit is that Heron cannot have lived before the first century B.C.

Again, if Heron lived at the beginning of the first century B.C., it is remarkable that he is nowhere mentioned by Vitruvius. The *De architectura* was apparently brought out in 14 B.C. and in the preface to Book VII Vitruvius gives a list of authorities on *machinationes* from whom he made extracts. The list contains twelve names and has every appearance of being scrupulously complete; but, while it includes Archytas (second), Archimedes (third), Ctesibius (fourth), and Philon of Byzantium (sixth), it does not mention Heron. Nor is it possible to establish interdependence between Heron and Vitruvius; the differences seem, on the whole, to be more numerous than the resemblances. A few of the differences may be mentioned. Vitruvius uses 3 as the value of π, whereas Heron always uses the Archimedean value $3\frac{1}{7}$. Both writers make extracts from the Aristotelian *Μηχανικὰ προβλήματα*, but their selections are different. The

machines used by the two for the same purpose frequently differ in details; e.g. in Vitruvius's hodometer a pebble drops into a box at the end of each Roman mile,[1] while in Heron's the distance completed is marked by a pointer.[2] It is indeed pointed out that the water-organ of Heron is in many respects more primitive than that of Vitruvius; but, as the instruments are altogether different, this can scarcely be said to prove anything.

On the other hand, there are points of contact between certain propositions of Heron and of the Roman *agrimensores*. Columella, about A.D. 62, gave certain measurements of plane figures which agree with the formulae used by Heron, notably those for the equilateral triangle, the regular hexagon (in this case not only the formula but the actual figures agree with Heron's) and the segment of a circle which is less than a semicircle, the formula in the last case being

$$\tfrac{1}{2}(s+h)h + \tfrac{1}{14}(\tfrac{1}{2}s)^2,$$

where s is the chord and h the height of the segment. Here there might seem to be dependence, one way or the other; but the possibility is not excluded that the two writers may merely have drawn from a common source; for Heron, in giving the formula for the area of the segment of a circle, states that it was the formula used by 'the more accurate investigators' (*οἱ ἀκριβέστερον ἐζητηκότες*).[3]

We have, lastly, to consider the relation between Ptolemy and Heron. If Heron lived about 100 B.C., he was 200 years earlier than Ptolemy (A.D. 100–178). The argument used to prove that Ptolemy came some time after Heron is based on a passage of Proclus where Ptolemy is said to have remarked on the untrustworthiness of the method in vogue among the 'more ancient' writers of measuring the apparent diameter of the sun by means of water-clocks.[4] Hipparchus, says Proclus, used his dioptra for the purpose, and Ptolemy followed him. Proclus proceeds:

'Let us then set out here not only the observations of the ancients but also the construction of the dioptra of

[1] Vitruvius, x. 14.
[2] Heron, *Dioptra*, c. 34.
[3] Heron, *Metrica*, i. 31, p. 74. 21.
[4] Proclus, *Hypotyposis*, pp. 120. 9–15, 124. 7–26.

Hipparchus. And first we will show how we can measure an interval of time by means of the regular efflux of water, a procedure which was explained by Heron the mechanician in his treatise on water-clocks.'

Theon of Alexandria has a passage to a similar effect.[1] He first says that the most ancient mathematicians contrived a vessel which would let water flow out uniformly through a small aperture at the bottom, and then adds at the end, almost in the same words as Proclus uses, that Heron showed how this is managed in the first book of his work on water-clocks. Theon's account is from Pappus's Commentary on the *Syntaxis*, and this is also Proclus's source, as is shown by the fact that Proclus gives a drawing of the water-clock which appears to have been lost in Theon's transcription from Pappus, but which Pappus must have reproduced from the work of Heron. Tittel infers that Heron must have ranked as one of the 'more ancient' writers as compared with Ptolemy. But this again does not seem to be a necessary inference. No doubt Heron's work was a convenient place to refer to for a description of a water-clock, but it does not necessarily follow that Ptolemy was referring to Heron's clock rather than some earlier form of the same instrument.

An entirely different conclusion from that of Tittel is reached in the article 'Ptolemaios and Heron' already alluded to.[2] The arguments are shortly these. (1) Ptolemy says in his *Geography* (c. 3) that his predecessors had only been able to measure the distance between two places (as an arc of a great circle on the earth's circumference) in the case where the two places are on the same meridian. He claims that he himself invented a way of doing this even in the case where the two places are neither on the same meridian nor on the same parallel circle, provided that the heights of the pole at the two places respectively, and the angle between the great circle passing through both and the meridian circle through one of the places, are known. Now Heron in his *Dioptra* deals with the problem of measuring the distance between two places by means of the dioptra, and takes as an example

[1] Theon, *Comm. on the Syntaxis*, Basel, 1538, pp. 261 sq. (quoted in Proclus, *Hypotyposis*, ed. Manitius, pp. 309–11).

[2] Hammer-Jensen, *op. cit.*

the distance between Rome and Alexandria.[1] Unfortunately the text is in places corrupt and deficient, so that the method cannot be reconstructed in detail. But it involved the observation of the same lunar eclipse at Rome and Alexandria respectively and the drawing of the *analemma* for Rome. That is to say, the mathematical method which Ptolemy claims to have invented is spoken of by Heron as a thing generally known to experts and not more remarkable than other technical matters dealt with in the same book. Consequently Heron must have been later than Ptolemy. (It is right to add that some hold that the chapter of the *Dioptra* in question is not germane to the subject of the treatise, and was probably not written by Heron but interpolated by some later editor; if this is so, the argument based upon it falls to the ground.) (2) The dioptra described in Heron's work is a fine and accurate instrument, very much better than anything Ptolemy had at his disposal. If Ptolemy had been aware of its existence, it is highly unlikely that he would have taken the trouble to make his separate and imperfect 'parallactic' instrument, since it could easily have been grafted on to Heron's dioptra. Not only, therefore, must Heron have been later than Ptolemy but, seeing that the technique of instrument-making had made such strides in the interval, he must have been considerably later. (3) In his work περὶ ῥοπῶν[2] Ptolemy, as we have seen, disputed the view of Aristotle that air has weight even when surrounded by air. Aristotle satisfied himself experimentally that a vessel full of air is heavier than the same vessel empty; Ptolemy, also by experiment, convinced himself that the former is actually the lighter. Ptolemy then extended his argument to water, and held that water with water round it has no weight, and that the diver, however deep he dives, does not feel the weight of the water above him. Heron[3] asserts that water has no appreciable weight and has no appreciable power of compressing the air in a vessel inverted and forced down into the water. In confirmation of this he cites the case of the diver, who is not prevented from breathing when far below

[1] Heron, *Dioptra*, c. 35 (vol. iii, pp. 302–6).
[2] Simplicius on *De caelo*, p. 710. 14, Heib. (Ptolemy, vol. ii, p. 263).
[3] Heron, *Pneumatica*, i. Pref. (vol. i, p. 22. 14 sq.).

the surface. He then inquires what is the reason why the diver is not oppressed though he has an unlimited weight of water on his back. He accepts, therefore, the view of Ptolemy as to the fact, however strange this may seem. But he is not satisfied with the explanation given: 'Some say', he goes on, 'it is because water in itself is uniformly heavy (*ἰσοβαρὲς αὐτὸ καθ' αὑτό*)'—this seems to be equivalent to Ptolemy's dictum that water in water has no weight—'but they give no explanation whatever why divers . . .' He himself attempts an explanation based on Archimedes. It is suggested, therefore, that Heron's criticism is directed specifically against Ptolemy and no one else. (4) It is suggested that the Dionysius to whom Heron dedicated his *Definitions* is a certain Dionysius who was *praefectus urbi* at Rome in A.D. 301. The grounds are these (*a*) Heron addresses Dionysius as *Διονύσιε λαμπρότατε*, where *λαμπρότατος* obviously corresponds to the Latin *clarissimus*, a title which in the third century and under Diocletian was not yet in common use. Further, this Dionysius was *curator aquarum* and *curator operum publicorum*, so that he was the sort of person who would have to do with the engineers, architects and craftsmen for whom Heron wrote. Lastly, he is mentioned in an inscription commemorating an improvement of water supply and dedicated 'to Tiberinus, father of all waters, and to the ancient inventors of marvellous constructions' (*repertoribus admirabilium fabricarum priscis viris*), an expression which is not found in any other inscription, but which recalls the sort of tribute that Heron frequently pays to his predecessors. This identification of the two persons named Dionysius is an ingenious conjecture, but the evidence is not such as to make it anything more.[1]

The result of the whole investigation just summarized is to place Heron in the third century A.D., and perhaps little, if anything, earlier than Pappus. Heiberg accepts this conclusion,[2] which may therefore, I suppose, be said to hold the field for the present.

[1] Dionysius was of course a very common name. Diophantus dedicated his *Arithmetica* to a person of this name (*τιμιώτατέ μοι Διονύσιε*), whom he praised for his ambition to learn the solutions of arithmetical problems. This Dionysius must have lived in the second half of the third century A.D., and if Heron also belonged to this time, is it not possible that Heron's Dionysius was the same person?

[2] Heron, vol. v, p. ix.

Heron was known as ὁ Ἀλεξανδρεύς (e.g. by Pappus) or ὁ μηχανικός (*mechanicus*), to distinguish him from other persons of the same name; Proclus and Damianus use the latter title, while Pappus also speaks of οἱ περὶ τὸν Ἥρωνα μηχανικοί.

Character of works.

Heron was an almost encyclopaedic writer on mathematical and physical subjects. Practical utility rather than theoretical completeness was the object aimed at; his environment in Egypt no doubt accounts largely for this. His *Metrica* begins with the old legend of the traditional origin of geometry in Egypt, and in the *Dioptra* we find one of the very problems which geometry was intended to solve, namely that of re-establishing boundaries of lands when the flooding of the Nile had destroyed the land-marks: 'When the boundaries of an area have become obliterated to such an extent that only two or three marks remain, in addition to a plan of the area, to supply afresh the remaining marks.'[1] Heron makes little or no claim to originality; he often quotes authorities, but, in accordance with Greek practice, he more frequently omits to do so, evidently without any idea of misleading any one; only when he has made what is in his opinion any slight improvement on the methods of his predecessors does he trouble to mention the fact, a habit which clearly indicates that, except in these cases, he is simply giving the best traditional methods in the form which seemed to him easiest of comprehension and application. The *Metrica* seems to be richest in definite references to the discoveries of predecessors; the names mentioned are Archimedes, Dionysodorus, Eudoxus, Plato; in the *Dioptra* Eratosthenes is quoted, and in the introduction to the *Catoptrica* Plato and Aristotle are mentioned.

The practical utility of Heron's manuals being so great, it was natural that they should have great vogue, and equally natural that the most popular of them at any rate should be re-edited, altered and added to by later writers; this was inevitable with books which, like the *Elements* of Euclid, were in regular use in Greek, Byzantine, Roman, and Arabian

[1] Heron, *Dioptra*, c. 25, p. 268. 17–19.

education for centuries. The geometrical or mensurational books in particular gave scope for expansion by multiplication of examples, so that it is difficult to disentangle the genuine Heron from the rest of the collections which have come down to us under his name. Hultsch's considered criterion is as follows: 'The Heron texts which have come down to our time are authentic in so far as they bear the author's name and have kept the original design and form of Heron's works, but are unauthentic in so far as, being constantly in use for practical purposes, they were repeatedly re-edited and, in the course of re-editing, were rewritten with a view to the particular needs of the time.'

List of Treatises.

Such of the works of Heron as have survived have reached us in very different ways. Those which have come down in the Greek are:

I. The *Metrica*, first discovered in 1896 in a manuscript of the eleventh (or twelfth) century at Constantinople by R. Schöne and edited by his son, H. Schöne (*Heronis Opera*, iii, Teubner, 1903).

II. *On the Dioptra*, edited in an Italian version by Venturi in 1814; the Greek text was first brought out by A. J. H. Vincent[1] in 1858, and the critical edition of it by H. Schöne is included in the Teubner vol. iii just mentioned.

III. The *Pneumatica*, in two Books, which appeared first in a Latin translation by Commandinus, published after his death in 1575; the Greek text was first edited by Thévenot in *Veterum mathematicorum opera Graece et Latine edita* (Paris, 1693), and is now available in *Heronis Opera*, i (Teubner, 1899), by W. Schmidt.

IV. *On the art of constructing automata* (περὶ αὐτοματο-ποιητικῆς), or *The automaton-theatre*, first edited in an Italian translation by B. Baldi in 1589; the Greek text was included in Thévenot's *Vet. math.*, and now forms part of *Heronis Opera*, vol. i, by W. Schmidt.

V. *Belopoeïca* (on the construction of engines of war), edited

[1] *Notices et extraits des manuscrits de la Bibliothèque impériale*, xix, pt. 2, pp. 157–337.

by B. Baldi (Augsburg, 1616), Thévenot (*Vet. math.*), Köchly and Rüstow (1853) and by Wescher (*Poliorcétique des Grecs*, 1867, the first critical edition).

VI. The *Cheirobalistra* (Ἥρωνος χειροβαλλίστρας κατασκευὴ καὶ συμμετρία (?)), edited by V. Prou, *Notices et extraits*, xxvi. 2 (Paris, 1877).

VII. The geometrical works, *Definitiones*, *Geometria*, *Geodaesia*, *Stereometrica* I and II, *Mensurae*, *Liber Geeponicus*, edited by Hultsch with *Variae collectiones* (*Heronis Alexandrini geometricorum et stereometricorum reliquiae*, 1864). This edition will now be replaced by that of Heiberg in the Teubner collection (vols. iv, v), which contains much additional matter from the Constantinople manuscript referred to, but omits the *Liber Geeponicus* (except a few extracts) and the *Geodaesia* (which contains only a few extracts from the *Geometry* of Heron).

Only fragments survive of the Greek text of the *Mechanics* in three Books, which, however, is extant in the Arabic (now edited, with German translation, in *Heronis Opera*, vol. ii, by L. Nix and W. Schmidt, Teubner, 1901).

A smaller separate mechanical treatise, the *Βαρουλκός*, is quoted by Pappus.[1] The object of it was 'to move a given weight by means of a given force', and the machine consisted of an arrangement of interacting toothed wheels with different diameters.

At the end of the *Dioptra* is a description of a *hodometer* for measuring distances traversed by a wheeled vehicle, a kind of taxameter, likewise made of a combination of toothed wheels.

A work on *Water-clocks* (περὶ ὑδρίων ὡροσκοπείων) is mentioned in the *Pneumatica* as having contained four Books, and is also alluded to by Pappus.[2] Fragments are preserved in Proclus (*Hypotyposis*, chap. 4) and in Pappus's commentary on Book V of Ptolemy's *Syntaxis* reproduced by Theon.

Of Heron's *Commentary on Euclid's Elements* only very meagre fragments survive in Greek (Proclus), but a large number of extracts are fortunately preserved in the Arabic commentary of an-Nairīzī, edited (1) in the Latin version of Gherard of Cremona by Curtze (Teubner, 1899), and (2) by

[1] Pappus, viii, p. 1060. 5.   [2] *Ib.*, p. 1026. 1.

Besthorn and Heiberg (*Codex Leidensis* 399. 1, five parts of which had appeared up to 1910). The commentary extended as far as *Elem.* VIII. 27 at least.

The *Catoptrica*, as above remarked under Ptolemy, exists in a Latin translation from the Greek, presumed to be by William of Moerbeke, and is included in vol. ii of *Heronis Opera*, edited, with introduction, by W. Schmidt.

Nothing is known of the *Camarica* ('on vaultings') mentioned by Eutocius (on Archimedes, *Sphere and Cylinder*), the *Zygia* (balancings) associated by Pappus with the *Automata*,[1] or of a work on the use of the astrolabe mentioned in the *Fihrist*.

We are in this work concerned with the treatises of mathematical content, and therefore can leave out of account such works as the *Pneumatica*, the *Automata*, and the *Belopoeïca*. The *Pneumatica* and *Automata* have, however, an interest to the historian of physics in so far as they employ the force of compressed air, water, or steam. In the *Pneumatica* the reader will find such things as siphons, 'Heron's fountain', 'penny-in-the-slot' machines, a fire-engine, a water-organ, and many arrangements employing the force of steam.

Geometry.

(α) *Commentary on Euclid's Elements.*

In giving an account of the geometry and mensuration (or geodesy) of Heron it will be well, I think, to begin with what relates to the *elements*, and first the Commentary on Euclid's *Elements*, of which we possess a number of extracts in an-Nairīzī and Proclus, enabling us to form a general idea of the character of the work. Speaking generally, Heron's comments do not appear to have contained much that can be called important. They may be classified as follows:

(1) A few general notes, e.g. that Heron would not admit more than three axioms.

(2) Distinctions of a number of particular *cases* of Euclid's propositions according as the figure is drawn in one way or another.

[1] Pappus, viii, p. 1024. 28.

Of this class are the different cases of I. 35, 36, III. 7, 8 (where the chords to be compared are drawn on different sides of the diameter instead of on the same side), III. 12 (which is not Euclid's at all but Heron's own, adding the case of external to that of internal contact in III. 11[1], VI. 19 (where the triangle in which an additional line is drawn is taken to be the *smaller* of the two), VII. 19 (where the particular case is given of *three* numbers in continued proportion instead of four proportionals).

(3) Alternative proofs.

It appears to be Heron who first introduced the easy but uninstructive semi-algebraical method of proving the propositions II. 2–10 which is now so popular. On this method the propositions are proved 'without figures' as consequences of II. 1 corresponding to the algebraical formula

$$a(b+c+d+\ldots) = ab+ac+ad+\ldots$$

Heron explains that it is not possible to prove II. 1 without drawing a number of lines (i.e. without actually drawing the rectangles), but that the following propositions up to II. 10 can be proved by merely drawing one line. He distinguishes two varieties of the method, one by *dissolutio*, the other by *compositio*, by which he seems to mean *splitting-up* of rectangles and squares and *combination* of them into others. But in his proofs he sometimes combines the two varieties.

Alternative proofs are given (*a*) of some propositions of Book III, namely III. 25 (placed after III. 30 and starting from the *arc* instead of the chord), III. 10 (proved by means of III. 9), III. 13 (a proof preceded by a lemma to the effect that a straight line cannot meet a circle in more than two points).

A class of alternative proof is (*b*) that which is intended to meet a particular objection (ἔνστασις) which had been or might be raised to Euclid's constructions. Thus in certain cases Heron avoids *producing* a certain straight line, where Euclid produces it, the object being to meet the objection of one who should deny our right to assume that there is *any space available*. Of this class are his proofs of I. 11, 20 and his note on I. 16. Similarly in I. 48 he supposes the right-angled

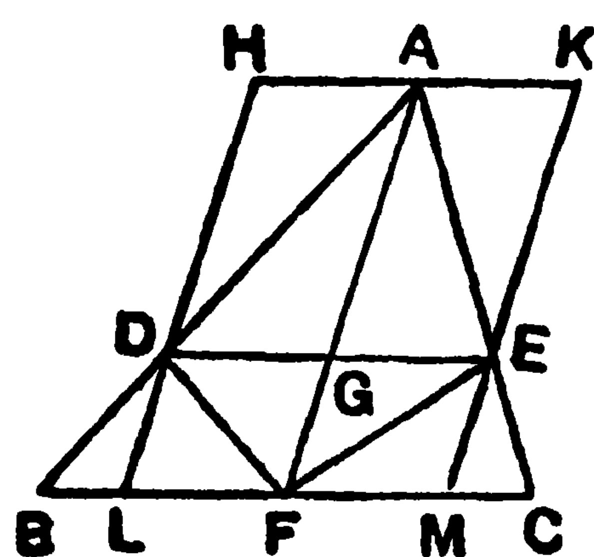
triangle which is constructed to be constructed on the same side of the common side as the given triangle is.

A third class (*c*) is that which avoids *reductio ad absurdum*, e.g. a direct proof of I. 19 (for which he requires and gives a preliminary lemma) and of I. 25.

(4) Heron supplies certain *converses* of Euclid's propositions e.g. of II. 12, 13 and VIII. 27.

(5) A few additions to, and extensions of, Euclid's propositions are also found. Some are unimportant, e.g. the construction of isosceles and scalene triangles in a note on I. 1 and the construction of *two* tangents in III. 17. The most important extension is that of III. 20 to the case where the angle at the circumference is greater than a right angle, which gives an easy way of proving the theorem of III. 22. Interesting also are the notes on I. 37 (on I. 24 in Proclus), where Heron proves that two triangles with two sides of the one equal to two sides of the other and with the included angles *supplementary* are equal in area, and compares the areas where the sum of the included angles (one being supposed greater than the other) is less or greater than two right angles, and on I. 47, where there is a proof (depending on preliminary lemmas) of the fact that, in the figure of Euclid's proposition (see next page), the straight lines AL, BG, CE meet in a point. This last proof is worth giving. First come the lemmas.

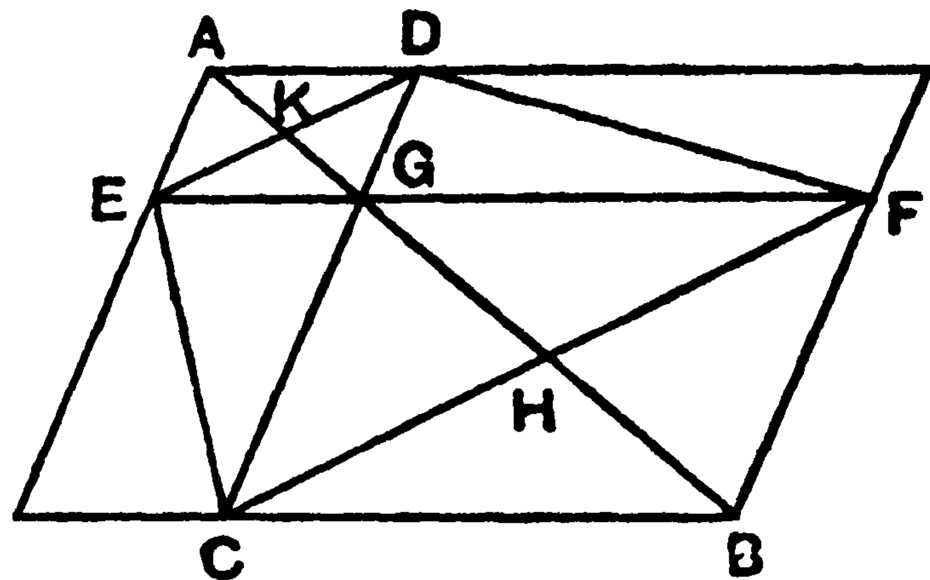
(1) If in a triangle ABC a straight line DE be drawn parallel to the base BC cutting the sides AB, AC or those sides produced in D, E, and if F be the middle point of BC, then the straight line AF (produced if necessary) will also bisect DE. (HK is drawn through A parallel to DE, and HDL, KEM through D, E parallel to AF meeting the base in L, M respectively. Then the triangles ABF, AFC between the same parallels are equal. So are the triangles DBF, EFC. Therefore the differences, the triangles ADF, AEF, are equal and so therefore are the parallelograms HF, KF. Therefore $LF = FM$, or $DG = GE$.)



(2) is the converse of Eucl. I. 43. If a parallelogram is

cut into four others $ADGE$, DF, $FGCB$, CE, so that DF, CE are equal, the common vertex G will lie on the diagonal AB.

Heron produces AG to meet CF in H, and then proves that AHB is a straight line.



Since DF, CE are equal, so are the triangles DGF, ECG. Adding the triangle GCF, we have the triangles ECF, DCF equal, and DE, CF are parallel.

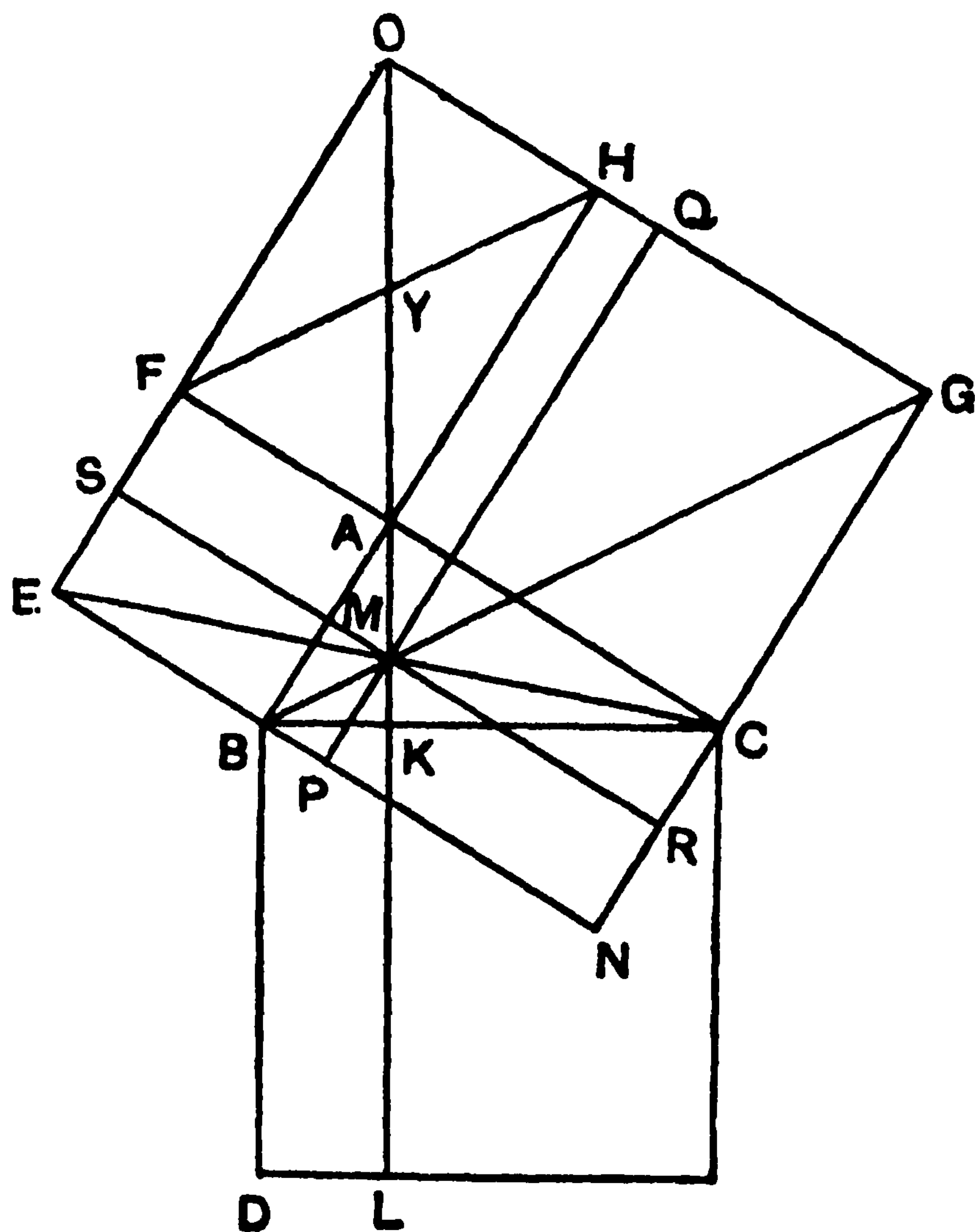
But (by I. 34, 29, 26) the triangles AKE, GKD are congruent, so that $EK = KD$; and by lemma (1) it follows that $CH = HF$.

Now, in the triangles FHB, CHG, two sides (BF, FH and GC, CH) and the included angles are equal; therefore the triangles are congruent, and the angles BHF, GHC are equal.

Add to each the angle GHF, and

$$\angle BHF + \angle FHG = \angle CHG + \angle GHF = \text{two right angles.}$$

To prove his substantive proposition Heron draws AKL perpendicular to BC, and joins EC meeting AK in M. Then we have only to prove that BMG is a straight line.



Complete the parallelogram $FAHO$, and draw the diagonals OA, FH meeting in Y. Through M draw PQ, SR parallel respectively to BA, AC.

Now the triangles FAH, BAC are equal in all respects;

therefore $\angle HFA = \angle ABC$

$= \angle CAK$ (since AK is at right angles to BC).

But, the diagonals of the rectangle FH cutting one another in Y, we have $FY = YA$ and $\angle HFA = \angle OAF$;

therefore $\angle OAF = \angle CAK$, and OA is in a straight line with AKL.

Therefore, OM being the diagonal of SQ, $SA = AQ$, and, if we add AM to each, $FM = MH$.

Also, since EC is the diagonal of FN, $FM = MN$.

Therefore the parallelograms MH, MN are equal; and hence, by the preceding lemma, BMG is a straight line. Q.E.D.

(β) The *Definitions*.

The elaborate collection of *Definitions* is dedicated to one Dionysius in a preface to the following effect:

'In setting out for you a sketch, in the shortest possible form, of the technical terms premised in the elements of geometry, I shall take as my point of departure, and shall base my whole arrangement upon, the teaching of Euclid, the author of the elements of theoretical geometry; for by this means I think that I shall give you a good general understanding not only of Euclid's doctrine but of many other works in the domain of geometry. I shall begin then with the *point*.'

He then proceeds to the definitions of the point, the line, the different sorts of lines, straight, circular, 'curved' and 'spiral-shaped' (the Archimedean spiral and the cylindrical helix), Defs. 1–7; surfaces, plane and not plane, solid body, Defs. 8–11; angles and their different kinds, plane, solid, rectilinear and not rectilinear, right, acute and obtuse angles, Defs. 12–22; figure, boundaries of figure, varieties of figure, plane, solid, composite (of homogeneous or non-homogeneous parts) and incomposite, Defs. 23–6. The incomposite plane figure is the circle, and definitions follow of its parts, segments (which are composite of non-homogeneous parts), the semicircle, the ἁψίς (less than a semicircle), and the segment greater than a semicircle, angles in segments, the sector,

'concave' and 'convex', lune, garland (these last two are composite of homogeneous parts) and *axe* (πέλεκυς), bounded by four circular arcs, two concave and two convex, Defs. 27–38. Rectilineal figures follow, the various kinds of triangles and of quadrilaterals, the gnomon in a parallelogram, and the gnomon in the more general sense of the figure which added to a given figure makes the whole into a similar figure, polygons, the parts of figures (side, diagonal, height of a triangle), perpendicular, parallels, the three figures which will fill up the space round a point, Defs. 39–73. Solid figures are next classified according to the surfaces bounding them, and lines on surfaces are divided into (1) simple and circular, (2) mixed, like the conic and spiric curves, Defs. 74, 75. The sphere is then defined, with its parts, and stated to be the figure which, of all figures having the same surface, is the greatest in content, Defs. 76–82. Next the cone, its different species and its parts are taken up, with the distinction between the three conics, the section of the acute-angled cone ('by some also called *ellipse*') and the sections of the right-angled and obtuse-angled cones (also called *parabola* and *hyperbola*), Defs. 83–94; the cylinder, a section in general, the *spire* or *tore* in its three varieties, open, continuous (or just closed) and 'crossing-itself', which respectively have sections possessing special properties, 'square rings' which are cut out of cylinders (i.e. presumably rings the cross-section of which through the centre is two squares), and various other figures cut out of spheres or mixed surfaces, Defs. 95–7; rectilineal solid figures, pyramids, the five regular solids, the semi-regular solids of Archimedes two of which (each with fourteen faces) were known to Plato, Defs. 98–104; prisms of different kinds, parallelepipeds, with the special varieties, the cube, the *beam*, δοκός (length longer than breadth and depth, which may be equal), the *brick*, πλινθίς (length less than breadth and depth), the σφηνίσκος or βωμίσκος with length, breadth and depth unequal, Defs. 105–14.

Lastly come definitions of relations, equality of lines, surfaces, and solids respectively, similarity of figures, 'reciprocal figures', Defs. 115–18; indefinite increase in magnitude, parts (which must be homogeneous with the wholes, so that e.g. the horn-like angle is not a part or submultiple of a right

or any angle), multiples, Defs. 119–21; proportion in magnitudes, what magnitudes can have a ratio to one another, magnitudes in the same ratio or magnitudes in proportion, definition of greater ratio, Defs. 122–5; transformation of ratios (*componendo, separando, convertendo, alternando, invertendo* and *ex aequali*), Defs. 126–7; commensurable and incommensurable magnitudes and straight lines, Defs. 128, 129. There follow two tables of measures, Defs. 130–2.

The *Definitions* are very valuable from the point of view of the historian of mathematics, for they give the different alternative definitions of the fundamental conceptions; thus we find the Archimedean 'definition' of a straight line, other definitions which we know from Proclus to be due to Apollonius, others from Posidonius, and so on. No doubt the collection may have been recast by some editor or editors after Heron's time, but it seems, at least in substance, to go back to Heron or earlier still. So far as it contains original definitions of Posidonius, it cannot have been compiled earlier than the first century B.C.; but its content seems to belong in the main to the period before the Christian era. Heiberg adds to his edition of the *Definitions* extracts from Heron's Geometry, postulates and axioms from Euclid, extracts from Geminus on the classification of mathematics, the principles of geometry, &c., extracts from Proclus or some early collection of scholia on Euclid, and extracts from Anatolius and Theon of Smyrna, which followed the actual definitions in the manuscripts. These various additions were apparently collected by some Byzantine editor, perhaps of the eleventh century.

Mensuration.

The *Metrica, Geometrica, Stereometrica, Geodaesia, Mensurae.*

We now come to the mensuration of Heron. Of the different works under this head the *Metrica* is the most important from our point of view because it seems, more than any of the others, to have preserved its original form. It is also more fundamental in that it gives the theoretical basis of the formulae used, and is not a mere application of rules to particular examples. It is also more akin to theory in that it

formed by the intersection of two cylinders with axes at right angles inscribed in a cube, also measured in the *Method* (15), the five regular solids (16–19). Book III deals with the division of figures into parts having given ratios to one another, first plane figures (1–19), then solids, a pyramid, a cone and a frustum, a sphere (20–3).

The *Geometria* or *Geometrumena* is a collection based upon Heron, but not his work in its present form. The addition of a theorem due to Patricius[1] and a reference to him in the *Stereometrica* (I. 22) suggest that Patricius edited both works, but the date of Patricius is uncertain. Tannery identifies him with a mathematical professor of the tenth century, Nicephorus Patricius; if this is correct, he would be contemporary with the Byzantine writer (erroneously called Heron) who is known to have edited genuine works of Heron, and indeed Patricius and the anonymous Byzantine might be one and the same person. The mensuration in the *Geometry* has reference almost entirely to the same figures as those measured in Book I of the *Metrica*, the difference being that in the *Geometry* (1) the rules are not explained but merely applied to examples, (2) a large number of numerical illustrations are given for each figure, (3) the Egyptian way of writing fractions as the sum of submultiples is followed, (4) lengths and areas are given in terms of particular measures, and the calculations are lengthened by a considerable amount of conversion from one measure into another. The first chapters (1–4) are of the nature of a general introduction, including certain definitions and ending with a table of measures. Chaps. 5–99, Hultsch (= 5–20, 14, Heib.), though for the most part corresponding in content to *Metrica* I, seem to have been based on a different collection, because chaps. 100–3 and 105 (= 21, 1–25, 22, 3–24, Heib.) are clearly modelled on the *Metrica*, and 101 is headed 'A definition (really 'measurement') of a circle in another book of Heron'. Heiberg transfers to the *Geometrica* a considerable amount of the content of the so-called *Liber Geeponicus*, a badly ordered collection consisting to a large extent of extracts from the other works. Thus it begins with 41 definitions identical with the same number of the *Definitiones*. Some sections

[1] *Geometrica*, 21 26 (vol. iv, p. 386. 23).

Heiberg puts side by side with corresponding sections of the *Geometrica* in parallel columns; others he inserts in suitable places; sections 78. 79 contain two important problems in indeterminate analysis (= *Geom.* 24, 1–2, Heib.). Heiberg adds, from the Constantinople manuscript containing the *Metrica*, eleven more sections (chap. 24, 3–13) containing indeterminate problems, and other sections (chap. 24, 14–30 and 37–51) giving the mensuration, mainly, of figures inscribed in or circumscribed to others, e.g. squares or circles in triangles, circles in squares, circles about triangles, and lastly of circles and segments of circles.

The *Stereometrica* I has at the beginning the title *Εἰσαγωγαὶ τῶν στερεομετρουμένων Ἥρωνος* but, like the *Geometrica*, seems to have been edited by Patricius. Chaps. 1–40 give the mensuration of the geometrical solid figures, the sphere, the cone, the frustum of a cone, the obelisk with circular base, the cylinder, the 'pillar', the cube. the *σφηνίσκος* (also called *ὄνυξ*), the *μείουρον προεσκαριφευμένον*, pyramids, and frusta. Some portions of this section of the book go back to Heron; thus in the measurement of the sphere chap. 1 = *Metrica* II. 11, and both here and elsewhere the ordinary form of fractions appears. Chaps. 41–54 measure the contents of certain buildings or other constructions, e.g. a theatre, an amphitheatre, a swimming-bath, a well, a ship, a wine-butt, and the like.

The second collection, *Stereometrica* II, appears to be of Byzantine origin and contains similar matter to *Stereometrica* I, parts of which are here repeated. Chap. 31 (27, Heib.) gives the problem of Thales. to find the height of a pillar or a tree by the measurement of shadows; the last sections measure various pyramids, a prism, a *βωμίσκος* (little altar).

The *Geodaesia* is not an independent work, but only contains extracts from the *Geometry*; thus chaps. 1–16 = *Geom.* 5–31, Hultsch (=5, 2–12, 32, Heib.); chaps. 17–19 give the methods of finding, in any scalene triangle the sides of which are given, the segments of the base made by the perpendicular from the vertex, and of finding the area direct by the well-known 'formula of Heron'; i.e. we have here the equivalent of *Metrica* I. 5–8.

Lastly, the *μετρήσεις*, or *Mensurae*, was attributed to Heron

in an Archimedes manuscript of the ninth century, but cannot in its present form be due to Heron, although portions of it have points of contact with the genuine works. Sects. 2-27 measure all sorts of objects, e.g. stones of different shapes, a pillar, a tower, a theatre, a ship, a vault, a hippodrome; but sects. 28-35 measure geometrical figures, a circle and segments of a circle (cf. *Metrica* I), and sects. 36-48 on spheres, segments of spheres, pyramids, cones and frusta are closely connected with *Stereom.* I and *Metrica* II; sects. 49-59, giving the mensuration of receptacles and plane figures of various shapes, seem to have a different origin. We can now take up the

Contents of the *Metrica*.

Book I. Measurement of Areas.

The preface records the tradition that the first geometry arose out of the practical necessity of measuring and distributing land (whence the name 'geometry'), after which extension to three dimensions became necessary in order to measure solid bodies. Heron then mentions Eudoxus and Archimedes as pioneers in the discovery of difficult measurements, Eudoxus having been the first to prove that a cylinder is three times the cone on the same base and of equal height, and that circles are to one another as the squares on their diameters, while Archimedes first proved that the surface of a sphere is equal to four times the area of a great circle in it, and the volume two-thirds of the cylinder circumscribing it.

(α) *Area of scalene triangle.*

After the easy cases of the rectangle, the right-angled triangle and the isosceles triangle, Heron gives two methods of finding the area of a scalene triangle (acute-angled or obtuse-angled) when the lengths of the three sides are given.

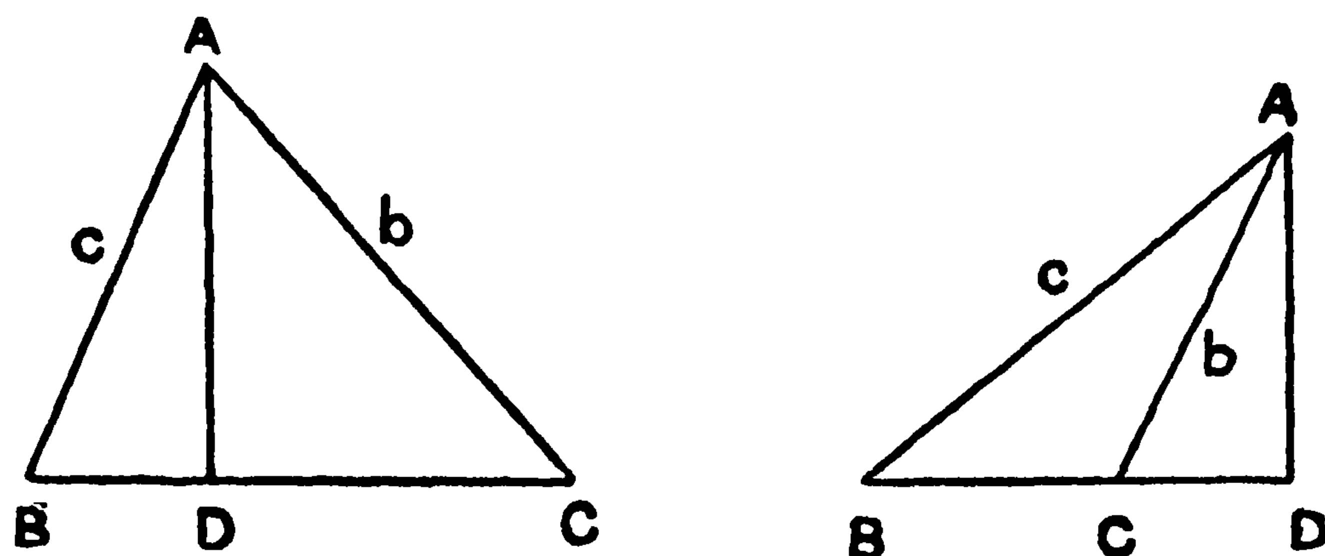
The first method is based on Eucl. II. 12 and 13. If a, b, c be the sides of the triangle opposite to the angles A, B, C respectively, Heron observes (chap. 4) that any angle, e.g. C, is acute, right or obtuse according as $c^2 < = \text{or} > a^2+b^2$, and this is the criterion determining which of the two propositions is applicable. The method is directed to determining, first the segments into which any side is divided by the perpendicular

from the opposite vertex, and thence the length of the perpendicular itself. We have, in the cases of the triangle acute-angled at C and the triangle obtuse-angled at C respectively,

$$c^2 = a^2 + b^2 \mp 2a \,.\, CD,$$

or

$$CD = \{(a^2 + b^2) \sim c^2\} \,/\, 2a,$$



whence $AD^2\,(= b^2 - CD^2)$ is found, so that we know the area $(= \frac{1}{2}a \,.\, AD)$.

In the cases given in *Metrica* I. 5, 6 the sides are (14, 15, 13) and (11, 13, 20) respectively, and AD is found to be rational (= 12). But of course both CD (or BD) and AD may be surds, in which case Heron gives approximate values. Cf. *Geom.* 53, 54, Hultsch (15. 1–4, Heib.), where we have a triangle in which $a = 8$, $b = 4$, $c = 6$, so that $a^2 + b^2 - c^2 = 44$ and $CD = 44/16 = 2\frac{1}{2}\,\frac{1}{4}$. Thus $AD^2 = 16 - (2\frac{1}{2}\,\frac{1}{4})^2 = 16 - 7\frac{1}{2}\,\frac{1}{16}$ $= 8\frac{1}{4}\,\frac{1}{8}\,\frac{1}{16}$, and $AD = \surd(8\frac{1}{4}\,\frac{1}{8}\,\frac{1}{16}) = 2\frac{2}{3}\,\frac{1}{4}$ approximately, whence the area $= 4 \times 2\frac{2}{3}\,\frac{1}{4} = 11\frac{2}{3}$. Heron then observes that we get a nearer result still if we multiply AD^2 by $(\frac{1}{2}a)^2$ before extracting the square root, for the area is then $\surd(16 \times 8\frac{1}{4}\,\frac{1}{8}\,\frac{1}{16})$ or $\surd(135)$, which is very nearly $11\frac{1}{2}\,\frac{1}{14}\,\frac{1}{21}$ or $11\frac{13}{21}$.

So in *Metrica* I. 9, where the triangle is 10, 8, 12 (10 being the base), Heron finds the perpendicular to be $\surd 63$, but he obtains the area as $\surd(\frac{1}{4}AD^2 \,.\, BC^2)$, or $\surd(1575)$, while observing that we *can*, of course, take the approximation to $\surd 63$, or $7\frac{1}{2}\,\frac{1}{4}\,\frac{1}{8}\,\frac{1}{16}$, and multiply it by half 10, obtaining $39\frac{1}{2}\,\frac{1}{8}\,\frac{1}{16}$ as the area.

Proof of the formula $\Delta = \surd\{s(s-a)(s-b)(s-c)\}$.

The second method is that known as the 'formula of Heron', namely, in our notation, $\Delta = \surd\{s(s-a)(s-b)(s-c)\}$. The proof of the formula is given in *Metrica* I. 8 and also in

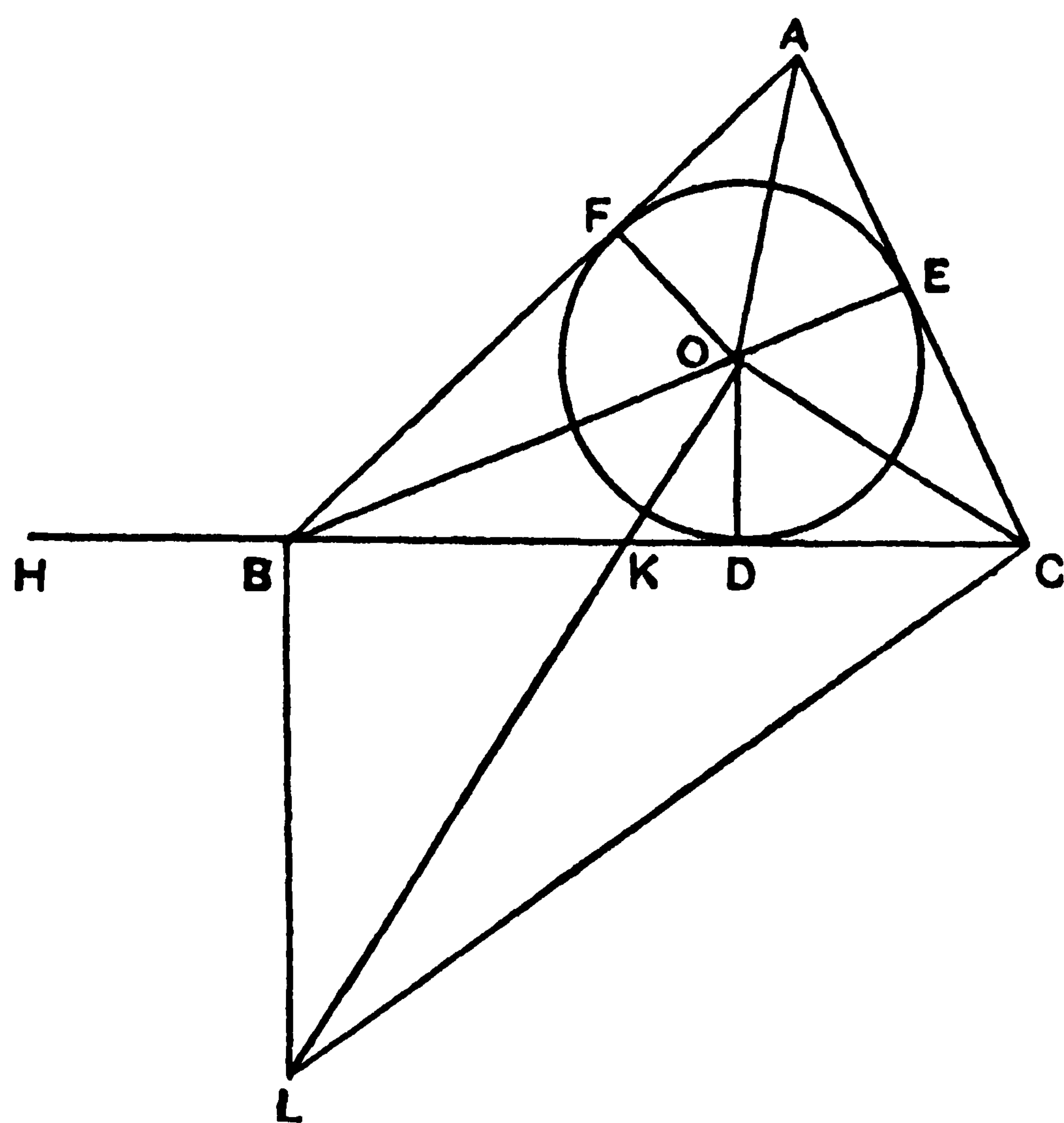
chap. 30 of the *Dioptra*; but it is now known (from Arabian sources) that the proposition is due to Archimedes.

Let the sides of the triangle ABC be given in length.

Inscribe the circle DEF, and let O be the centre.

Join AO, BO, CO, DO, EO, FO.

Then
$$BC \,.\, OD = 2\triangle BOC,$$
$$CA \,.\, OE = 2\triangle COA,$$
$$AB \,.\, OF = 2\triangle AOB;$$



whence, by addition,
$$p \,.\, OD = 2\triangle ABC,$$
where p is the perimeter.

Produce CB to H, so that $BH = AF$.

Then, since $AE = AF$, $BF = BD$, and $CE = CD$, we have $CH = \frac{1}{2}p = s$.

Therefore
$$CH \,.\, OD = \triangle ABC.$$

But $CH \,.\, OD$ is the 'side' of the product $CH^2 \,.\, OD^2$, i.e. $\sqrt{(CH^2 \,.\, OD^2)}$,

so that
$$(\triangle ABC)^2 = CH^2 \,.\, OD^2.$$

Draw OL at right angles to OC cutting BC in K, and BL at right angles to BC meeting OL in L. Join CL.

Then, since each of the angles COL, CBL is right, $COBL$ is a quadrilateral in a circle.

Therefore $$\angle COB + \angle CLB = 2R.$$

But $\angle COB + \angle AOF = 2R$, because AO, BO, CO bisect the angles round O, and the angles COB, AOF are together equal to the angles AOC, BOF, while the sum of all four angles is equal to $4R$.

Consequently $$\angle AOF = \angle CLB.$$

Therefore the right-angled triangles AOF, CLB are similar;

therefore $$BC : BL = AF : FO$$
$$= BH : OD,$$

and, alternately, $$CB : BH = BL : OD$$
$$= BK : KD;$$

whence, *componendo*, $$CH : HB = BD : DK.$$

It follows that

$$CH^2 : CH \,.\, HB = BD \,.\, DC : CD \,.\, DK$$
$$= BD \,.\, DC : OD^2, \text{ since the angle } COK \text{ is right.}$$

Therefore $$(\Delta ABC)^2 = CH^2 \,.\, OD^2 \text{ (from above)}$$
$$= CH \,.\, HB \,.\, BD \,.\, DC$$
$$= s(s-a)(s-b)(s-c).$$

(β) *Method of approximating to the square root of a non-square number.*

It is à propos of the triangle 7, 8, 9 that Heron gives the important statement of his method of approximating to the value of a surd, which before the discovery of the passage of the *Metrica* had been a subject of unlimited conjecture as bearing on the question how Archimedes obtained his approximations to $\sqrt{3}$.

In this case $s = 12$, $s-a = 5$, $s-b = 4$, $s-c = 3$, so that

$$\Delta = \sqrt{(12 \,.\, 5 \,.\, 4 \,.\, 3)} = \sqrt{(720)}.$$

'Since', says Heron,[1] '720 has not its side rational, we can obtain its side within a very small difference as follows. Since the next succeeding square number is 729, which has 27 for its side, divide 720 by 27. This gives $26\frac{2}{3}$. Add 27 to this, making $53\frac{2}{3}$, and take half of this or $26\frac{1}{2}\,\frac{1}{3}$. The side of 720 will therefore be very nearly $26\frac{1}{2}\,\frac{1}{3}$. In fact, if we multiply $26\frac{1}{2}\,\frac{1}{3}$ by itself, the product is $720\frac{1}{36}$, so that the difference (in the square) is $\frac{1}{36}$.

'If we desire to make the difference still smaller than $\frac{1}{36}$, we shall take $720\frac{1}{36}$ instead of 729 [or rather we should take $26\frac{1}{2}\,\frac{1}{3}$ instead of 27], and by proceeding in the same way we shall find that the resulting difference is much less than $\frac{1}{36}$.'

In other words, if we have a non-square number A, and a^2 is the nearest square number to it, so that $A = a^2 \pm b$, then we have, as the first approximation to $\sqrt{A}$,

$$\alpha_1 = \tfrac{1}{2}\left(a + \frac{A}{a}\right); \qquad (1)$$

for a second approximation we take

$$\alpha_2 = \tfrac{1}{2}\left(\alpha_1 + \frac{A}{\alpha_1}\right), \qquad (2)$$

and so on.[2]

[1] *Metrica*, i. 8, pp. 18. 22–20. 5.

[2] The method indicated by Heron was known to Barlaam and Nicolas Rhabdas in the fourteenth century. The equivalent of it was used by Luca Paciuolo (fifteenth–sixteenth century), and it was known to the other Italian algebraists of the sixteenth century. Thus Luca Paciuolo gave $2\frac{1}{2}$, $2\frac{9}{20}$ and $2\frac{881}{1960}$ as successive approximations to $\sqrt{6}$. He obtained the first as $2 + \frac{2}{2 \cdot 2}$, the second as $2\frac{1}{2} - \frac{(2\frac{1}{2})^2 - 6}{2 \cdot 2\frac{1}{2}}$, and the third as $\frac{49}{20} - \frac{(\frac{49}{20})^2 - 6}{2 \cdot \frac{49}{20}}$. The above rule gives $\frac{1}{2}(2 + \frac{6}{2}) = 2\frac{1}{2}$, $\frac{1}{2}(\frac{5}{2} + \frac{12}{5}) = 2\frac{9}{20}$, $\frac{1}{2}(\frac{49}{20} + \frac{120}{49}) = 2\frac{881}{1960}$.

The formula of Heron was again put forward, in modern times, by Buzengeiger as a means of accounting for the Archimedean approximation to $\sqrt{3}$, apparently without knowing its previous history. Bertrand also stated it in a treatise on arithmetic (1853). The method, too, by which Oppermann and Alexeieff sought to account for Archimedes's approximations is in reality the same. The latter method depends on the formula

$$\tfrac{1}{2}(\alpha + \beta) : \sqrt{(\alpha\beta)} = \sqrt{(\alpha\beta)} : \frac{2\alpha\beta}{\alpha + \beta}.$$

Alexeieff separated A into two factors a_0, b_0, and pointed out that if, say,

$$a_0 > \sqrt{A} > b_0,$$

then, $$\tfrac{1}{2}(a_0 + b_0) > \sqrt{A} > \frac{2A}{a_0 + b_0} \text{ or } \frac{2a_0 b_0}{a_0 + b_0},$$

Substituting in (1) the value $a^2 \pm b$ for A, we obtain

$$\alpha_1 = a \pm \frac{b}{2a}.$$

Heron does not seem to have used this formula with a negative sign, unless in *Stereom.* I. 33 (34, Hultsch), where $\sqrt{(63)}$

and again, if $\quad \frac{1}{2}(a_0 + b_0) = a_1, \;\; 2A/(a_0+b_0) = b_1,$

$$\tfrac{1}{2}(a_1 + b_1) > \sqrt{A} > \frac{2A}{a_1 + b_1},$$

and so on.

Now suppose that, in Heron's formulae, we put $a = X_0$, $A/a = x_0$, $\alpha_1 = X_1$, $A/\alpha_1 = x_1$, and so on. We then have

$$X_1 = \tfrac{1}{2}\left(a + \frac{A}{a}\right) = \tfrac{1}{2}(X_0 + x_0), \quad x_1 = \frac{A}{X_1} = \frac{A}{\frac{1}{2}(X_0 + x_0)} \text{ or } \frac{2X_0 x_0}{X_0 + x_0};$$

that is, X_1, x_1 are, respectively, the arithmetic and harmonic means between X_0, x_0; X_2, x_2 are the arithmetic and harmonic means between X_1, x_1, and so on, exactly as in Alexeieff's formulae.

Let us now try to apply the method to Archimedes's case, $\sqrt{3}$, and we shall see to what extent it serves to give what we want. Suppose we begin with $3 > \sqrt{3} > 1$. We then have

$$\tfrac{1}{2}(3+1) > \sqrt{3} > 3/\tfrac{1}{2}(3+1), \text{ or } 2 > \sqrt{3} > \tfrac{3}{2},$$

and from this we derive successively

$$\tfrac{7}{4} > \sqrt{3} > \tfrac{12}{7}, \;\; \tfrac{97}{56} > \sqrt{3} > \tfrac{168}{97}, \;\; \tfrac{18817}{10864} > \sqrt{3} > \tfrac{32592}{18817}.$$

But, if we start from $\frac{5}{3}$, obtained by the formula $a + \dfrac{b}{2a+1} < \sqrt{(a^2+b)}$, we obtain the following approximations by excess,

$$\tfrac{1}{2}(\tfrac{5}{3} + \tfrac{9}{5}) = \tfrac{26}{15}, \;\; \tfrac{1}{2}(\tfrac{26}{15} + \tfrac{45}{26}) = \tfrac{1351}{780}.$$

The second process then gives one of Archimedes's results, $\frac{1351}{780}$, but neither of the two processes gives the other, $\frac{265}{153}$, directly. The latter can, however, be obtained by using the formula that, if $\dfrac{a}{b} < \dfrac{c}{d}$, then $\dfrac{a}{b} < \dfrac{ma+nc}{mb+nd} < \dfrac{c}{d}$.

For we can obtain $\frac{265}{153}$ from $\frac{97}{56}$ and $\frac{168}{97}$ thus: $\dfrac{97+168}{56+97} = \dfrac{265}{153}$, or from $\frac{97}{56}$ and $\frac{7}{4}$ thus: $\dfrac{11.97-7}{11.56-4} = \dfrac{1060}{612} = \dfrac{265}{153}$; and so on. Or again $\frac{1351}{780}$ can be obtained from $\frac{18817}{10864}$ and $\frac{97}{56}$ thus: $\dfrac{18817+97}{10864+56} = \dfrac{18914}{10920} = \dfrac{1351}{780}$.

The advantage of the method is that, as compared with that of continued fractions, it is a very rapid way of arriving at a close approximation. Günther has shown that the $(m+1)$th approximation obtained by Heron's formula is the 2^mth obtained by continued fractions. ('Die quadratischen Irrationalitäten der Alten und deren Entwickelungsmethoden' in *Abhandlungen zur Gesch. d. Math.* iv. 1882, pp. 83-6.)

is given as approximately $8 - \frac{1}{16}$. In *Metrica* I. 9, as we have seen, $\sqrt{(63)}$ is given as $7\frac{1}{2}\ \frac{1}{4}\ \frac{1}{8}\ \frac{1}{16}$, which was doubtless obtained from the formula (1) as

$$\tfrac{1}{2}(8+\tfrac{63}{8}) = \tfrac{1}{2}(8+7\tfrac{7}{8}) = 7\tfrac{1}{2}\ \tfrac{1}{4}\ \tfrac{1}{8}\ \tfrac{1}{16}.$$

The above seems to be the only *classical* rule which has been handed down for finding second and further approximations to the value of a surd. But, although Heron thus shows how to obtain a second approximation, namely by formula (2), he does not seem to make any direct use of this method himself, and consequently the question how the approximations closer than the first which are to be found in his works were obtained still remains an open one.

(γ) *Quadrilaterals.*

It is unnecessary to give in detail the methods of measuring the areas of quadrilaterals (chaps. 11–16). Heron deals with the following kinds, the parallel-trapezium (isosceles or non-isosceles), the rhombus and rhomboid, and the quadrilateral which has one angle right and in which the four sides have given lengths. Heron points out that in the rhombus or rhomboid, and in the general case of the quadrilateral, it is necessary to know a diagonal as well as the four sides. The mensuration in all the cases reduces to that of the rectangle and triangle.

(δ) *The regular polygons with 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12 sides.*

Beginning with the *equilateral triangle* (chap. 17), Heron proves that, if a be the side and p the perpendicular from a vertex on the opposite side, $a^2 : p^2 = 4 : 3$, whence

$$a^4 : p^2 a^2 = 4 : 3 = 16 : 12,$$

so that
$$a^4 : (\Delta\, ABC)^2 = 16 : 3,$$

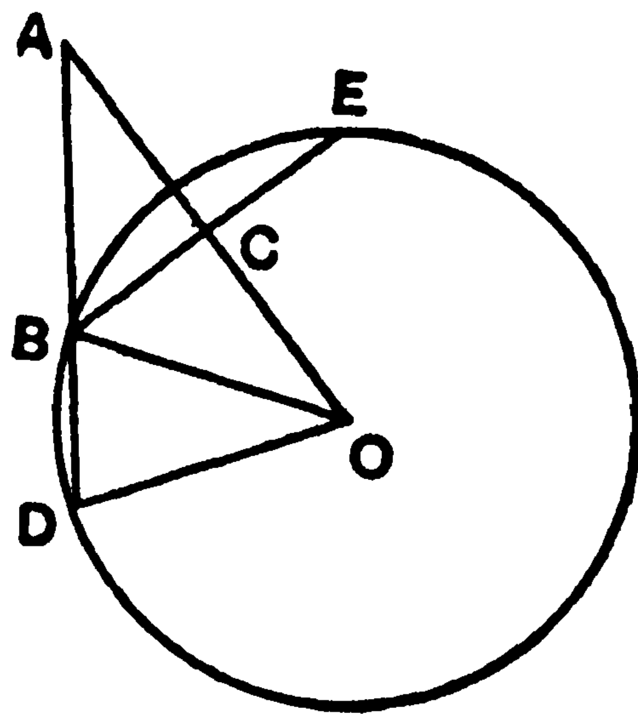
and $(\Delta\, ABC)^2 = \frac{3}{16}a^4$. In the particular case taken $a = 10$ and $\Delta^2 = 1875$, whence $\Delta = 43\frac{1}{3}$ nearly.

Another method is to use an approximate value for $\sqrt{3}$ in the formula $\sqrt{3}\,.\,a^2/4$. This is what is done in the *Geometrica* 14 (10, Heib.), where we are told that the area is $(\frac{1}{3}+\frac{1}{10})a^2$;

now $\frac{1}{3}+\frac{1}{10}=\frac{13}{30}=\frac{1}{4}(\frac{26}{15})$, so that the approximation used by Heron for $\sqrt{3}$ is here $\frac{26}{15}$. For the side 10, the method gives the same result as above, for $\frac{13}{30}\cdot 100 = 43\frac{1}{3}$.

The regular *pentagon* is next taken (chap. 18). Heron premises the following lemma.

Let ABC be a right-angled triangle, with the angle A equal to $\frac{2}{5}R$. Produce AC to O so that $CO = AC$. If now AO is divided in extreme and mean ratio, AB is equal to the greater segment. (For produce AB to D so that $AD = AO$, and join BO, DO. Then, since ADO is isosceles and the angle at $A = \frac{2}{5}R$, $\angle ADO = \angle AOD = \frac{4}{5}R$, and, from the equality of the triangles ABC, OBC, $\angle AOB = \angle BAO = \frac{2}{5}R$. It follows that the triangle ADO is the isosceles triangle of Eucl. IV. 10, and AD is divided in extreme and mean ratio in B.) Therefore, says Heron, $(BA+AC)^2 = 5AC^2$. [This is Eucl. XIII. 1.]



Now, since $\angle BOC = \frac{2}{5}R$, if BC be produced to E so that $CE = BC$, BE subtends at O an angle equal to $\frac{4}{5}R$, and therefore BE is the side of a regular pentagon inscribed in the circle with O as centre and OB as radius. (This circle also passes through D, and BD is the side of a regular decagon in the same circle.) If now $BO = AB = r$, $OC = p$, $BE = a$, we have from above, $(r+p)^2 = 5p^2$, whence, since $\sqrt{5}$ is approximately $\frac{9}{4}$, we obtain approximately $r = \frac{5}{4}p$, and $\frac{1}{2}a = \frac{3}{4}p$, so that $p = \frac{2}{3}a$. Hence $\frac{1}{2}pa = \frac{1}{3}a^2$, and the area of the pentagon $= \frac{5}{3}a^2$. Heron adds that, if we take a closer approximation to $\sqrt{5}$ than $\frac{9}{4}$, we shall obtain the area still more exactly. In the *Geometry*[1] the formula is given as $\frac{12}{7}a^2$.

The regular *hexagon* (chap. 19) is simply 6 times the equilateral triangle with the same side. If Δ be the area of the equilateral triangle with side a, Heron has proved that $\Delta^2 = \frac{3}{16}a^4$ (*Metrica* I. 17), hence $(\text{hexagon})^2 = \frac{27}{4}a^4$. If, e.g. $a = 10$, $(\text{hexagon})^2 = 67500$, and (hexagon) $= 259$ nearly. In the *Geometry*[2] the formula is given as $\frac{13}{5}a^2$, while 'another book' is quoted as giving $6(\frac{1}{3}+\frac{1}{10})a^2$; it is added that the latter formula, obtained from the area of the triangle, $(\frac{1}{3}+\frac{1}{10})a^2$, represents the more accurate procedure, and is fully set out by

[1] *Geom.* 102 (21, 14, Heib.). [2] *Ib.* 102 (21, 16, 17, Heib.).

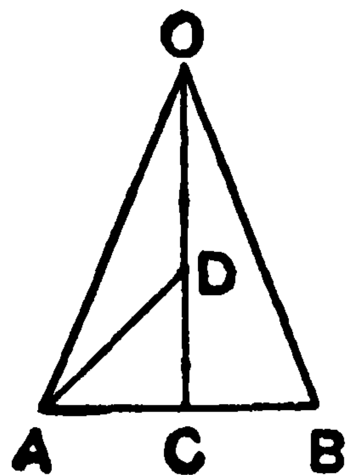
Heron. As a matter of fact, however, $6\left(\frac{1}{3}+\frac{1}{10}\right)=\frac{13}{5}$ exactly, and only the *Metrica* gives the more accurate calculation.

The regular *heptagon*.

Heron assumes (chap. 20) that, if a be the side and r the radius of the circumscribing circle, $a=\frac{7}{8}r$, being approximately equal to the perpendicular from the centre of the circle to the side of the regular hexagon inscribed in it (for $\frac{7}{8}$ is the approximate value of $\frac{1}{2}\sqrt{3}$). This theorem is quoted by Jordanus Nemorarius (d. 1237) as an 'Indian rule'; he probably obtained it from Abū'l Wafā (940–98). The *Metrica* shows that it is of Greek origin, and, if Archimedes really wrote a book on the heptagon in a circle, it may be due to him. If then p is the perpendicular from the centre of the circle on the side (a) of the inscribed heptagon, $r/(\frac{1}{2}a)=8/3\frac{1}{2}$ or $16/7$, whence $p^2/(\frac{1}{2}a)^2=\frac{207}{49}$, and $p/\frac{1}{2}a=$ (approximately) $14\frac{1}{3}/7$ or $43/21$. Consequently the area of the heptagon $=7\,.\,\frac{1}{2}pa=7\,.\,\frac{43}{84}a^2=\frac{43}{12}a^2$.

The regular *octagon*, *decagon* and *dodecagon*.

In these cases (chaps. 21, 23, 25) Heron finds p by drawing the perpendicular OC from O, the centre of the circumscribed circle, on a side AB, and then making the angle OAD equal to the angle AOD.



For the octagon,

$\angle ADC=\frac{1}{2}R$, and $p=\frac{1}{2}a\,(1+\sqrt{2})=\frac{1}{2}a\,(1+\frac{17}{12})$ or $\frac{1}{2}a\,.\,\frac{29}{12}$ approximately.

For the decagon,

$\angle ADC=\frac{2}{5}R$, and $AD:DC=5:4$ nearly (see preceding page);

hence $AD:AC=5:3$, and $p=\frac{1}{2}a\,(\frac{5}{3}+\frac{4}{3})=\frac{3}{2}a$.

For the dodecagon,

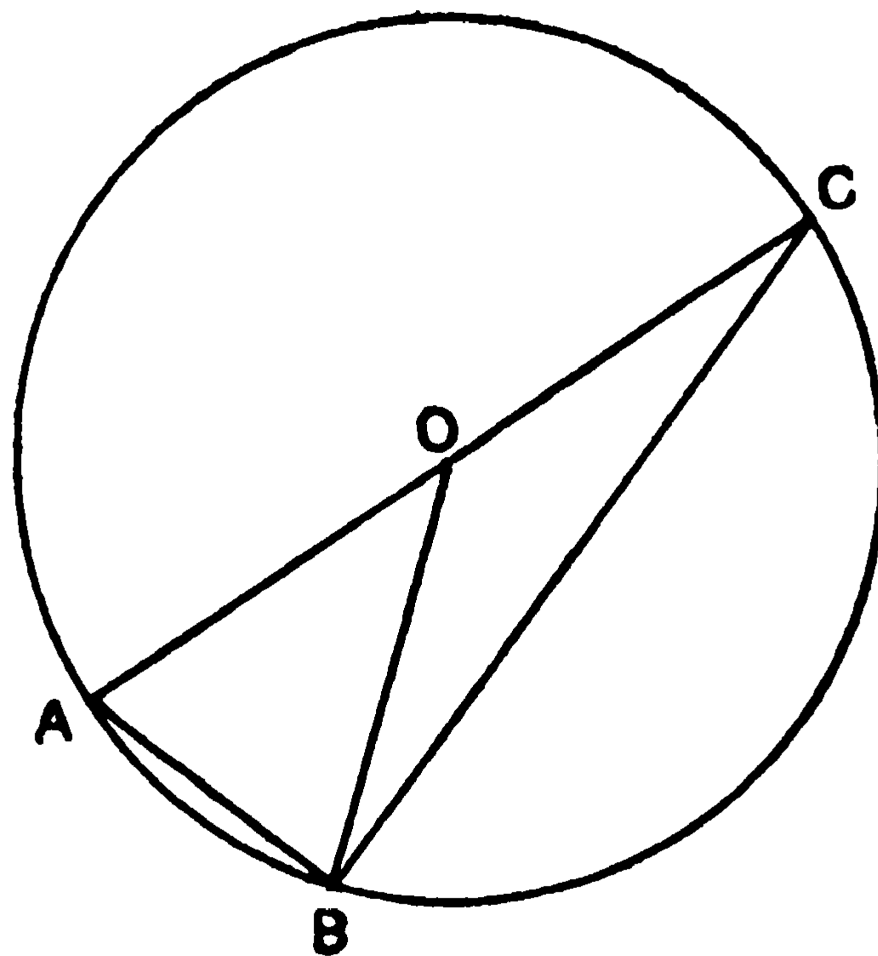
$\angle ADC=\frac{1}{3}R$, and $p=\frac{1}{2}a\,(2+\sqrt{3})=\frac{1}{2}a\,(2+\frac{7}{4})=\frac{15}{8}a$ approximately.

Accordingly $A_8=\frac{29}{6}a^2$, $A_{10}=\frac{15}{2}a^2$, $A_{12}=\frac{45}{4}a^2$, where a is the side in each case.

The regular *enneagon* and *hendecagon*.

In these cases (chaps. 22, 24) the Table of Chords (i.e.

presumably Hipparchus's Table) is appealed to. If AB be the side (a) of an enneagon or hendecagon inscribed in a circle, AC the diameter through A, we are told that the Table of Chords gives $\frac{1}{3}$ and $\frac{7}{25}$ as the respective approximate values of the ratio AB/AC. The angles subtended at the centre O by the side AB are $40°$ and $32\frac{8}{11}°$ respectively, and Ptolemy's Table gives, as the chords subtended by angles of $40°$ and $33°$ respectively, $41^p\ 2'\ 33''$ and $34^p\ 4'\ 55''$ (expressed in 120th parts of the diameter); Heron's figures correspond to 40^p and $33^p\ 36'$ respectively. For the *enneagon* $AC^2 = 9\,AB^2$, whence $BC^2 = 8\,AB^2$ or approximately $\frac{289}{36}AB^2$, and $BC = \frac{17}{6}a$; therefore (area of enneagon) $= \frac{9}{2}\,.\,\Delta ABC = \frac{51}{8}a^2$. For the *hendecagon* $AC^2 = \frac{625}{49}AB^2$ and $BC^2 = \frac{576}{49}AB^2$, so that $BC = \frac{24}{7}a$, and area of hendecagon $= \frac{11}{2}\,.\,\Delta ABC = \frac{66}{7}a^2$.



An ancient formula for the ratio between the side of any regular polygon and the diameter of the circumscribing circle is preserved in *Geëpon.* 147 sq. (= Pseudo-Dioph. 23–41), namely $d_n = n\frac{a_n}{3}$. Now the ratio na_n/d_n tends to π as the number (n) of sides increases, and the formula indicates a time when π was generally taken as $= 3$.

(ϵ) *The Circle.*

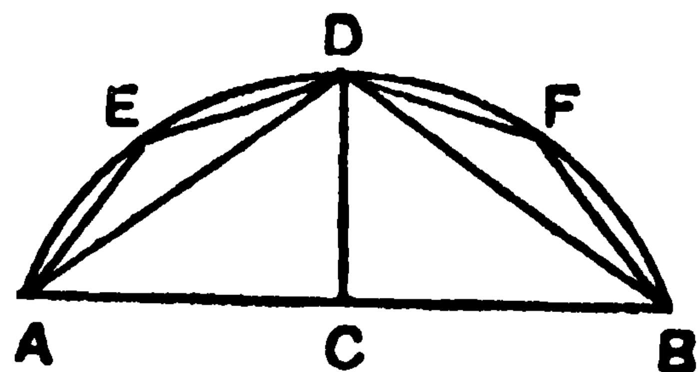
Coming to the circle (*Metrica* I. 26) Heron uses Archimedes's value for π, namely $\frac{22}{7}$, making the circumference of a circle $\frac{44}{7}r$ and the area $\frac{11}{14}d^2$, where r is the radius and d the diameter. It is here that he gives the more exact limits for π which he says that Archimedes found in his work *On Plinthides and Cylinders*, but which are not convenient for calculations. The limits, as we have seen, are given in the text as $\frac{211875}{67441} < \pi < \frac{197888}{62351}$, and with Tannery's alteration to $\frac{211872}{67441} < \pi < \frac{195882}{62351}$ are quite satisfactory.[1]

[1] See vol. i, pp. 232–3.

(ζ) *Segment of a circle.*

According to Heron (*Metrica* I. 30) the ancients measured the area of a segment rather inaccurately, taking the area to be $\frac{1}{2}(b+h)h$, where b is the base and h the height. He conjectures that it arose from taking $\pi = 3$, because, if we apply the formula to the semicircle, the area becomes $\frac{1}{2} \cdot 3r^2$, where r is the radius. Those, he says (chap. 31), who have investigated the area more accurately have added $\frac{1}{14}(\frac{1}{2}b)^2$ to the above formula, making it $\frac{1}{2}(b+h)h + \frac{1}{14}(\frac{1}{2}b)^2$, and this seems to correspond to the value $3\frac{1}{7}$ for π, since, when applied to the semicircle, the formula gives $\frac{1}{2}(3r^2 + \frac{1}{7}r^2)$. He adds that this formula should only be applied to segments of a circle less than a semicircle, and not even to all of these, but only in cases where b is not greater than $3h$. Suppose e.g. that $b = 60$, $h = 1$; in that case even $\frac{1}{14}(\frac{1}{2}b)^2 = \frac{1}{14} \cdot 900 = 64\frac{2}{7}$, which is greater even than the parallelogram with 60, 1 as sides, which again is greater than the segment. Where therefore $b > 3h$, he adopts another procedure.

This is exactly modelled on Archimedes's quadrature of a segment of a parabola. Heron proves (*Metrica* I. 27–29, 32) that, if ADB be a segment of a circle, and D the middle point of the arc, and if the arcs AD, DB be similarly bisected at E, F,



$$\Delta ADB < 4(\Delta AED + \Delta DFB).$$

Similarly, if the same construction be made for the segments AED, BFD, each of them is less than 4 times the sum of the two small triangles in the segments left over. It follows that

$$(\text{area of segmt. } ADB) > \Delta ADB\{1 + \tfrac{1}{4} + (\tfrac{1}{4})^2 + \ldots\}$$
$$> \tfrac{4}{3}\Delta ADB.$$

'If therefore we measure the triangle, and add one-third of it, we shall obtain the area of the segment as nearly as possible.' That is, for segments in which $b > 3h$, Heron takes the area to be equal to that of the parabolic segment with the same base and height, or $\frac{2}{3}bh$.

In addition to these three formulae for S, the area of a segment, there are yet others, namely

$$S = \tfrac{1}{2}(b+h)h(1 + \tfrac{1}{21}), \quad \textit{Mensurae } 29,$$
$$S = \tfrac{1}{2}(b+h)h(1 + \tfrac{1}{16}), \quad \text{,, } \quad 31.$$

The first of these formulae is applied to a segment greater than a semicircle, the second to a segment less than a semicircle.

In the *Metrica* the area of a segment greater than a semicircle is obtained by subtracting the area of the complementary segment from the area of the circle.

From the *Geometrica*[1] we find that the circumference of the segment less than a semicircle was taken to be $\sqrt{(b^2 + 4h^2)} + \frac{1}{4}h$ or alternatively $\sqrt{(b^2 + 4h^2)} + \{\sqrt{(b^2 + 4h^2)} - b\}\frac{h}{b}$.

(η) *Ellipse, parabolic segment, surface of cylinder, right cone, sphere and segment of sphere.*

After the area of an ellipse (*Metrica* I. 34) and of a parabolic segment (chap. 35), Heron gives the surface of a cylinder (chap. 36) and a right cone (chap. 37); in both cases he unrolls the surface on a plane so that the surface becomes that of a parallelogram in the one case and a sector of a circle in the other. For the surface of a sphere (chap. 38) and a segment of it (chap. 39) he simply uses Archimedes's results.

Book I ends with a hint how to measure irregular figures, plane or not. If the figure is plane and bounded by an irregular curve, neighbouring points are taken on the curve such that, if they are joined in order, the contour of the polygon so formed is not much different from the curve itself, and the polygon is then measured by dividing it into triangles. If the surface of an irregular solid figure is to be found, you wrap round it pieces of very thin paper or cloth, enough to cover it, and you then spread out the paper or cloth and measure that.

Book II. Measurement of volumes.

The preface to Book II is interesting as showing how vague the traditions about Archimedes had already become.

'After the measurement of surfaces, rectilinear or not, it is proper to proceed to the solid bodies, the surfaces of which we have already measured in the preceding book, surfaces plane and spherical, conical and cylindrical, and irregular surfaces as well. The methods of dealing with these solids are, in

[1] Cf. *Geom.*, 94, 95 (19. 2, 4, Heib.), 97. 4 (20. 7, Heib.).

view of their surprising character, referred to Archimedes by certain writers who give the traditional account of their origin. But whether they belong to Archimedes or another, it is necessary to give a sketch of these methods as well.'

The Book begins with generalities about figures all the sections of which parallel to the base are equal to the base and similarly situated, while the centres of the sections are on a straight line through the centre of the base, which may be either obliquely inclined or perpendicular to the base; whether the said straight line ('the axis') is or is not perpendicular to the base, the volume is equal to the product of the area of the base and the *perpendicular* height of the top of the figure from the base. The term 'height' is thenceforward restricted to the length of the perpendicular from the top of the figure on the base.

(α) *Cone, cylinder, parallelepiped (prism), pyramid, and frustum.*

II. 1–7 deal with a cone, a cylinder, a 'parallelepiped' (the base of which is not restricted to the parallelogram but is in the illustration given a regular hexagon, so that the figure is more properly a prism with polygonal bases), a triangular prism, a pyramid with base of any form, a frustum of a triangular pyramid; the figures are in general *oblique*.

(β) *Wedge-shaped solid* (βωμίσκος *or* σφηνίσκος).

II. 8 is a case which is perhaps worth giving. It is that of a rectilineal solid, the base of which is a rectangle $ABCD$ and has opposite to it another rectangle $EFGH$, the sides of which are respectively parallel but not necessarily proportional to those of $ABCD$. Take AK equal to EF, and BL equal to FG. Bisect BK, CL in V, W, and draw $KRPU$, $VQOM$ parallel to AD, and $LQRN$, $WOPT$ parallel to AB. Join FK, GR, LG, GU, HN.

Then the solid is divided into (1) the parallelepiped with AR, EG as opposite faces, (2) the prism with KL as base and FG as the opposite edge, (3) the prism with NU as base and GH as opposite edge, and (4) the pyramid with $RLCU$ as base and G as vertex. Let h be the 'height' of the figure. Now

the parallelepiped (1) is on AR as base and has height h; the prism (2) is equal to a parallelepiped on KQ as base and with height h; the prism (3) is equal to a parallelepiped with NP as base and height h; and finally the pyramid (4) is equal to a parallelepiped of height h and one-third of RC as base.

Therefore the whole solid is equal to one parallelepiped with height h and base equal to $(AR+KQ+NP+RO+\frac{1}{3}RO)$ or $AO+\frac{1}{3}RO$.

Now, if $AB = a$, $BC = b$, $EF = c$, $FG = d$,

$$AV = \tfrac{1}{2}(a+c), AT = \tfrac{1}{2}(b+d), RQ = \tfrac{1}{2}(a-c), RP = \tfrac{1}{2}(b-d).$$

Therefore volume of solid

$$= \{\tfrac{1}{4}(a+c)\,(b+d)+\tfrac{1}{12}(a-c)\,(b-d)\}\,h.$$

The solid in question is evidently the true βωμίσκος ('little altar'), for the formula is used to calculate the content of a βωμίσκος in *Stereom.* II. 40 (68, Heib.) It is also, I think, the σφηνίσκος ('little wedge'), a measurement of which is given in *Stereom.* I. 26 (25, Heib.) It is true that the second term of the first factor $\frac{1}{12}(a-c)\,(b-d)$ is there neglected, perhaps because in the case taken ($a = 7$, $b = 6$, $c = 5$, $d = 4$) this term ($= \frac{1}{3}$) is small compared with the other ($= 30$). A particular σφηνίσκος, in which either $c = a$ or $d = b$, was called ὄνυξ; the second term in the factor of the content vanishes in this case, and, if e.g. $c = a$, the content is $\frac{1}{2}(b+d)ah$. Another βωμίσκος is measured in *Stereom.* I. 35 (34, Heib.), where the solid is inaccurately called 'a pyramid oblong (ἑτερομήκης) and truncated (κόλουρος) or half-perfect'.

The method is the same *mutatis mutandis* as that used in II. 6 for the frustum of a pyramid with any triangle for base, and it is applied in II. 9 to the case of a frustum of a pyramid with a square base, the formula for which is

$$[\{\tfrac{1}{2}(a+a')\}^2+\tfrac{1}{3}\{\tfrac{1}{2}(a-a')\}^2]h,$$

where a, a' are the sides of the larger and smaller bases respectively, and h the height; the expression is of course easily reduced to $\frac{1}{3}h(a^2+aa'+a'^2)$.

(γ) *Frustum of cone, sphere, and segment thereof.*

A *frustum of a cone* is next measured in two ways, (1) by comparison with the corresponding frustum of the circumscribing pyramid with square base, (2) directly as the difference between two cones (chaps. 9, 10). The volume of the frustum of the cone is to that of the frustum of the circumscribing pyramid as the area of the base of the cone to that of the base of the pyramid; i.e. the volume of the frustum of the cone is $\frac{1}{4}\pi$, or $\frac{11}{14}$, times the above expression for the frustum of the pyramid with a^2, a'^2 as bases, and it reduces to $\frac{1}{12}\pi h\ (a^2+aa'+a'^2)$, where a, a' are the *diameters* of the two bases. For the *sphere* (chap. 11) Heron uses Archimedes's proposition that the circumscribing cylinder is $1\frac{1}{2}$ times the sphere, whence the volume of the sphere $= \frac{2}{3}.d.\frac{11}{14}d^2$ or $\frac{11}{21}d^3$; for a *segment of a sphere* (chap. 12) he likewise uses Archimedes's result (*On the Sphere and Cylinder*, II. 4).

(δ) *Anchor-ring or tore.*

The anchor-ring or *tore* is next measured (chap. 13) by means of a proposition which Heron quotes from Dionysodorus, and which is to the effect that, if a be the radius of either circular section of the *tore* through the axis of revolution, and c the distance of its centre from that axis,

$$\pi a^2 : ac = (\text{volume of tore}) : \pi c^2 . 2a$$

[whence volume of tore $= 2\pi^2 ca^2$]. In the particular case taken $a = 6$, $c = 14$, and Heron obtains, from the proportion $113\frac{1}{7} : 84 = V : 7392$, $V = 9956\frac{4}{7}$. But he shows that he is aware that the volume is the product of the area of the

describing circle and the length of the path of its centre. For, he says, since 14 is a radius (of the path of the centre), 28 is its diameter and 88 its circumference. 'If then the tore be straightened out and made into a cylinder, it will have 88 for its length, and the diameter of the base of the cylinder is 12; so that the solid content of the cylinder is, as we have seen, $9956\frac{4}{7}$' $(= 88 \cdot \frac{11}{14} \cdot 144)$.

(ϵ) *The two special solids of Archimedes's 'Method'.*

Chaps. 14, 15 give the measurement of the two remarkable solids of Archimedes's *Method*, following Archimedes's results.

(ζ) *The five regular solids.*

In chaps. 16–18 Heron measures the content of the five regular solids after the cube. He has of course in each case to find the perpendicular from the centre of the circumscribing sphere on any face. Let p be this perpendicular, a the edge of the solid, r the radius of the circle circumscribing any face. Then (1) for the *tetrahedron*

$$a^2 = 3r^2,\ p^2 = a^2 - \tfrac{1}{3}a^2 = \tfrac{2}{3}a^2.$$

(2) In the case of the *octahedron*, which is the sum of two equal pyramids on a square base, the content is one-third of that base multiplied by the diagonal of the figure, i.e. $\frac{1}{3} \cdot a^2 \cdot \sqrt{2}\,a$ or $\frac{1}{3}\sqrt{2} \cdot a^3$; in the case taken $a = 7$, and Heron takes 10 as an approximation to $\sqrt{(2 \cdot 7^2)}$ or $\sqrt{98}$, the result being $\frac{1}{3} \cdot 10 \cdot 49$ or $163\frac{1}{3}$. (3) In the case of the *icosahedron* Heron merely says that

$$p : a = 93 : 127 \left(\text{the real value of the ratio is } \tfrac{1}{2}\sqrt{\frac{7 + 3\sqrt{5}}{6}}\right).$$

(4) In the case of the *dodecahedron*, Heron says that $p : a = 9 : 8$ (the true value is $\frac{1}{2}\sqrt{\frac{25 + 11\sqrt{5}}{10}}$, and, if $\sqrt{5}$ is put equal to $\frac{9}{4}$, Heron's ratio is readily obtained).

Book II ends with an allusion to the method attributed to Archimedes for measuring the contents of irregular bodies by immersing them in water and measuring the amount of fluid displaced.

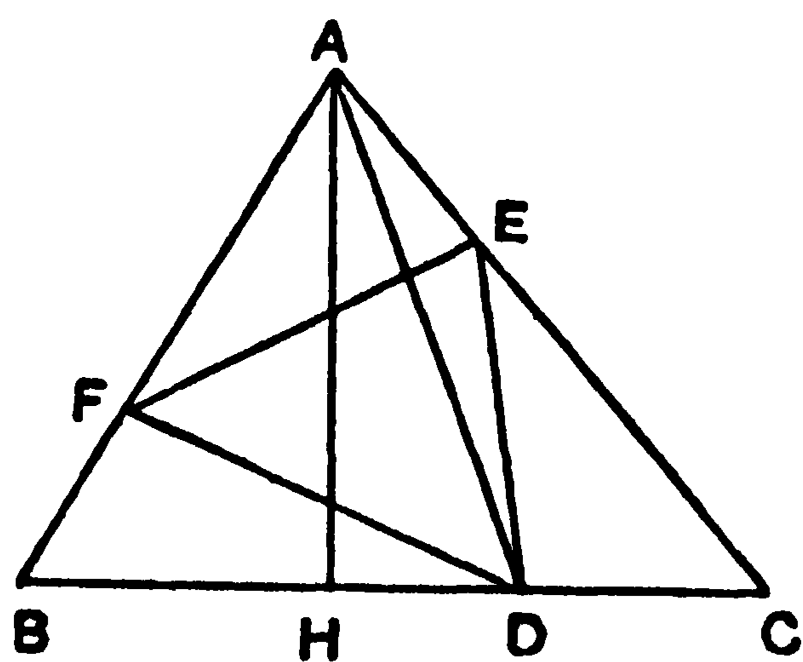
Book III. Divisions of figures.

This book has much in common with Euclid's book *On divisions (of figures)*, the problem being to divide various figures, plane or solid, by a straight line or plane into parts having a given ratio. In III. 1–3 a triangle is divided into two parts in a given ratio by a straight line (1) passing through a vertex, (2) parallel to a side, (3) through any point on a side. III. 4 is worth description: 'Given a triangle ABC, to cut out of it a triangle DEF (where D, E, F are points on the sides respectively) given in magnitude and such that the triangles AEF, BFD, CED may be equal in area.' Heron *assumes* that, if D, E, F divide the sides so that

$$AF : FB = BD : DC = CE : EA,$$

the latter three triangles are equal in area.

He then has to find the value of each of the three ratios which will result in the triangle DEF having a given area.



Join AD.

Since $BD : CD = CE : EA$,

$$BC : CD = CA : AE,$$

and $\Delta ABC : \Delta ADC = \Delta ADC : \Delta ADE$.

Also $\Delta ABC : \Delta ABD = \Delta ADC : \Delta EDC$.

But (since the area of the triangle DEF is given) ΔEDC is given, as well as ΔABC. Therefore $\Delta ABD \times \Delta ADC$ is given.

Therefore, if AH be perpendicular to BC,

$$AH^2 . BD . DC \text{ is given;}$$

therefore $BD . DC$ is given, and, since BC is given, D is given in position (we have to apply to BC a rectangle equal to $BD . DC$ and falling short by a square).

As an example Heron takes $AB = 13$, $BC = 14$, $CA = 15$, $\Delta DEF = 24$. ΔABC is then 84, and $AH = 12$.

Thus $\Delta EDC = 20$, and $AH^2 . BD . DC = 4 . 84 . 20 = 6720$;

therefore $BD . DC = 6720/144$ or $46\frac{2}{3}$ (the text omits the $\frac{2}{3}$).

Therefore, says Heron, $BD = 8$ approximately. For 8 we

should apparently have $8\frac{1}{2}$, since DC is immediately stated to be $5\frac{1}{2}$ (not 6). That is, in solving the equation

$$x^2 - 14x + 46\tfrac{2}{3} = 0,$$

which gives $x = 7 \pm \sqrt{(2\frac{1}{3})}$, Heron apparently substituted $2\frac{1}{4}$ or $\frac{9}{4}$ for $2\frac{1}{3}$, thereby obtaining $1\frac{1}{2}$ as an approximation to the surd.

(The lemma assumed in this proposition is easily proved. Let $m:n$ be the ratio $AF:FB = BD:DC = CE:EA$.

Then $AF = mc/(m+n)$, $FB = nc/(m+n)$, $CE = mb/(m+n)$,
$EA = nb/(m+n)$, &c.

Hence

$$\triangle AFE/\triangle ABC = \frac{mn}{(m+n)^2} = \triangle BDF/\triangle ABC = \triangle CDE/\triangle ABC,$$

and the triangles AFE, BDF, CDE are equal.

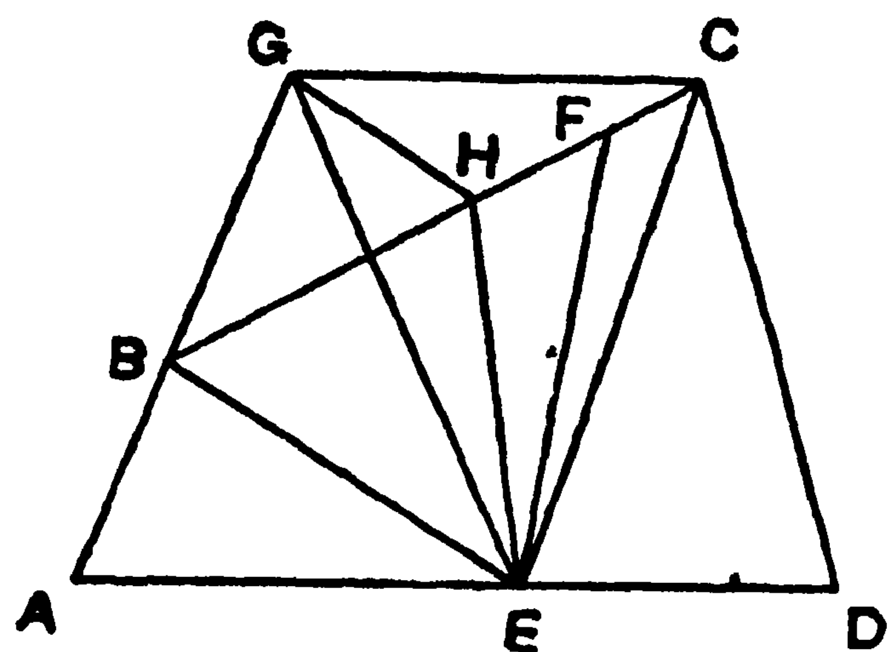
Pappus[1] has the proposition that the triangles ABC, DEF have the same centre of gravity.)

Heron next shows how to divide a parallel-trapezium into two parts in a given ratio by a straight line (1) through the point of intersection of the non-parallel sides, (2) through a given point on one of the parallel sides, (3) parallel to the parallel sides, (4) through a point on one of the non-parallel sides (III. 5–8). III. 9 shows how to divide the area of a circle into parts which have a given ratio by means of an inner circle with the same centre. For the problems beginning with III. 10 Heron says that numerical calculation alone no longer suffices, but geometrical methods must be applied. Three problems are reduced to problems solved by Apollonius in his treatise *On cutting off an area*. The first of these is III. 10, to cut off from the angle of a triangle a given proportion of the triangle by a straight line through a point on the opposite side produced. III. 11, 12, 13 show how to cut any quadrilateral into parts in a given ratio by a straight line through a point (1) on a side (*a*) dividing the side in the given ratio, (*b*) not so dividing it, (2) not on any side, (*a*) in the case where the quadrilateral is a trapezium, i.e. has two sides parallel, (*b*) in the case where it is not; the last case (*b*) is reduced (like III. 10) to the 'cutting-off of an

[1] Pappus, viii, pp. 1034–8. Cf. pp. 430–2 *post*.

area'. These propositions are ingenious and interesting. III. 11 shall be given as a specimen.

Given any quadrilateral $ABCD$ and a point E on the side AD, to draw through E a straight line EF which shall cut the quadrilateral into two parts in the ratio of AE to ED. (We omit the analysis.) Draw CG parallel to DA to meet AB produced in G.



Join BE, and draw GH parallel to BE meeting BC in H.

Join CE, EH, EG.

Then $\Delta GBE = \Delta HBE$ and, adding ΔABE to each, we have

$$\Delta AGE = (\text{quadrilateral } ABHE).$$

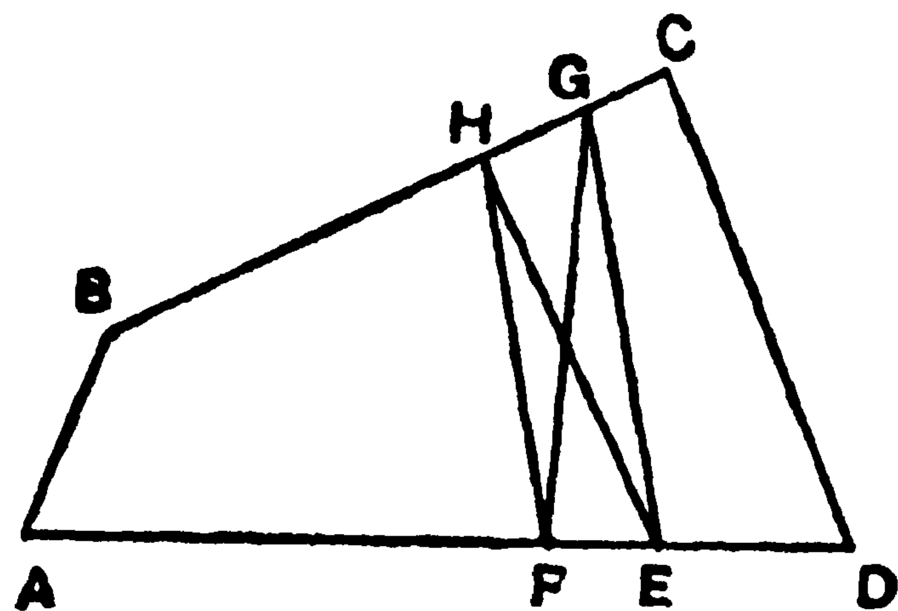
Therefore (quadr. $ABHE$) : $\Delta CED = \Delta GAE : \Delta CED$

$$= AE : ED.$$

But (quadr. $ABHE$) and ΔCED are parts of the quadrilateral, and they leave over only the triangle EHC. We have therefore only to divide ΔEHC in the same ratio $AE : ED$ by the straight line EF. This is done by dividing HC at F in the ratio $AE : ED$ and joining EF.

The next proposition (III. 12) is easily reduced to this.

If $AE : ED$ is not equal to the given ratio, let F divide AD in the given ratio, and through F draw FG dividing the quadrilateral in the given ratio (III. 11).



Join EG, and draw FH parallel to EG. Let FH meet BC in H, and join EH.

Then is EH the required straight line through E dividing the quadrilateral in the given ratio.

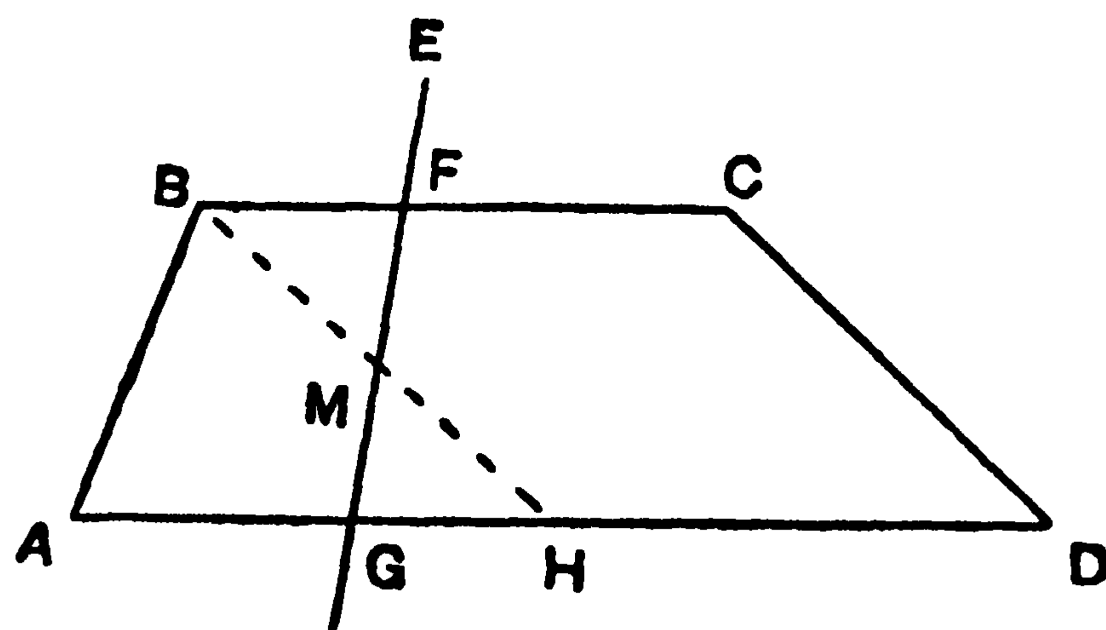
For $\Delta FGE = \Delta HGE$. Add to each (quadr. $GEDC$).

Therefore (quadr. $CGFD$) = (quadr. $CHED$).

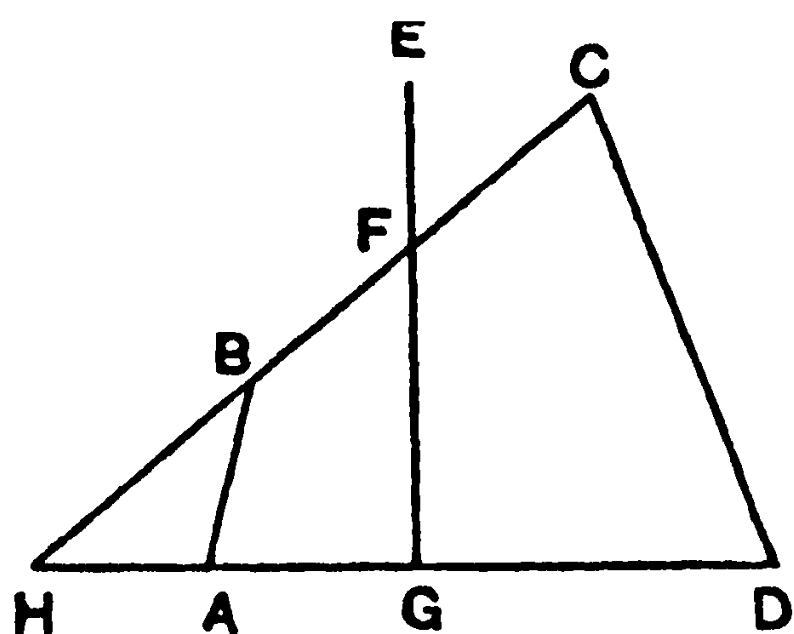
Therefore EH divides the quadrilateral in the given ratio, just as FG does.

The case (III. 13) where E is not on a side of the quadrilateral [(2) above] takes two different forms according as the

two opposite sides which the required straight line cuts are (*a*) parallel or (*b*) not parallel. In the first case (*a*) the problem reduces to drawing a straight line through *E* intersecting the parallel sides in points *F*, *G* such that $BF+AG$



is equal to a given length. In the second case (*b*) where *BC*, *AD* are not parallel Heron supposes them to meet in *H*. The angle at *H* is then given, and the area *ABH*. It is then a question of cutting off from a triangle with vertex *H* a triangle *HFG* of given area by a straight line drawn from *E*, which is again a problem in Apollonius's *Cutting-off of an*

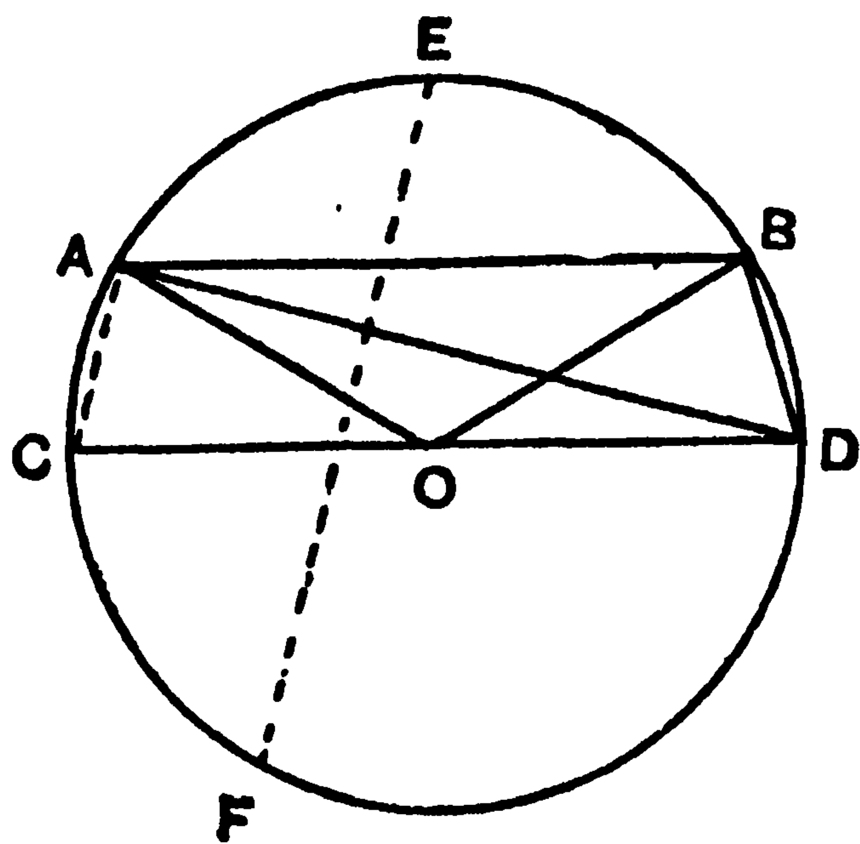


area. The auxiliary problem in case (*a*) is easily solved in III. 16. Measure *AH* equal to the given length. Join *BH* and bisect it at *M*. Then *EM* meets *BC*, *AD* in points such that $BF+AG=$ the given length. For, by congruent triangles, $BF = GH$.

The same problems are solved for the case of any polygon in III. 14, 15. A sphere is then divided (III. 17) into segments such that their surfaces are in a given ratio, by means of Archimedes, *On the Sphere and Cylinder*, II. 3, just as, in III. 23, Prop. 4 of the same Book is used to divide a sphere into segments having their volumes in a given ratio.

III. 18 is interesting because it recalls an ingenious proposition in Euclid's book *On Divisions*. Heron's problem is 'To divide a given circle into three equal parts by two straight

lines', and he observes that, 'as the problem is clearly not rational, we shall, for practical convenience, make the division, as exactly as possible, in the following way.' AB is the side of an equilateral triangle inscribed in the circle. Let CD be the parallel diameter, O the centre of the circle, and join AO, BO, AD, DB. Then shall the segment ABD be very nearly one-third of the circle. For, since AB is the side of an equilateral triangle in the circle, the sector $OAEB$ is one-third of the circle. And the triangle AOB forming part of the sector is equal to the triangle ADB; therefore the segment AEB *plus* the triangle ABD is equal to one-third of the circle, and the segment ABD only differs from this by the small segment on BD as base, which may be neglected. Euclid's proposition is to cut off one-third (or any fraction) of a circle between two parallel chords (see vol. i, pp. 429–30).



III. 19 finds a point D within any triangle ABC such that the triangles DBC, DCA, DAB are all equal; and then Heron passes to the division of solid figures.

The solid figures divided in a given ratio (besides the sphere) are the pyramid with base of any form (III. 20), the cone (III. 21) and the frustum of a cone (III. 22), the cutting planes being parallel to the base in each case. These problems involve the extraction of the cube root of a number which is in general not an exact cube, and the point of interest is Heron's method of approximating to the cube root in such a case. Take the case of the cone, and suppose that the portion to be cut off at the top is to the rest of the cone as m to n. We have to find the ratio in which the height or the edge is cut by the plane parallel to the base which cuts the cone in the given ratio. The volume of a cone being $\frac{1}{3}\pi c^2 h$, where c is the radius of the base and h the height, we have to find the height of the cone the volume of which is $\frac{m}{m+n}\cdot\frac{1}{3}\pi c^2 h$, and, as the height h' is to the radius c' of its base as h is to c, we have simply to find h' where

$h'^3/h^3 = m/(m+n)$. Or, if we take the edges e, e' instead of the heights, $e'^3/e^3 = m/(m+n)$. In the case taken by Heron $m:n = 4:1$, and $e = 5$. Consequently $e'^3 = \frac{4}{5} . 5^3 = 100$. Therefore, says Heron, $e' = 4\frac{9}{14}$ approximately, and in III. 20 he shows how this is arrived at.

Approximation to the cube root of a non-cube number.

'Take the nearest cube numbers to 100 both above and below; these are 125 and 64.

Then $125 - 100 = 25$,

and $100 - 64 = 36$.

Multiply 5 into 36; this gives 180. Add 100, making 280. ⟨Divide 180 by 280⟩; this gives $\frac{9}{14}$. Add this to the side of the smaller cube: this gives $4\frac{9}{14}$. This is as nearly as possible the cube root ("cubic side") of 100 units.'

We have to conjecture Heron's formula from this example. Generally, if $a^3 < A < (a+1)^3$, suppose that $A - a^3 = d_1$, and $(a+1)^3 - A = d_2$. The best suggestion that has been made is Wertheim's,[1] namely that Heron's formula for the approximate cube root was $a + \frac{(a+1)d_1}{(a+1)d_1 + ad_2}$. The 5 multiplied into the 36 might indeed have been the square root of 25 or $\sqrt{d_2}$, and the 100 added to the 180 in the denominator of the fraction might have been the original number 100 (A) and not 4.25 or ad_2, but Wertheim's conjecture is the more satisfactory because it can be evolved out of quite elementary considerations. This is shown by G. Eneström as follows.[2] Using the same notation, Eneström further supposes that x is the exact value of $\sqrt[3]{A}$, and that $(x-a)^3 = \delta_1$, $(a+1-x)^3 = \delta_2$.

Thus

$$\delta_1 = x^3 - 3x^2a + 3xa^2 - a^3, \text{ and } 3ax(x-a) = x^3 - a^3 - \delta_1 = d_1 - \delta_1.$$

Similarly from $\delta_2 = (a+1-x)^3$ we derive

$$3(a+1)x(a+1-x) = (a+1)^3 - x^3 - \delta_2 = d_2 - \delta_2.$$

Therefore

$$\frac{d_2 - \delta_2}{d_1 - \delta_1} = \frac{3(a+1)x(a+1-x)}{3ax(x-a)} = \frac{(a+1)\{1-(x-a)\}}{a(x-a)}$$

$$= \frac{a+1}{a(x-a)} - \frac{a+1}{a};$$

[1] *Zeitschr. f. Math. u. Physik*, xliv, 1899, hist.-litt. Abt., pp. 1–3.

[2] *Bibliotheca Mathematica*, viii$_3$, 1907–8, pp. 412–13.

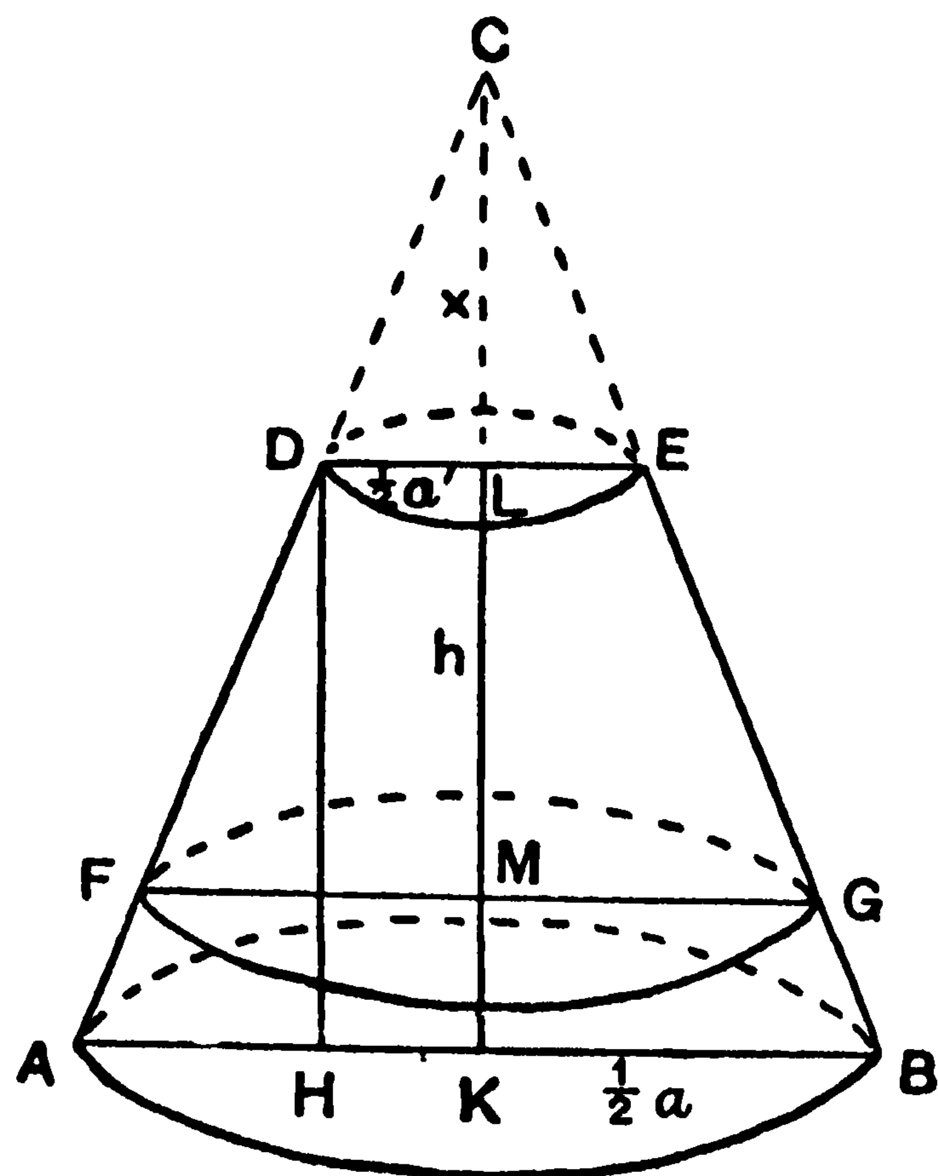
and, solving for $x-a$, we obtain

$$x-a=\frac{(a+1)(d_1-\delta_1)}{(a+1)(d_1-\delta_1)+a(d_2-\delta_2)},$$

or
$$\sqrt[3]{A}=a+\frac{(a+1)(d_1-\delta_1)}{(a+1)(d_1-\delta_1)+a(d_2-\delta_2)}.$$

Since δ_1, δ_2 are in any case the cubes of fractions, we may neglect them for a first approximation, and we have

$$\sqrt[3]{A}=a+\frac{(a+1)d_1}{(a+1)d_1+ad_2}.$$



III. 22, which shows how to cut a frustum of a cone in a given ratio by a section parallel to the bases, shall end our account of the *Metrica*. I shall give the general formulae on the left and Heron's case on the right. Let $ABED$ be the frustum, let the *diameters* of the bases be a, a', and the height h. Complete the cone, and let the height of CDE be x.

Suppose that the frustum has to be cut by a plane FG in such a way that

$$(\text{frustum } DG):(\text{frustum } FB)=m:n.$$

In the case taken by Heron

$$a=28,\ a'=21,\ h=12,\ m=4,\ n=1.$$

Draw DH perpendicular to AB.

Since $(DG):(FB) = m:n$,

$$(DB):(DG) = (m+n):m.$$

Now

$$(DB) = \tfrac{1}{12}\pi h(a^2+aa'+a'^2),$$

and $(DG) = \dfrac{m}{m+n}(DB)$.

Let y be the height (CM) of the cone CFG.

Then $DH:AH = CK:KA$,

or $h:\frac{1}{2}(a-a') = (x+h):\frac{1}{2}a$,

whence x is known.

Cone $CDE = \frac{1}{12}\pi a'^2 x$,

cone $CFG = (CDE) + \dfrac{m}{m+n}(DB)$,

cone $CAB = (CDE) + (DB)$.

Now, says Heron,

$$\frac{(CAB)+(CDE)}{(CFG)} = \frac{(x+h)^3+x^3}{y^3}.$$

[He might have said simply

$(CDE):(CFG) = x^3:y^3$.]

This gives y or CM,

whence LM is known.

Now $AD^2 = AH^2+DH^2$

$$= \{\tfrac{1}{2}(a-a')\}^2+h^2,$$

so that AD is known.

Therefore $DF = \dfrac{y-x}{h}.AD$ is known.

$$(DG):(FB) = 4:1,$$

$$(DB):(DG) = 5:4.$$

$$(DB) = 5698,$$

$$(DG) = 4558\tfrac{2}{5}.$$

$$x+h = \frac{14.12}{3\frac{1}{2}} = 48,$$

and $x = 48-12 = 36$.

(cone CDE) $= 4158$,

(cone CFG) $= 4158+4558\frac{2}{5} = 8716\frac{2}{5}$,

(cone CAB) $= 4158+5698 = 9856$.

$$y^3 = \frac{8716\frac{2}{5}}{9856+4158}\cdot(48^3+36^3)$$

$$= 8716\tfrac{2}{5}\cdot\tfrac{157248}{14014} = 97805,$$

whence $y = 46$ approximately.

Therefore $LM = y-x = 10$.

$$AD^2 = (3\tfrac{1}{2})^2+12^2$$

$$= 156\tfrac{1}{4},$$

and $AD = 12\frac{1}{2}$.

Therefore $DF = \frac{10}{12}.12\frac{1}{2}$

$$= 10\tfrac{5}{12}.$$

Quadratic equations solved in Heron.

We have already met with one such equation (in *Metrica* III. 4), namely $x^2 - 14x + 46\frac{2}{3} = 0$, the result only ($x = 8\frac{1}{2}$) being given. There are others in the *Geometrica* where the process of solution is shown.

(1) *Geometrica* 24, 3 (Heib.). 'Given a square such that the sum of its area and perimeter is 896 feet: to separate the area from the perimeter': i.e. $x^2 + 4x = 896$. Heron takes half of 4 and adds its square, completing the square on the left side.

(2) *Geometrica* 21, 9 and 24, 46 (Heib.) give one and the same equation, *Geom.* 24, 47 another like it. 'Given the sum of the diameter, perimeter and area of a circle, to find each of them.'

The two equations are

$$\tfrac{11}{14}d^2 + \tfrac{29}{7}d = 212,$$

and

$$\tfrac{11}{14}d^2 + \tfrac{29}{7}d = 67\tfrac{1}{2}.$$

Our usual method is to begin by dividing by $\frac{11}{14}$ throughout, so as to leave d^2 as the first term. Heron's is to *multiply* by such a number as will leave a square as the first term. In this case he multiplies by 154, giving $11^2d^2 + 58 \cdot 11d = 212 \cdot 154$ or $67\frac{1}{2} \cdot 154$ as the case may be. Completing the square, he obtains $(11d + 29)^2 = 32648 + 841$ or $10395 + 841$. Thus $11d + 29 = \sqrt{(33489)}$ or $\sqrt{(11236)}$, that is, 183 or 106. Thus $11d = 154$ or 77, and $d = 14$ or 7, as the case may be.

Indeterminate problems in the *Geometrica*.

Some very interesting indeterminate problems are now included by Heiberg in the *Geometrica*.[1] Two of them (chap. 24, 1–2) were included in the *Geëponicus* in Hultsch's edition (sections 78, 79); the rest are new, having been found in the Constantinople manuscript from which Schöne edited the *Metrica*. As, however, these problems, to whatever period they belong, are more akin to algebra than to mensuration, they will be more properly described in a later chapter on Algebra.

[1] *Heronis Alexandrini opera*, vol. iv, p. 414. 28 sq.

The *Dioptra* (περὶ διόπτρας).

This treatise begins with a careful description of the *dioptra*, an instrument which served with the ancients for the same purpose as a theodolite with us (chaps. 1–5). The problems with which the treatise goes on to deal are (*a*) problems of 'heights and distances', (*b*) engineering problems, (*c*) problems of mensuration, to which is added (chap. 34) a description of a 'hodometer', or taxameter, consisting of an arrangement of toothed wheels and endless screws on the same axes working on the teeth of the next wheels respectively. The book ends with the problem (chap. 37), 'With a given force to move a given weight by means of interacting toothed wheels', which really belongs to mechanics, and was apparently added, like some other problems (e.g. 31, 'to measure the outflow of, i.e. the volume of water issuing from, a spring'), in order to make the book more comprehensive. The essential problems dealt with are such as the following. To determine the difference of level between two given points (6), to draw a straight line connecting two points the one of which is not visible from the other (7), to measure the least breadth of a river (9), the distance of two inaccessible points (10), the height of an inaccessible point (12), to determine the difference between the heights of two inaccessible points and the position of the straight line joining them (13), the depth of a ditch (14); to bore a tunnel through a mountain going straight from one mouth to the other (15), to sink a shaft through a mountain perpendicularly to a canal flowing underneath (16); given a subterranean canal of any form, to find on the ground above a point from which a vertical shaft must be sunk in order to reach a given point on the canal (for the purpose e.g. of removing an obstruction) (20); to construct a harbour on the model of a given segment of a circle, given the ends (17), to construct a vault so that it may have a spherical surface modelled on a given segment (18). The mensuration problems include the following: to measure an irregular area, which is done by inscribing a rectilineal figure and then drawing perpendiculars to the sides at intervals to meet the contour (23), or by drawing one straight line across the area and erecting perpendiculars from

that to meet the contour on both sides (24); given that all the boundary stones of a certain area have disappeared except two or three, but that the plan of the area is forthcoming, to determine the position of the lost boundary stones (25). Chaps. 26–8 remind us of the *Metrica*: to divide a given area into given parts by straight lines drawn from one point (26); to measure a given area without entering it, whether because it is thickly covered with trees, obstructed by houses, or entry is forbidden! (27); chaps. 28–30 = *Metrica* III. 7, III. 1, and I. 7, the last of these three propositions being the proof of the 'formula of Heron' for the area of a triangle in terms of the sides. Chap. 35 shows how to find the distance between Rome and Alexandria along a great circle of the earth by means of the observation of the same eclipse at the two places, the *analemma* for Rome, and a concave hemisphere constructed for Alexandria to show the position of the sun at the time of the said eclipse. It is here mentioned that the estimate by Eratosthenes of the earth's circumference in his book *On the Measurement of the Earth* was the most accurate that had been made up to date.[1] Some hold that the chapter, like some others which have no particular connexion with the real subject of the *Dioptra* (e.g. chaps. 31, 34, 37–8) were probably inserted by a later editor, 'in order to make the treatise as complete as possible'.[2]

The *Mechanics*.

It is evident that the *Mechanics*, as preserved in the Arabic, is far from having kept its original form, especially in Book I. It begins with an account of the arrangement of toothed wheels designed to solve the problem of moving a given weight by a given force; this account is the same as that given at the end of the Greek text of the *Dioptra*, and it is clearly the same description as that which Pappus[3] found in the work of Heron entitled Βαρουλκός ('weight-lifter') and himself reproduced with a ratio of force to weight altered from 5 : 1000 to 4 : 160 and with a ratio of 2 : 1 substituted for 5 : 1 in the diameters of successive wheels. It would appear that the chapter from the Βαρουλκός was inserted in place of

[1] Heron, vol. iii, p. 302. 13–17. [2] *Ib.*, p. 302. 9.
[3] Pappus, viii, p. 1060 sq.

the first chapter or chapters of the real *Mechanics* which had been lost. The treatise would doubtless begin with generalities introductory to mechanics such as we find in the (much interpolated) beginning of Pappus, Book VIII. It must then apparently have dealt with the properties of circles, cylinders, and spheres with reference to their importance in mechanics; for in Book II. 21 Heron says that the circle is of all figures the most movable and most easily moved, the same thing applying also to the cylinder and sphere, and he adds in support of this a reference to a proof 'in the preceding Book'. This reference may be to I. 21, but at the end of that chapter he says that 'cylinders, even when heavy, if placed on the ground so that they touch it in one line only, are easily moved, and the same is true of spheres also, a matter *which we have already discussed*'; the discussion may have come earlier in the Book, in a chapter now lost.

The treatise, beginning with chap. 2 after the passage interpolated from the Βαρουλκός, is curiously disconnected. Chaps. 2–7 discuss the motion of circles or wheels, equal or unequal, moving on different axes (e.g. interacting toothed wheels), or fixed on the same axis, much after the fashion of the Aristotelian *Mechanical problems.*

Aristotle's Wheel.

In particular (chap. 7) Heron attempts to explain the puzzle of the 'Wheel of Aristotle', which remained a puzzle up to quite modern times, and gave rise to the proverb, 'rotam Aristotelis magis torquere, quo magis torqueretur'.[1] 'The question is', says the Aristotelian problem 24, 'why does the greater circle roll an equal distance with the lesser circle when they are placed about the same centre, whereas, when they roll separately, as the size of one is to the size of the other, so are the straight lines traversed by them to one another?'[2] Let AC, BD be quadrants of circles with centre O bounded by the same radii, and draw tangents AE, BF at A and B. In the first case suppose the circle BD to roll along BF till D takes the position H; then the radius ODC will be at right angles to AE, and C will be at G, a point such that AG is equal to BH. In the second

[1] See Van Capelle, *Aristotelis quaestiones mechanicae*, 1812, p. 263 sq.
[2] Arist. *Mechanica*, 855 a 28.

case suppose the circle AC to roll along AE till ODC takes the position $O'FE$; then D will be at F where $AE = BF$. And similarly if a whole revolution is performed and OBA is again perpendicular to AE. Contrary, therefore, to the principle that the greater circle moves quicker than the smaller on the same axis, it would appear that the movement of the smaller in this case is as quick as that of the greater, since $BH = AG$, and $BF = AE$. Heron's explanation is that, e.g. in the case where the larger circle rolls on AE, the lesser circle maintains the same speed as the greater because it has *two* motions; for if we regard the smaller circle as merely fastened to the larger, and not rolling at all, its centre O will move to O' traversing a distance OO' equal to AE and BF; hence the greater circle will take the lesser with it over an equal distance, the rolling of the lesser circle having no effect upon this.

The parallelogram of velocities.

Heron next proves the parallelogram of velocities (chap. 8); he takes the case of a rectangle, but the proof is applicable generally.

The way it is put is this. A point moves with uniform velocity along a straight line AB, from A to B, while at the same time AB moves with uniform velocity always parallel to itself with its extremity A describing the straight line AC. Suppose that, when the point arrives at B, the straight line

reaches the position CD. Let EF be any intermediate position of AB, and G the position at the same instant of the moving point on it. Then clearly $AE:AC = EG:EF$; therefore $AE:EG = AC:EF = AC:CD$, and it follows that G lies on the diagonal AD, which is therefore the actual path of the moving point.

Chaps. 9–19 contain a digression on the construction of plane and solid figures similar to given figures but greater or less in a given ratio. Heron observes that the case of plane figures involves the finding of a mean proportional between two straight lines, and the case of solid figures the finding of *two* mean proportionals; in chap. 11 he gives his solution of the latter problem, which is preserved in Pappus and Eutocius as well, and has already been given above (vol. i, pp. 262–3).

The end of chap. 19 contains, quite inconsequently, the construction of a toothed wheel to move on an endless screw, after which chap. 20 makes a fresh start with some observations on weights in equilibrium on a horizontal plane but tending to fall when the plane is inclined, and on the ready mobility of objects of cylindrical form which touch the plane in one line only.

Motion on an inclined plane.

When a weight is hanging freely by a rope over a pulley, no force applied to the other end of the rope less than the weight itself will keep it up, but, if the weight is placed on an inclined plane, and both the plane and the portion of the weight in contact with it are smooth, the case is different. Suppose, e.g., that a weight in the form of a cylinder is placed on an inclined plane so that the line in which they touch is horizontal; then the force required to be applied to a rope parallel to the line of greatest slope in the plane in order to keep the weight in equilibrium is less than the weight. For the vertical plane passing through the line of contact between the cylinder and the plane divides the cylinder into two unequal parts, that on the downward side of the plane being the greater, so that the cylinder will tend to roll down; but the force required to support the cylinder is the 'equivalent', not of the weight of the whole cylinder, but of the difference

between the two portions into which the vertical plane cuts it (chap. 23).

On the centre of gravity.

This brings Heron to the centre of gravity (chap. 24). Here a definition by Posidonius, a Stoic, of the 'centre of gravity' or 'centre of inclination' is given, namely 'a point such that, if the body is hung up at it, the body is divided into two equal parts' (he should obviously have said 'divided *by any vertical plane through the point of suspension* into two equal parts'). But, Heron says, Archimedes distinguished between the 'centre of gravity' and the 'point of suspension', defining the latter as a point on the body such that, if the body is hung up at it, all the parts of the body remain in equilibrium and do not oscillate or incline in any direction. '"Bodies", said Archimedes, "may rest (without inclining one way or another) with either a line, or only one point, in the body fixed".' The 'centre of inclination', says Heron, 'is one single point in any particular body to which all the vertical lines through the points of suspension converge.' Comparing Simplicius's quotation of a definition by Archimedes in his Κεντροβαρικά, to the effect that the centre of gravity is a certain point in the body such that, if the body is hung up by a string attached to that point, it will remain in its position without inclining in any direction,[1] we see that Heron directly used a certain treatise of Archimedes. So evidently did Pappus, who has a similar definition. Pappus also speaks of a body supported at a point by a vertical stick: if, he says, the body is in equilibrium, the line of the stick produced upwards must pass through the centre of gravity.[2] Similarly Heron says that the same principles apply when the body is supported as when it is suspended. Taking up next (chaps. 25–31) the question of 'supports', he considers cases of a heavy beam or a wall supported on a number of pillars, equidistant or not, even or not even in number, and projecting or not projecting beyond one or both of the extreme pillars, and finds how much of the weight is supported on each pillar. He says that Archimedes laid down the principles in his 'Book on

[1] Simplicius on *De caelo*, p. 543. 31–4, Heib.
[2] Pappus, viii, p. 1032. 5–24.

Supports'. As, however, the principles are the same whether the body is supported or hung up, it does not follow that this was a different work from that known as περὶ ζυγῶν. Chaps. 32–3, which are on the principles of the lever or of weighing, end with an explanation amounting to the fact that 'greater circles overpower smaller when their movement is about the same centre', a proposition which Pappus says that Archimedes proved in his work περὶ ζυγῶν.[1] In chap. 32, too, Heron gives as his authority a proof given by Archimedes in the same work. With I. 33 may be compared II. 7, where Heron returns to the same subject of the greater and lesser circles moving about the same centre and states the fact that weights reciprocally proportional to their radii are in equilibrium when suspended from opposite ends of the horizontal diameters, observing that Archimedes proved the proposition in his work 'On the equalization of inclination' (presumably ἰσορροπίαι).

Book II. The five mechanical powers.

Heron deals with the wheel and axle, the lever, the pulley, the wedge and the screw, and with combinations of these powers. The description of the powers comes first, chaps. 1–6, and then, after II. 7, the proposition above referred to, and the theory of the several powers based upon it (chaps. 8–20). Applications to specific cases follow. Thus it is shown how to move a weight of 1000 talents by means of a force of 5 talents, first by the system of wheels described in the *Βαρουλκός*, next by a system of pulleys, and thirdly by a combination of levers (chaps. 21–5). It is possible to combine the different powers (other than the wedge) to produce the same result (chap. 29). The wedge and screw are discussed with reference to their angles (chaps. 30–1), and chap. 32 refers to the effect of friction.

Mechanics in daily life; queries and answers.

After a prefatory chapter (33), a number of queries resembling the Aristotelian problems are stated and answered (chap. 34), e.g. 'Why do waggons with two wheels carry a weight more easily than those with four wheels?', 'Why

[1] Pappus, viii, p. 1068. 20-3.

do great weights fall to the ground in a shorter time than lighter ones?', 'Why does a stick break sooner when one puts one's knee against it in the middle?', 'Why do people use pincers rather than the hand to draw a tooth?', 'Why is it easy to move weights which are suspended?', and 'Why is it the more difficult to move such weights the farther the hand is away from them, right up to the point of suspension or a point near it?', 'Why are great ships turned by a rudder although it is so small?', 'Why do arrows penetrate armour or metal plates but fail to penetrate cloth spread out?'

Problems on the centre of gravity, &c.

II. 35, 36, 37 show how to find the centre of gravity of a triangle, a quadrilateral and a pentagon respectively. Then, assuming that a triangle of uniform thickness is supported by a prop at each angle, Heron finds what weight is supported by each prop, (*a*) when the props support the triangle only, (*b*) when they support the triangle plus a given weight placed at any point on it (chaps. 38, 39). Lastly, if known weights are put on the triangle at each angle, he finds the centre of gravity of the system (chap. 40); the problem is then extended to the case of any polygon (chap. 41).

Book III deals with the practical construction of engines for all sorts of purposes, machines employing pulleys with one, two, or more supports for lifting weights, oil-presses, &c.

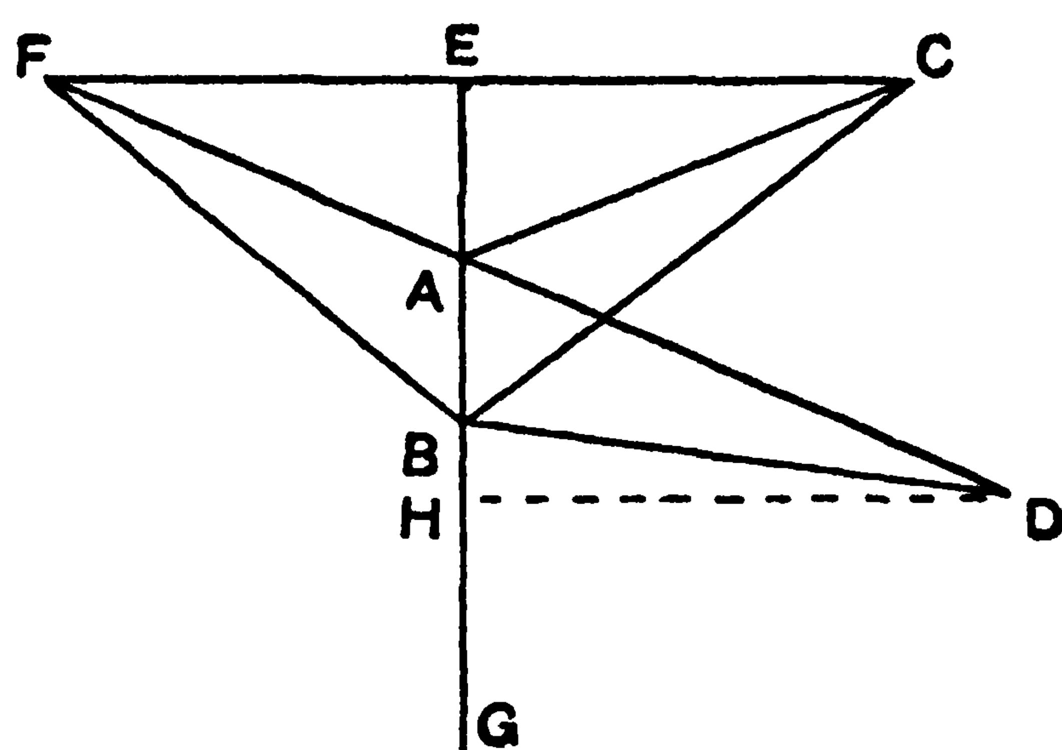
The *Catoptrica*.

This work need not detain us long. Several of the theoretical propositions which it contains are the same as propositions in the so-called *Catoptrica* of Euclid, which, as we have seen, was in all probability the work of Theon of Alexandria and therefore much later in date. In addition to theoretical propositions, it contains problems the purpose of which is to construct mirrors or combinations of mirrors of such shape as will reflect objects in a particular way, e.g. to make the right side appear as the right in the picture (instead of the reverse), to enable a person to see his back or to appear in the mirror head downwards, with face distorted, with three eyes or two noses, and so forth. Concave and convex

cylindrical mirrors play a part in these arrangements. The whole theory of course ultimately depends on the main propositions 4 and 5 that the angles of incidence and reflection are equal whether the mirror is plane or circular.

Heron's proof of equality of angles of incidence and reflection.

Let AB be a plane mirror, C the eye, D the object seen. The argument rests on the fact that nature 'does nothing in vain'. Thus light travels in a straight line, that is, by the quickest road. Therefore, even when the ray is a line broken at a point by reflection, it must mark the shortest broken line of the kind connecting the eye and the object. Now, says Heron, I maintain that the shortest of the broken lines (broken at the mirror) which connect C and D is the line, as CAD, the parts of which make equal angles with the mirror. Join DA and produce it to meet in F the perpendicular from C to AB. Let B be any point on the mirror other than A, and join FB, BD.



Now $\qquad \angle EAF = \angle BAD$

$$= \angle CAE, \text{ by hypothesis.}$$

Therefore the triangles AEF, AEC, having two angles equal and AE common, are equal in all respects.

Therefore $CA = AF$, and $CA + AD = DF$.

Since $FE = EC$, and BE is perpendicular to FC, $BF = BC$.

Therefore $\qquad CB + BD = FB + BD$

$$> FD,$$

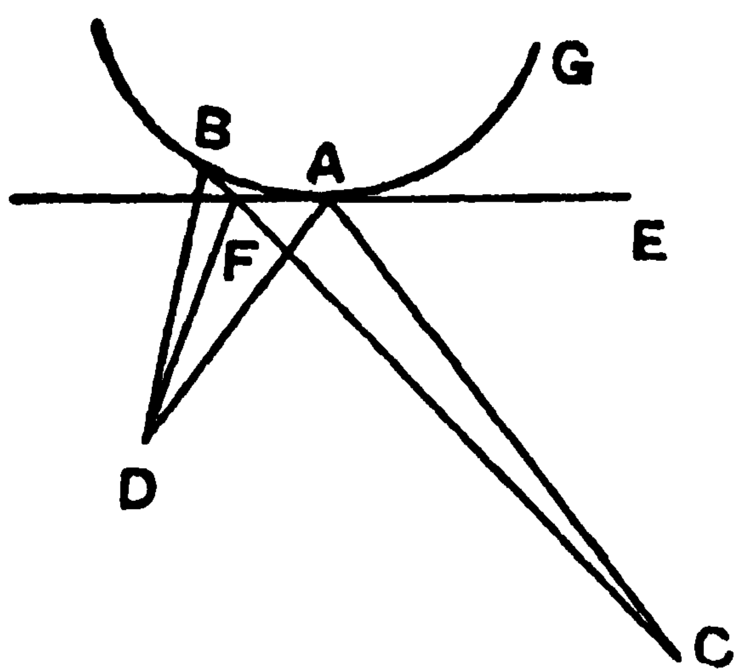
$$\text{i.e.} > CA + AD.$$

The proposition was of course known to Archimedes. We gather from a scholium to the Pseudo-Euclidean *Catoptrica* that he proved it in a different way, namely by *reductio ad absurdum*, thus: Denote the angles CAE, DAB by α, β respectively. Then, α is $>$ $=$ or $<$ β. Suppose $\alpha > \beta$. Then,

reversing the ray so that the eye is at D instead of C, and the object at C instead of D, we must have $\beta > \alpha$. But β was less than α, which is impossible. (Similarly it can be proved that α is not less than β.) Therefore $\alpha = \beta$.

In the Pseudo-Euclidean *Catoptrica* the proposition is practically assumed; for the third assumption or postulate at the beginning states in effect that, in the above figure, if A be the point of incidence, $CE:EA = DH:HA$ (where DH is perpendicular to AB). It follows instantaneously (Prop. 1) that $\angle CAE = \angle DAH$.

If the mirror is the convex side of a circle, the same result follows *a fortiori*. Let CA, AD meet the arc at equal angles, and CB, BD at unequal angles. Let AE be the tangent at A, and complete the figure. Then, says Heron, (the angles GAC, BAD being by hypothesis equal), if we subtract the equal angles GAE, BAF from the equal angles GAC, BAD (both pairs of angles being 'mixed', be it observed), we have $\angle EAC = \angle FAD$. Therefore $CA + AD < CF + FD$ and *a fortiori* $< CB + BD$.



The problems solved (though the text is so corrupt in places that little can be made of it) were such as the following: 11, To construct a right-handed mirror (i.e. a mirror which makes the right side right and the left side left instead of the opposite); 12, to construct the mirror called *polytheoron* ('with many images'); 16, to construct a mirror inside the window of a house, so that you can see in it (while inside the room) everything that passes in the street; 18, to arrange mirrors in a given place so that a person who approaches cannot actually see either himself or any one else but can see any image desired (a 'ghost-seer').

XIX

PAPPUS OF ALEXANDRIA

We have seen that the Golden Age of Greek geometry ended with the time of Apollonius of Perga. But the influence of Euclid, Archimedes and Apollonius continued, and for some time there was a succession of quite competent mathematicians who, although not originating anything of capital importance, kept up the tradition. Besides those who were known for particular investigations, e.g. of new curves or surfaces, there were such men as Geminus who, it cannot be doubted, were thoroughly familiar with the great classics. Geminus, as we have seen, wrote a comprehensive work of almost encyclopaedic character on the classification and content of mathematics, including the history of the development of each subject. But the beginning of the Christian era sees quite a different state of things. Except in sphaeric and astronomy (Menelaus and Ptolemy), production was limited to elementary text-books of decidedly feeble quality. In the meantime it would seem that the study of higher geometry languished or was completely in abeyance, until Pappus arose to revive interest in the subject. From the way in which he thinks it necessary to describe the contents of the classical works belonging to the *Treasury of Analysis*, for example, one would suppose that by his time many of them were, if not lost, completely forgotten, and that the great task which he set himself was the re-establishment of geometry on its former high plane of achievement. Presumably such interest as he was able to arouse soon flickered out, but for us his work has an inestimable value as constituting, after the works of the great mathematicians which have actually survived, the most important of all our sources.

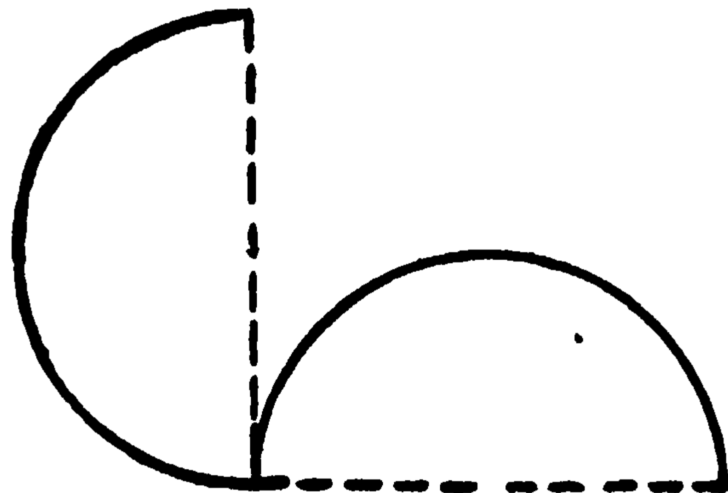
Date of Pappus.

Pappus lived at the end of the third century A.D. The authority for this date is a marginal note in a Leyden manuscript of chronological tables by Theon of Alexandria, where, opposite to the name of Diocletian, a scholium says, 'In his time Pappus wrote'. Diocletian reigned from 284 to 305, and this must therefore be the period of Pappus's literary activity. It is true that Suidas makes him a contemporary of Theon of Alexandria, adding that they both lived under Theodosius I (379–395). But Suidas was evidently not well acquainted with the works of Pappus; though he mentions a description of the earth by him and a commentary on four Books of Ptolemy's *Syntaxis*, he has no word about his greatest work, the *Synagoge*. As Theon also wrote a commentary on Ptolemy and incorporated a great deal of the commentary of Pappus, it is probable that Suidas had Theon's commentary before him and from the association of the two names wrongly inferred that they were contemporaries.

Works (commentaries) other than the *Collection*.

Besides the *Synagoge*, which is the main subject of this chapter, Pappus wrote several commentaries, now lost except for fragments which have survived in Greek or Arabic. One was a commentary on the *Elements* of Euclid. This must presumably have been pretty complete, for, while Proclus (on Eucl. I) quotes certain things from Pappus which may be assumed to have come in the notes on Book I, fragments of his commentary on Book X actually survive in the Arabic (see above, vol. i, pp. 154–5, 209), and again Eutocius in his note on Archimedes, *On the Sphere and Cylinder*, I. 13, says that Pappus explained in his commentary on the *Elements* how to inscribe in a circle a polygon similar to a polygon inscribed in another circle, which problem would no doubt be solved by Pappus, as it is by a scholiast, in a note on XII. 1. Some of the references by Proclus deserve passing mention. (1) Pappus said that the converse of Post. 4 (equality of all right angles) is not true, i.e. it is not true that all angles equal to a right angle are themselves right, since the 'angle' between the conterminous arcs of two semicircles which are equal and have their

diameters at right angles and terminating at one point is equal to, but is not, a right angle.[1] (2) Pappus said that, in addition to the genuine axioms of Euclid, there were others on record about unequals added to equals and equals added to unequals. Others given by Pappus are (says Proclus) involved by the definitions, e.g. that 'all parts of the plane and of the straight line coincide with one another', that 'a point divides a line, a line a surface, and a surface a solid', and that 'the infinite is (obtained) in magnitudes both by addition and diminution'.[2] (3) Pappus gave a pretty proof of Eucl. I. 5, which modern editors have spoiled when introducing it into text-books. If *AB*, *AC* are the equal sides in an isosceles triangle, Pappus compares the triangles *ABC* and *ACB* (i.e. as if he were comparing the triangle *ABC* seen from the front with the same triangle seen from the back), and shows that they satisfy the conditions of I. 4, so that they are equal in all respects, whence the result follows.[3]



Marinus at the end of his commentary on Euclid's *Data* refers to a commentary by Pappus on that book.

Pappus's commentary on Ptolemy's *Syntaxis* has already been mentioned (p. 274); it seems to have extended to six Books, if not to the whole of Ptolemy's work. The *Fihrist* says that he also wrote a commentary on Ptolemy's *Planisphaerium*, which was translated into Arabic by Thābit b. Qurra. Pappus himself alludes to his own commentary on the *Analemma* of Diodorus, in the course of which he used the conchoid of Nicomedes for the purpose of trisecting an angle.

We come now to Pappus's great work.

The *Synagoge* or *Collection*.

(*a*) *Character of the work; wide range.*

Obviously written with the object of reviving the classical Greek geometry, it covers practically the whole field. It is,

[1] Proclus on Eucl. I, pp. 189-90. [2] *Ib.*, pp. 197. 6-198. 15.
[3] *Ib.*, pp. 249. 20-250. 12.

however, a handbook or guide to Greek geometry rather than an encyclopaedia; it was intended, that is, to be read with the original works (where still extant) rather than to enable them to be dispensed with. Thus in the case of the treatises included in the *Treasury of Analysis* there is a general introduction, followed by a general account of the contents, with lemmas, &c., designed to facilitate the reading of the treatises themselves. On the other hand, where the history of a subject is given, e.g. that of the problem of the duplication of the cube or the finding of the two mean proportionals, the various solutions themselves are reproduced, presumably because they were not easily accessible, but had to be collected from various sources. Even when it is some accessible classic which is being described, the opportunity is taken to give alternative methods, or to make improvements in proofs, extensions, and so on. Without pretending to great originality, the whole work shows, on the part of the author, a thorough grasp of all the subjects treated, independence of judgement, mastery of technique; the style is terse and clear; in short, Pappus stands out as an accomplished and versatile mathematician, a worthy representative of the classical Greek geometry.

(β) *List of authors mentioned.*

The immense range of the *Collection* can be gathered from a mere enumeration of the names of the various mathematicians quoted or referred to in the course of it. The greatest of them, Euclid, Archimedes and Apollonius, are of course continually cited, others are mentioned for some particular achievement, and in a few cases the mention of a name by Pappus is the whole of the information we possess about the person mentioned. In giving the list of the names occurring in the book, it will, I think, be convenient and may economize future references if I note in brackets the particular occasion of the reference to the writers who are mentioned for one achievement or as the authors of a particular book or investigation. The list in alphabetical order is: Apollonius of Perga, Archimedes, Aristaeus the elder (author of a treatise in five Books on the Elements of Conics or of 'five Books on Solid Loci connected with the conics'), Aristarchus of Samos (*On the*

sizes and distances of the sun and moon), Autolycus (*On the moving sphere*), Carpus of Antioch (who is quoted as having said that Archimedes wrote only one mechanical book, that on sphere-making, since he held the mechanical appliances which made him famous to be nevertheless unworthy of written description: Carpus himself, who was known as *mechanicus*, applied geometry to other arts of this practical kind), Charmandrus (who added three simple and obvious loci to those which formed the beginning of the *Plane Loci* of Apollonius), Conon of Samos, the friend of Archimedes (cited as the propounder of a theorem about the spiral in a plane which Archimedes proved: this would, however, seem to be a mistake, as Archimedes says at the beginning of his treatise that he sent certain theorems, without proofs, to Conon, who would certainly have proved them had he lived), Demetrius of Alexandria (mentioned as the author of a work called 'Linear considerations', γραμμικαὶ ἐπιστάσεις, i.e. considerations on curves, as to which nothing more is known), Dinostratus, the brother of Menaechmus (cited, with Nicomedes, as having used the curve of Hippias, to which they gave the name of *quadratrix*, τετραγωνίζουσα, for the squaring of the circle), Diodorus (mentioned as the author of an *Analemma*), Eratosthenes (whose *mean-finder*, an appliance for finding two or any number of geometric means, is described, and who is further mentioned as the author of two Books 'On means' and of a work entitled 'Loci with reference to means'), Erycinus (from whose *Paradoxa* are quoted various problems seeming at first sight to be inconsistent with Eucl. I. 21, it being shown that straight lines can be drawn from two points on the base of a triangle to a point within the triangle which are together greater than the other two sides, provided that the points in the base may be points other than the extremities), Euclid, Geminus the mathematician (from whom is cited a remark on Archimedes contained in his book 'On the classification of the mathematical sciences', see above, p. 223), Heraclitus (from whom Pappus quotes an elegant solution of a νεῦσις with reference to a square), Hermodorus (Pappus's son, to whom he dedicated Books VII, VIII of his *Collection*), Heron of Alexandria (whose mechanical works are extensively quoted from), Hierius the philosopher (a contemporary of Pappus,

who is mentioned as having asked Pappus's opinion on the attempted solution by 'plane' methods of the problem of the two means, which actually gives a method of approximating to a solution[1]), Hipparchus (quoted as practically adopting three of the hypotheses of Aristarchus of Samos), Megethion (to whom Pappus dedicated Book V of his *Collection*), Menelaus of Alexandria (quoted as the author of *Sphaerica* and as having applied the name παράδοξος to a certain curve), Nicomachus (on three means additional to the first three), Nicomedes, Pandrosion (to whom Book III of the *Collection* is dedicated), Pericles (editor of Euclid's *Data*), Philon of Byzantium (mentioned along with Heron), Philon of Tyana (mentioned as the discoverer of certain complicated curves derived from the interweaving of plectoid and other surfaces), Plato (with reference to the five regular solids), Ptolemy, Theodosius (author of the *Sphaerica* and *On Days and Nights*).

(γ) *Translations and editions.*

The first published edition of the *Collection* was the Latin translation by Commandinus (Venice 1589, but dated at the end 'Pisauri apud Hieronymum Concordiam 1588'; reissued with only the title-page changed 'Pisauri ... 1602'). Up to 1876 portions only of the Greek text had appeared, namely Books VII, VIII in Greek and German, by C. J. Gerhardt, 1871 chaps. 33–105 of Book V, by Eisenmann, Paris 1824, chaps 45–52 of Book IV in *Iosephi Torelli Veronensis Geometrica* 1769, the remains of Book II, by John Wallis (in *Opera mathematica*, III, Oxford 1699); in addition, the restorers of works of Euclid and Apollonius from the indications furnished by Pappus give extracts from the Greek text relating to the particular works, Breton le Champ on Euclid's *Porisms*, Halley in his edition of the *Conics* of Apollonius (1710) and in his translation from the Arabic and restoration respectively of the *De sectione rationis* and *De sectione spatii* of Apollonius (1706), Camerer on Apollonius's *Tactiones* (1795), Simson and Horsley in their restorations of Apollonius's *Plana Loci* and *Inclinationes* published in the years 1749 and 177[illegible] respectively. In the years 1876–8 appeared the only com

[1] See vol. i, pp. 268–70.

plete Greek text, with apparatus, Latin translation, commentary, appendices and indices, by Friedrich Hultsch; this great edition is one of the first monuments of the revived study of the history of Greek mathematics in the last half of the nineteenth century, and has properly formed the model for other definitive editions of the Greek text of the other classical Greek mathematicians, e.g. the editions of Euclid, Archimedes, Apollonius, &c., by Heiberg and others. The Greek index in this edition of Pappus deserves special mention because it largely serves as a dictionary of mathematical terms used not only in Pappus but by the Greek mathematicians generally.

(δ) *Summary of contents.*

At the beginning of the work, Book I and the first 13 propositions (out of 26) of Book II are missing. The first 13 propositions of Book II evidently, like the rest of the Book, dealt with Apollonius's method of working with very large numbers expressed in successive powers of the myriad, 10000. This system has already been described (vol. i, pp. 40, 54–7). The work of Apollonius seems to have contained 26 propositions (25 leading up to, and the 26th containing, the final continued multiplication).

Book III consists of four sections. Section (1) is a sort of history of the problem of *finding two mean proportionals, in continued proportion, between two given straight lines.*

It begins with some general remarks about the distinction between theorems and problems. Pappus observes that, whereas the ancients called them all alike by one name, some regarding them all as problems and others as theorems, a clear distinction was drawn by those who favoured more exact terminology. According to the latter a problem is that in which it is proposed to *do* or *construct* something, a theorem that in which, given certain hypotheses, we investigate that which follows from and is necessarily implied by them. Therefore he who propounds a theorem, no matter how he has become aware of the fact which is a necessary consequence of the premisses, must state, as the object of inquiry, the right result and no other. On the other hand, he who propounds

a problem may bid us do something which is in fact impossible, and that without necessarily laying himself open to blame or criticism. For it is part of the solver's duty to determine the conditions under which the problem is possible or impossible, and, 'if possible, when, how, and in how many ways it is possible'. When, however, a man professes to know mathematics and yet commits some elementary blunder, he cannot escape censure. Pappus gives, as an example, the case of an unnamed person 'who was thought to be a great geometer' but who showed ignorance in that he claimed to know how to solve the problem of the two mean proportionals by 'plane' methods (i.e. by using the straight line and circle only). He then reproduces the argument of the anonymous person, for the purpose of showing that it does not solve the problem as its author claims. We have seen (vol. i, pp. 269–70) how the method, though not actually solving the problem, does furnish a series of successive approximations to the real solution. Pappus adds a few simple lemmas assumed in the exposition.

Next comes the passage[1], already referred to, on the distinction drawn by the ancients between (1) *plane* problems or problems which can be solved by means of the straight line and circle, (2) *solid* problems, or those which require for their solution one or more conic sections, (3) *linear* problems, or those which necessitate recourse to higher curves still, curves with a more complicated and indeed a forced or unnatural origin (βεβιασμένην) such as spirals, quadratrices, cochloids and cissoids, which have many surprising properties of their own. The problem of the two mean proportionals, being a *solid* problem, required for its solution either conics or some equivalent, and, as conics could not be constructed by purely geometrical means, various mechanical devices were invented such as that of Eratosthenes (the *mean-finder*), those described in the *Mechanics* of Philon and Heron, and that of Nicomedes (who used the 'cochloidal' curve). Pappus proceeds to give the solutions of Eratosthenes, Nicomedes and Heron, and then adds a fourth which he claims as his own, but which is practically the same as that attributed by Eutocius to Sporus. All these solutions have been given above (vol. i, pp. 258–64, 266–8).

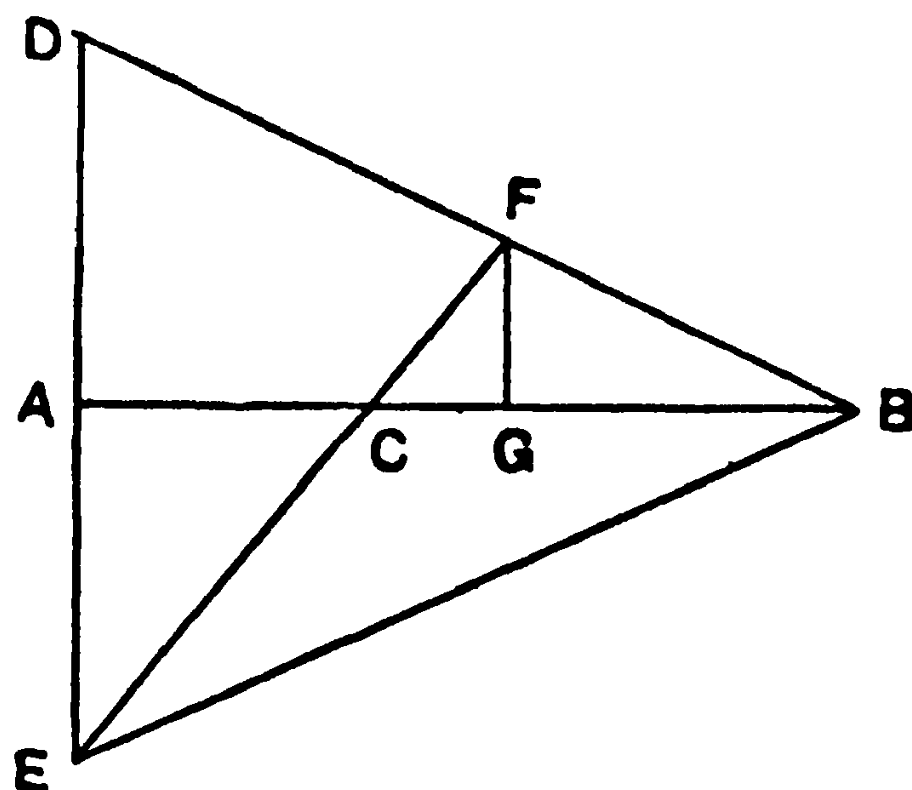
[1] Pappus, iii, p. 54. 7–22.

Section (2). *The theory of means.*

Next follows a section (pp. 69–105) on the theory of the different kinds of *means.* The discussion takes its origin from the statement of the 'second problem', which was that of 'exhibiting the three means' (i.e. the arithmetic, geometric and harmonic) 'in a semicircle'. Pappus first gives a construction by which another geometer (ἄλλος τις) claimed to have solved this problem, but he does not seem to have understood it, and returns to the same problem later (pp. 80–2).

In the meantime he begins with the definitions of the three means and then shows how, given any two of three terms a, b, c in arithmetical, geometrical or harmonical progression, the third can be found. The definition of the mean (b) of three terms a, b, c in harmonic progression being that it satisfies the relation $a : c = a - b : b - c$, Pappus gives alternative definitions for the arithmetic and geometric means in corresponding form, namely for the arithmetic mean $a : a = a - b : b - c$ and for the geometric $a : b = a - b : b - c$.

The construction for the harmonic mean is perhaps worth giving. Let AB, BG be two given straight lines. At A draw DAE perpendicular to AB, and make DA, AE equal. Join DB, BE. From G draw GF at right angles to AB meeting DB in F. Join EF meeting AB in C. Then BC is the required harmonic mean.



For

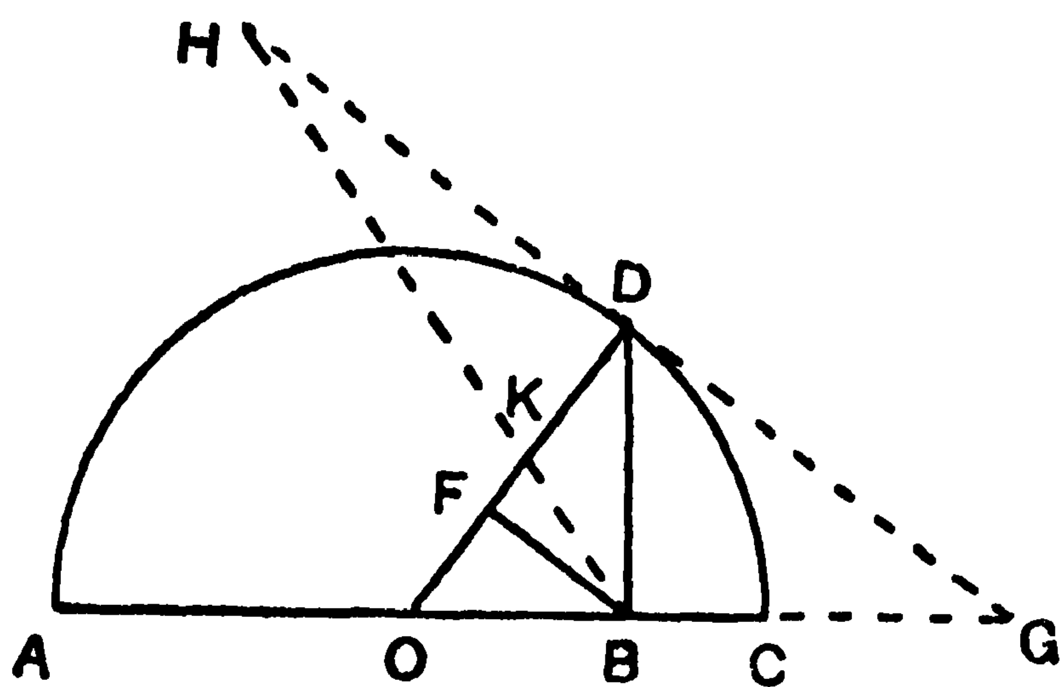
$$\begin{aligned} AB : BG &= DA : FG \\ &= EA : FG \\ &= AC : CG \\ &= (AB - BC) : (BC - BG). \end{aligned}$$

Similarly, by means of a like figure, we can find BG when AB, BC are given, and AB when BC, BG are given (in the latter case the perpendicular DE is drawn through G instead of A).

Then follows a proposition that, if the three means and the several extremes are represented in one set of lines, there must be five of them at least, and, after a set of five such lines have been found in the smallest possible integers, Pappus passes to

the problem of representing the three means with the respective extremes by *six* lines drawn in a semicircle.

Given a semicircle on the diameter AC, and B any point on the diameter, draw BD at right angles to AC. Let the tangent



at D meet AC produced in G, and measure DH along the tangent equal to DG. Join HB meeting the radius OD in K. Let BF be perpendicular to OD.

Then, exactly as above, it is shown that OK is a harmonic mean between OF and OD. Also BD is the geometric mean between AB, BC, while OC $(= OD)$ is the arithmetic mean between AB, BC.

Therefore the *six* lines DO $(= OC)$, OK, OF, AB, BC, BD supply the three means with the respective extremes.

But Pappus seems to have failed to observe that the 'certain other geometer', who has the same figure excluding the dotted lines, supplied the same in *five* lines. For he said that DF is 'a harmonic mean'. It is in fact the harmonic mean between AB, BC, as is easily seen thus.

Since ODB is a right-angled triangle, and BF perpendicular to OD,

$$DF : BD = BD : DO,$$

or
$$DF . DO = BD^2 = AB . BC.$$

But
$$DO = \tfrac{1}{2}(AB + BC);$$

therefore
$$DF . (AB + BC) = 2 AB . BC.$$

Therefore
$$AB . (DF - BC) = BC . (AB - DF),$$

that is,
$$AB : BC = (AB - DF) : (DF - BC),$$

and DF is the harmonic mean between AB, BC.

Consequently the *five* lines DO $(= OC)$, DF, AB, BC, BD exhibit all the three means with the extremes.

Pappus does not seem to have seen this, for he observes that the geometer in question, though saying that DF is a harmonic mean, does not say how it is a harmonic mean or between what straight lines.

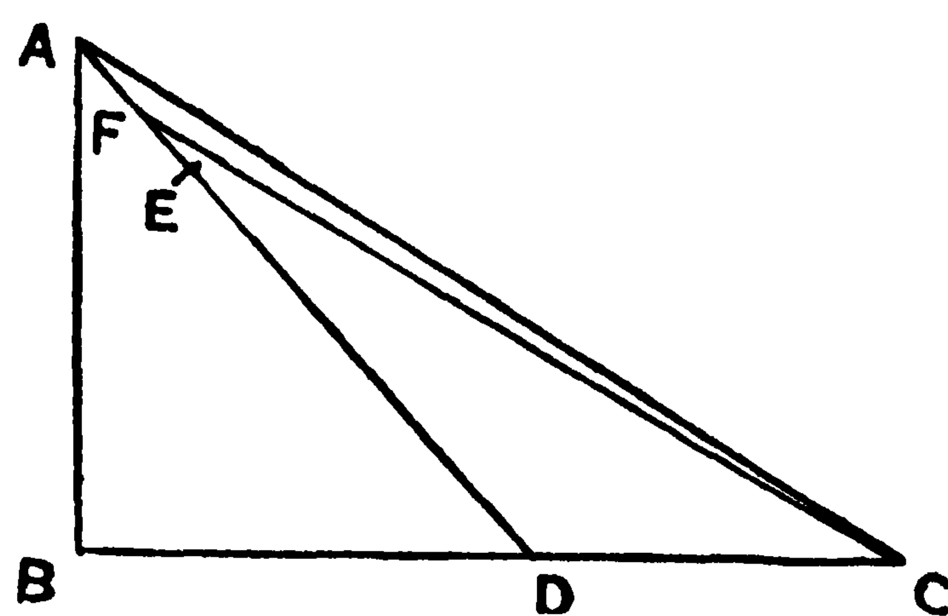
In the next chapters (pp. 84–104) Pappus, following Nicomachus and others, defines seven more means, three of which were ancient and the last four more modern, and shows how we can form all ten means as linear functions of α, β, γ, where α, β, γ are in geometrical progression. The exposition has already been described (vol. i, pp. 86–9).

Section (3). *The 'Paradoxes' of Erycinus.*

The third section of Book III (pp. 104–30) contains a series of propositions, all of the same sort, which are curious rather than geometrically important. They appear to have been taken direct from a collection of *Paradoxes* by one Erycinus.[1] The first set of these propositions (Props. 28–34) are connected with Eucl. I. 21, which says that, if from the extremities of the base of any triangle two straight lines be drawn meeting at any point within the triangle, the straight lines are together less than the two sides of the triangle other than the base, but contain a greater angle. It is pointed out that, if the straight lines are allowed to be drawn from points in the base other than the extremities, their sum may be greater than the other two sides of the triangle.

The first case taken is that of a right-angled triangle ABC right-angled at B. Draw AD to any point D on BC. Measure on it DE equal to AB, bisect AE in F, and join FC. Then shall

$$DF + FC \text{ be } > BA + AC.$$



For $EF + FC = AF + FC > AC$.

Add DE and AB respectively, and we have

$$DF + FC > BA + AC.$$

More elaborate propositions are next proved, such as the following.

1. In any triangle, except an equilateral triangle or an isosceles

[1] Pappus, iii, p. 106. 5–9.

triangle with base less than one of the other sides, it is possible to construct on the base and within the triangle two straight lines meeting at a point, the sum of which is *equal* to the sum of the other two sides of the triangle (Props. 29, 30).

2. In any triangle in which it is possible to construct two straight lines from the base to one internal point the sum of which is equal to the sum of the two sides of the triangle, it is also possible to construct two other such straight lines the sum of which is *greater* than that sum (Prop. 31).

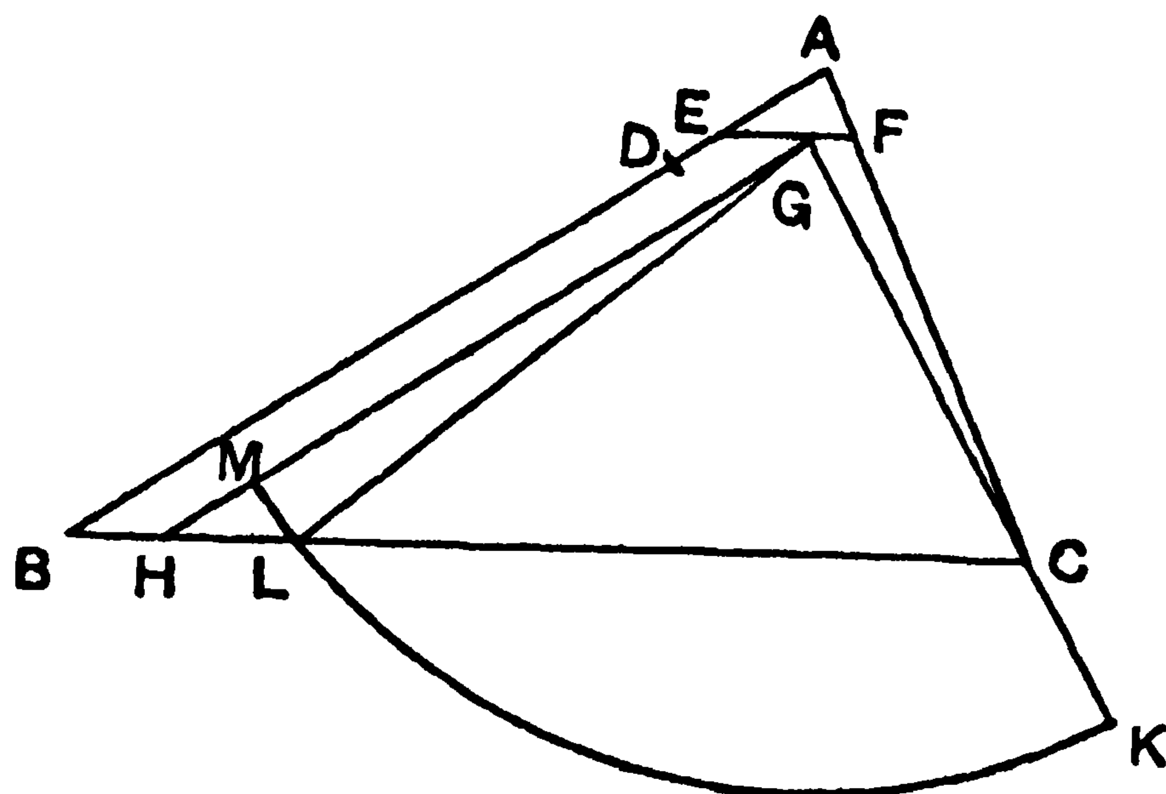
3. Under the same conditions, if the base is greater than either of the other two sides, two straight lines can be so constructed from the base to an internal point which are *respectively* greater than the other two sides of the triangle; and the lines may be constructed so as to be respectively *equal* to the two sides, if one of those two sides is less than the other and each of them is less than the base (Props. 32, 33).

4. The lines may be so constructed that their sum will bear to the sum of the two sides of the triangle any ratio less than 2 : 1 (Prop. 34).

As examples of the proofs, we will take the case of the scalene triangle, and prove the first and Part 1 of the third of the above propositions for such a triangle.

In the triangle ABC with base BC let AB be greater than AC.

Take D on BA such that $BD = \frac{1}{2}(BA + AC)$.



On DA between D and A take any point E, and draw EF parallel to BC. Let G be any point on EF; draw GH parallel to AB and join GC.

Now $EA+AC > EF+FC$

$$> EG+GC \text{ and } > GC, \textit{ a fortiori}.$$

Produce GC to K so that $GK = EA+AC$, and with G as centre and GK as radius describe a circle. This circle will meet HC and HG, because $GH = EB > BD$ or $DA+AC$ and $> GK$, *a fortiori.*

Then $$HG+GL = BE+EA+AC = BA+AC.$$

To obtain two straight lines HG', $G'L$ such that $HG'+G'L > BA+AC$, we have only to choose G' so that HG', $G'L$ enclose the straight lines HG, GL completely.

Next suppose that, given a triangle ABC in which $BC > BA > AC$, we are required to draw from two points on BC to an internal point two straight lines greater *respectively* than BA, AC.

With B as centre and BA as radius describe the arc AEF. Take any point E on it, and any point D on BE produced but within the triangle. Join DC, and produce it to G so that $DG = AC$. Then with D as centre and DG as radius describe a circle. This will meet both BC and BD because $BA > AC$, and *a fortiori* $DB > DG$.

Then, if L be any point on BH, it is clear that BD, DL are two straight lines satisfying the conditions.

A point L' on BH can be found such that DL' is *equal* to AB by marking off DN on DB equal to AB and drawing with D as centre and DN as radius a circle meeting BH in L'. Also, if DH be joined, $DH = AC$.

Propositions follow (35–9) having a similar relation to the Postulate in Archimedes, *On the Sphere and Cylinder*, I, about conterminous broken lines one of which wholly encloses

the other, i.e. it is shown that broken lines, consisting of several straight lines, can be drawn with two points on the base of a triangle or parallelogram as extremities, and of greater total length than the remaining two sides of the triangle or three sides of the parallelogram.

Props. 40–2 show that triangles or parallelograms can be constructed with sides respectively greater than those of a given triangle or parallelogram but having a less area.

Section (4). *The inscribing of the five regular solids in a sphere.*

The fourth section of Book III (pp. 132–62) solves the problems of inscribing each of the five regular solids in a given sphere. After some preliminary lemmas (Props. 43–53), Pappus attacks the substantive problems (Props. 54–8), using the method of analysis followed by synthesis in the case of each solid.

(*a*) In order to inscribe a regular pyramid or tetrahedron in the sphere, he finds two circular sections equal and parallel to one another, each of which contains one of two opposite edges as its diameter. If d be the diameter of the sphere, the parallel circular sections have d' as diameter, where $d^2 = \frac{3}{2}d'^2$.

(*b*) In the case of the cube Pappus again finds two parallel circular sections with diameter d' such that $d^2=\frac{3}{2}d'^2$; a square inscribed in one of these circles is one face of the cube and the square with sides parallel to those of the first square inscribed in the second circle is the opposite face.

(*c*) In the case of the octahedron the same two parallel circular sections with diameter d' such that $d^2 = \frac{3}{2}d'^2$ are used; an equilateral triangle inscribed in one circle is one face, and the opposite face is an equilateral triangle inscribed in the other circle but placed in exactly the opposite way.

(*d*) In the case of the icosahedron Pappus finds four parallel circular sections each passing through three of the vertices of the icosahedron; two of these are small circles circumscribing two opposite triangular faces respectively, and the other two circles are between these two circles, parallel to them, and equal to one another. The pairs of circles are determined in

this way. If d be the diameter of the sphere, set out two straight lines x, y such that d, x, y are in the ratio of the sides of the regular pentagon, hexagon and decagon respectively described in one and the same circle. The smaller pair of circles have r as radius where $r^2 = \frac{1}{3}y^2$, and the larger pair have r' as radius where $r'^2 = \frac{1}{3}x^2$.

(*e*) In the case of the dodecahedron the *same* four parallel circular sections are drawn as in the case of the icosahedron. Inscribed pentagons set the opposite way are inscribed in the two smaller circles; these pentagons form opposite faces. Regular pentagons inscribed in the larger circles with vertices at the proper points (and again set the opposite way) determine ten more vertices of the inscribed dodecahedron.

The constructions are quite different from those in Euclid XIII. 13, 15, 14, 16, 17 respectively, where the problem is first to construct the particular regular solid and then to 'comprehend it in a sphere', i.e. to determine the circumscribing sphere in each case. I have set out Pappus's propositions in detail elsewhere.[1]

Book IV.

At the beginning of Book IV the title and preface are missing, and the first section of the Book begins immediately with an enunciation. The first section (pp. 176–208) contains Propositions 1–12 which, with the exception of Props. 8–10, seem to be isolated propositions given for their own sakes and not connected by any general plan.

Section (1). *Extension of the theorem of Pythagoras.*

The first proposition is of great interest, being the generalization of Eucl. I. 47, as Pappus himself calls it, which is by this time pretty widely known to mathematicians. The enunciation is as follows.

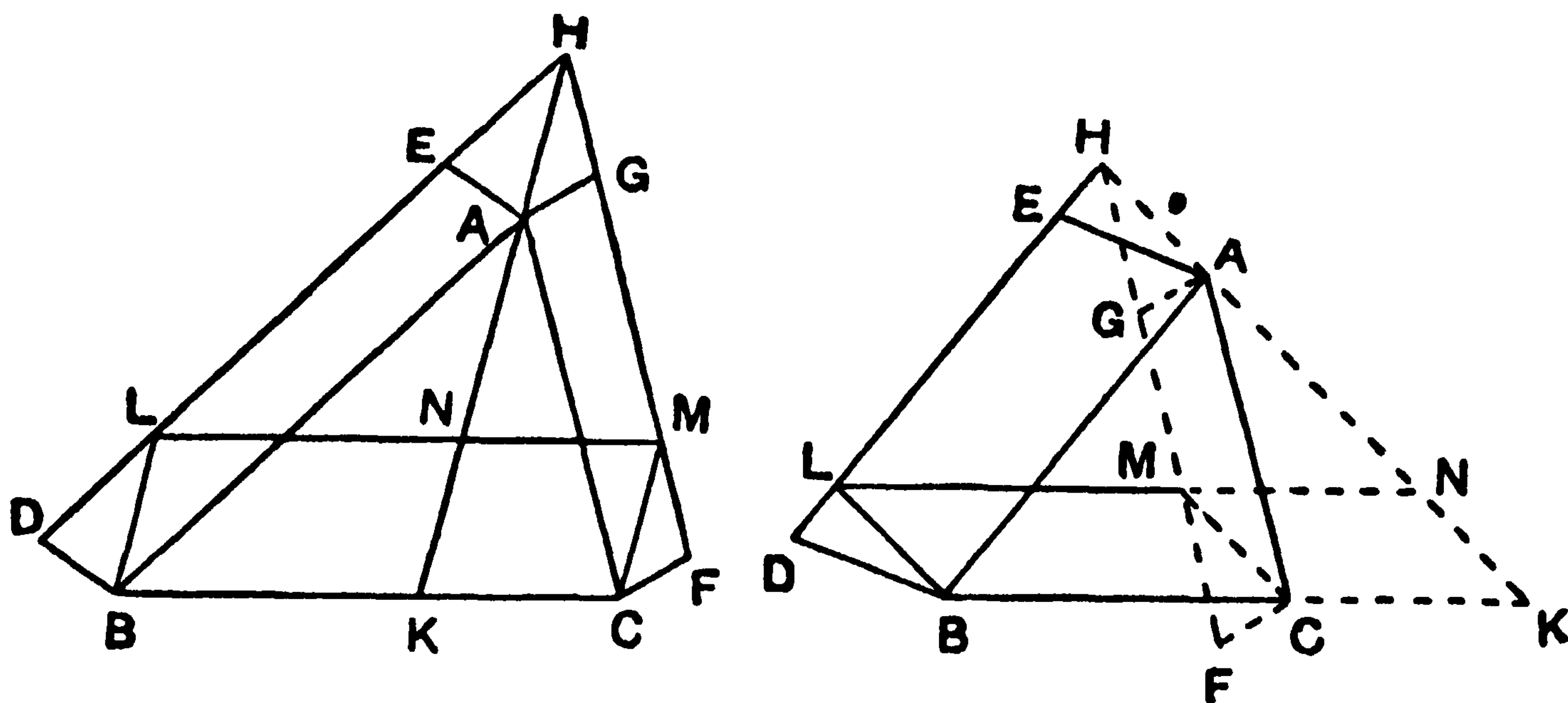
'If ABC be a triangle and on AB, AC any parallelograms whatever be described, as $ABDE$, $ACFG$, and if DE, FG produced meet in H and HA be joined, then the parallelograms $ABDE$, $ACFG$ are together equal to the parallelogram

[1] *Vide* notes to Euclid's propositions in *The Thirteen Books of Euclid's Elements*, pp. 473, 480, 477, 489–91, 501–3.

contained by *BC*, *HA* in an angle which is equal to the sum of the angles *ABC*, *DHA*.'

Produce *HA* to meet *BC* in *K*, draw *BL*, *CM* parallel to *KH* meeting *DE* in *L* and *FG* in *M*, and join *LNM*.

Then *BLHA* is a parallelogram, and *HA* is equal and parallel to *BL*.



Similarly *HA*, *CM* are equal and parallel; therefore *BL*, *CM* are equal and parallel.

Therefore *BLMC* is a parallelogram; and its angle *LBK* is equal to the sum of the angles *ABC*, *DHA*.

Now $\square\, ABDE = \square\, BLHA$, in the same parallels,

$= \square\, BLNK$, for the same reason.

Similarly $\square\, ACFG = \square\, ACMH = \square\, NKCM$.

Therefore, by addition, $\square\, ABDE + \square\, ACFG = \square\, BLMC$.

It has been observed (by Professor Cook Wilson[1]) that the parallelograms on *AB*, *AC* need not necessarily be erected *outwards* from *AB*, *AC*. If one of them, e.g. that on *AC*, be drawn inwards, as in the second figure above, and Pappus's construction be made, we have a similar result with a negative sign, namely,

$$\square\, BLMC = \square\, BLNK - \square\, CMNK$$
$$= \square\, ABDE - \square\, ACFG.$$

Again, if both *ABDE* and *ACFG* were drawn inwards, their sum would be equal to *BLMC* drawn *outwards*. Generally, if the areas of the parallelograms described outwards are regarded as of opposite sign to those of parallelograms drawn inwards,

[1] *Mathematical Gazette*, vii, p. 107 (May 1913).

we may say that the algebraic sum of the three parallelograms is equal to zero.

Though Pappus only takes one case, as was the Greek habit, I see no reason to doubt that he was aware of the results in the other possible cases.

Props. 2, 3 are noteworthy in that they use the method and phraseology of Eucl. X, proving that a certain line in one figure is the irrational called *minor* (see Eucl. X. 76), and a certain line in another figure is 'the excess by which the *binomial* exceeds the *straight line which produces with a rational area a medial whole*' (Eucl. X. 77). The propositions 4-7 and 11–12 are quite interesting as geometrical exercises, but their bearing is not obvious: Props. 4 and 12 are remarkable in that they are cases of analysis followed by synthesis applied to the proof of *theorems*. Props. 8–10 belong to the subject of *tangencies*, being the sort of propositions that would come as particular cases in a book such as that of Apollonius *On Contacts*; Prop. 8 shows that, if there are two equal circles and a given point outside both, the diameter of the circle passing through the point and touching both circles is 'given'; the proof is in many places obscure and assumes lemmas of the same kind as those given later à propos of Apollonius's treatise; Prop. 10 purports to show how, given three unequal circles touching one another two and two, to find the diameter of the circle including them and touching all three.

Section (2). *On circles inscribed in the ἄρβηλος ('shoemaker's knife').*

The next section (pp. 208–32), directed towards the demonstration of a theorem about the relative sizes of successive circles inscribed in the ἄρβηλος (shoemaker's knife), is extremely interesting and clever, and I wish that I had space to reproduce it completely. The ἄρβηλος, which we have already met with in Archimedes's 'Book of Lemmas', is formed thus. *BC* is the diameter of a semicircle *BGC* and *BC* is divided into two parts (in general unequal) at *D*; semicircles are described on *BD, DC* as diameters on the same side of *BC* as *BGC* is; the figure included between the three semicircles is the ἄρβηλος.

There is, says Pappus, on record an ancient proposition to the following effect. Let successive circles be inscribed in the ἄρβηλος touching the semicircles and one another as shown in the figure on p. 376, their centres being A, P, O .... Then, if p_1, p_2, p_3 ... be the perpendiculars from the centres A, P, O... on BC and d_1, d_2, d_3 ... the diameters of the corresponding circles,

$$p_1 = d_1, \quad p_2 = 2d_2, \quad p_3 = 3d_3 \ldots.$$

He begins by some lemmas, the course of which I shall reproduce as shortly as I can.

I. If (Fig. 1) two circles with centres A, C of which the former is the greater touch externally at B, and another circle with centre G touches the two circles at K, L respectively, then KL produced cuts the circle BL again in D and meets AC produced in a point E such that $AB : BC = AE : EC$. This is easily proved, because the circular segments DL, LK are similar, and CD is parallel to AG. Therefore

$$AB : BC = AK : CD = AE : EC.$$

Also $$KE . EL = EB^2.$$

For $AE : EC = AB : BC = AB : CF = (AE - AB) : (EC - CF)$

$$= BE : EF.$$

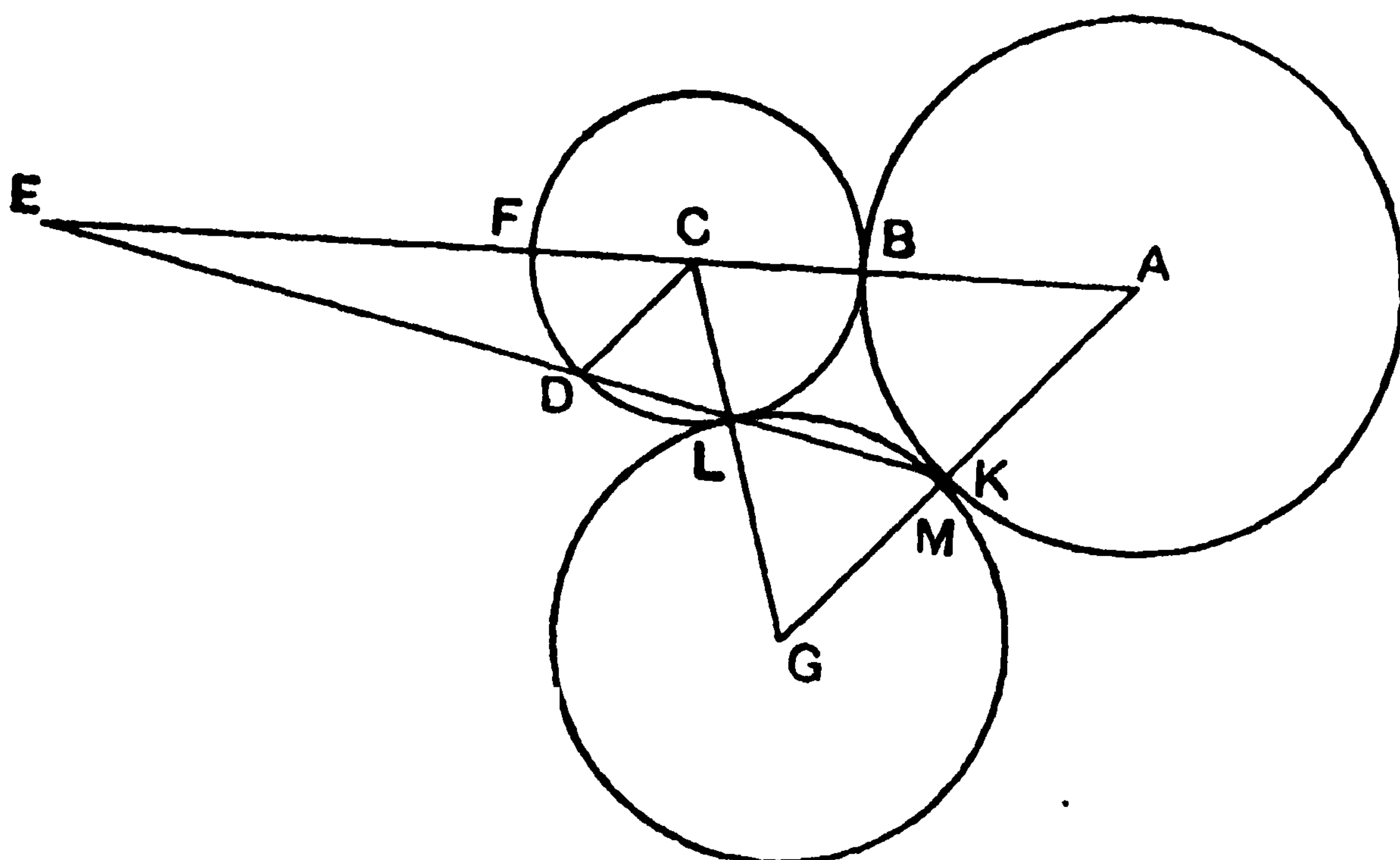


Fig 1.

But $AE : EC = KE : ED$; therefore $KE : ED = BE : EF$.

Therefore $KE . EL : EL . ED = BE^2 : BE . EF$.

And $EL . ED = BE . EF$; therefore $KE . EL = EB^2$.

II. Let (Fig. 2) BC, BD, being in one straight line, be the diameters of two semicircles BGC, BED, and let any circle as FGH touch both semicircles, A being the centre of the circle. Let M be the foot of the perpendicular from A on BC, r the radius of the circle FGH. There are two cases according as BD lies along BC or B lies between D and C; i.e. in the first case the two semicircles are the outer and one of the inner semicircles of the ἄρβηλος, while in the second case they are the two inner semicircles; in the latter case the circle FGH may either include the two semicircles or be entirely external to them. Now, says Pappus, it is to be proved that

in case (1) $BM : r = (BC+BD) : (BC-BD)$,

and in case (2) $BM : r = (BC-BD) : (BC+BD)$.

FIG. 2.

We will confine ourselves to the first case, represented in the figure (Fig. 2).

Draw through A the diameter HF parallel to BC. Then, since the circles BGC, HGF touch at G, and BC, HF are parallel diameters, GHB, GFC are both straight lines.

Let E be the point of contact of the circles FGH and BED; then, similarly, BEF, HED are straight lines.

Let HK, FL be drawn perpendicular to BC.

By the similar triangles BGC, BKH we have

$$BC : BG = BH : BK, \text{ or } CB \,.\, BK = GB \,.\, BH;$$

and by the similar triangles BLF, BED

$$BF : BL = BD : BE, \text{ or } DB \,.\, BL = FB \,.\, BE.$$

But $GB.BH = FB.BE$;

therefore $$CB.BK = DB.BL,$$

or $$BC:BD = BL:BK.$$

Therefore $$(BC+BD):(BC-BD) = (BL+BK):(BL-BK)$$
$$= 2BM:KL.$$

And $KL = HF = 2r$;

therefore $$BM:r = (BC+BD):(BC-BD). \qquad (a)$$

It is next proved that $BK.LC = AM^2$.

For, by similar triangles BKH, FLC,

$$BK:KH = FL:LC, \text{ or } BK.LC = KH.FL$$
$$= AM^2. \qquad (b)$$

Lastly, since $BC:BD = BL:BK$, from above,

$$BC:CD = BL:KL, \text{ or } BL.CD = BC.KL$$
$$= BC.2r. \qquad (c)$$

Also $BD:CD = BK:KL$, or $BK.CD = BD.KL$
$$= BD.2r. \qquad (d)$$

III. We now (Fig. 3) take any two circles touching the semicircles BGC, BED and one another. Let their centres be A and P, H their point of contact, d, d' their diameters respectively. Then, if AM, PN are drawn perpendicular to BC, Pappus proves that

$$(AM+d):d = PN:d'.$$

Draw BF perpendicular to BC and therefore touching the semicircles BGC, BED at B. Join AP, and produce it to meet BF in F.

Now, by II. (a) above,

$$(BC+BD):(BC-BD) = BM:AH,$$

and for the same reason $= BN:PH$;

it follows that $$AH:PH = BM:BN$$
$$= FA:FP.$$

Therefore (Lemma I), if the two circles touch the semicircle BED in R, E respectively, FRE is a straight line and $EF \cdot FR = FH^2$.

But $EF \cdot FR = FB^2$; therefore $FH = FB$.

If now BH meets PN in O and MA produced in S, we have, by similar triangles, $FH : FB = PH : PO = AH : AS$, whence $PH = PO$ and $SA = AH$, so that O, S are the intersections of PN, AM with the respective circles.

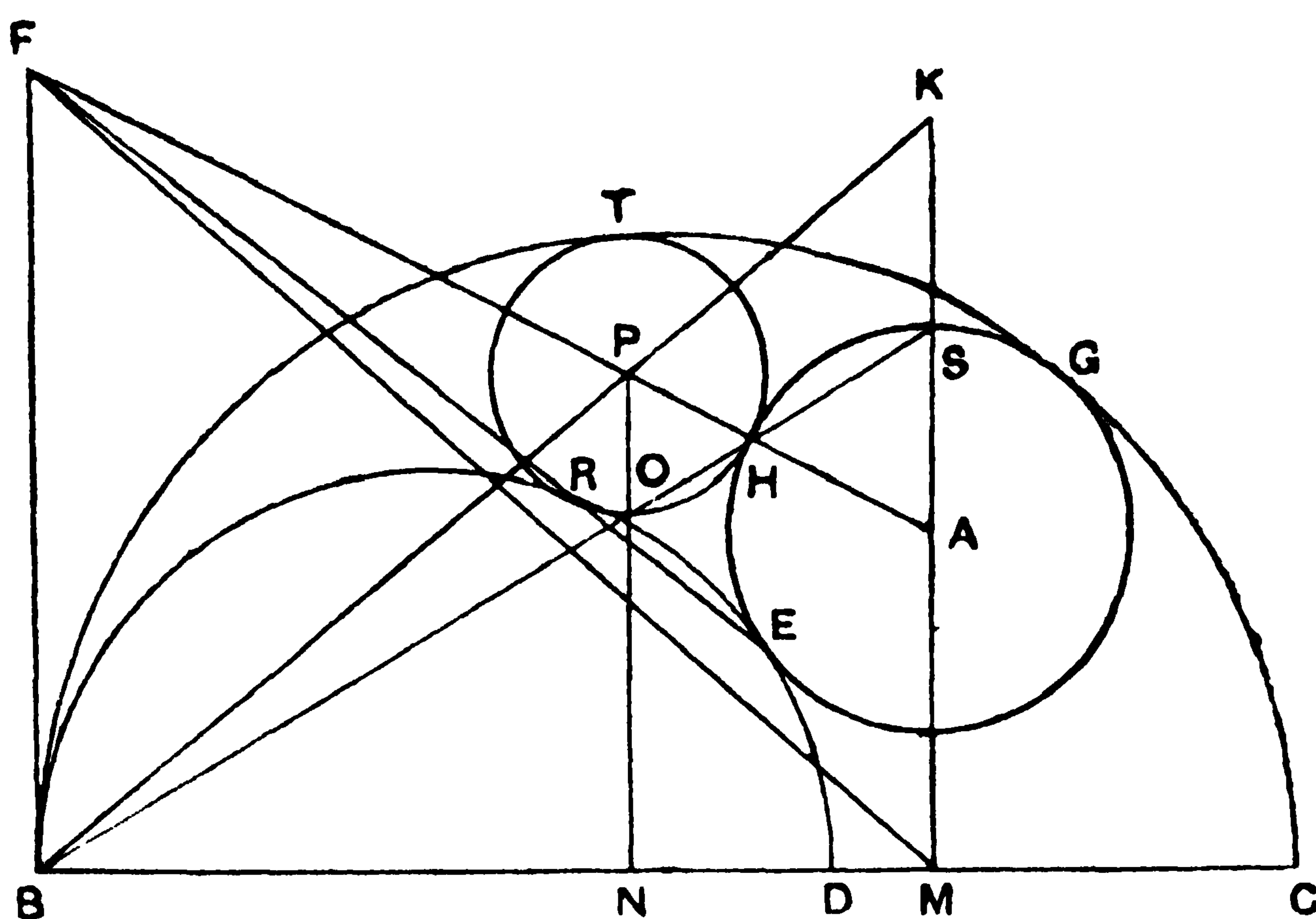


Fig. 3.

Join BP, and produce it to meet MA in K.

Now $$BM : BN = FA : FP$$
$$= AH : PH, \text{ from above,}$$
$$= AS : PO.$$

And $$BM : BN = BK : BP$$
$$= KS : PO.$$

Therefore $KS = AS$, and $KA = d$, the diameter of the circle EHG.

Lastly, $$MK : KS = PN : PO,$$

that is, $$(AM + d) : \tfrac{1}{2}d = PN : \tfrac{1}{2}d',$$

or $$(AM + d) : d = PN : d'.$$

IV. We now come to the substantive theorem.

Let FGH be the circle touching all three semicircles (Fig. 4). We have then, as in Lemma II,

$$BC \,.\, BK = BD \,.\, BL,$$

and for the same reason (regarding FGH as touching the semicircles BGC, DUC)

$$BC \,.\, CL = CD \,.\, CK.$$

From the first relation we have

$$BC : BD = BL : BK,$$

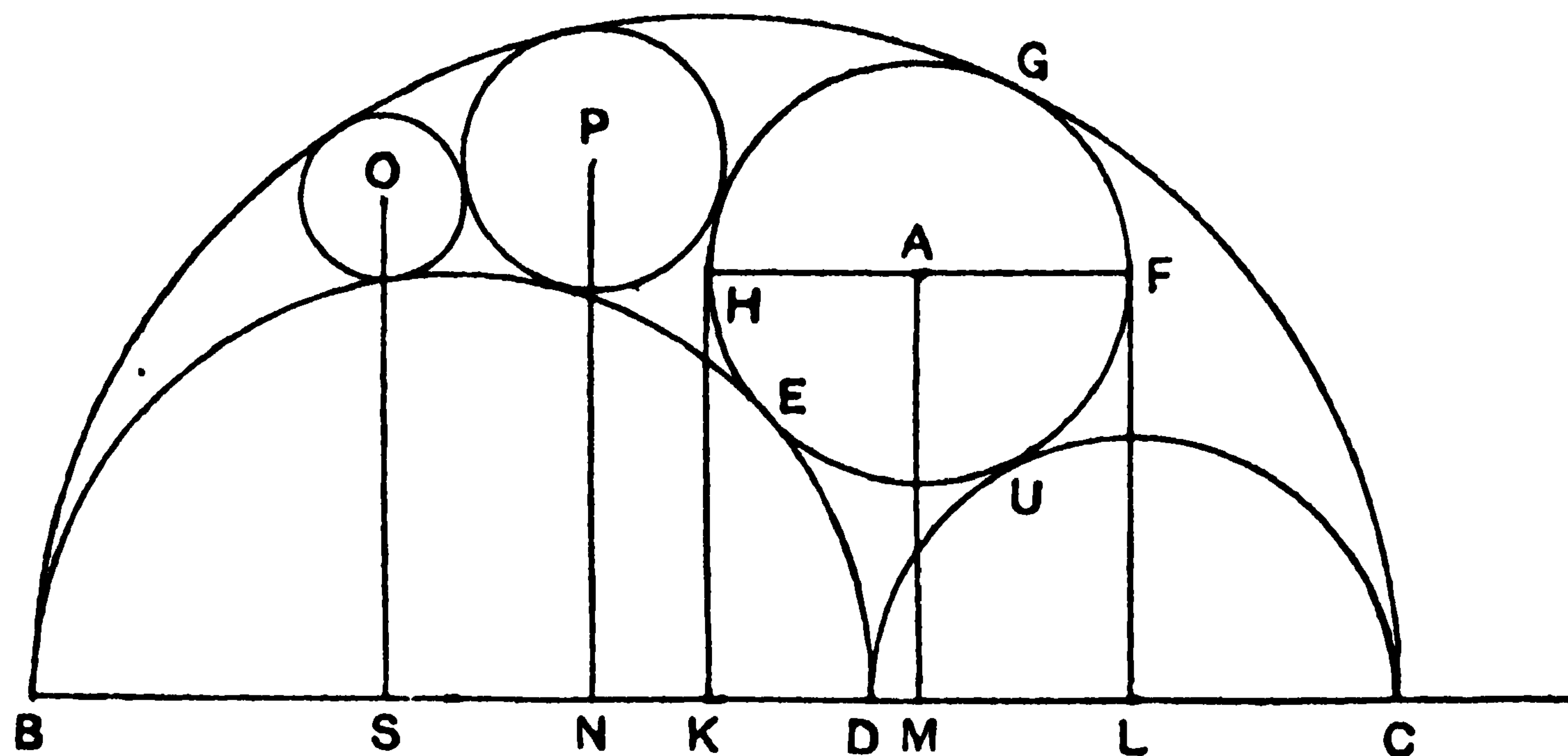


FIG. 4.

whence $DC : BD = KL : BK$, and inversely $BD : DC = BK : KL$, while, from the second relation, $BC : CD = CK : CL$,

whence $\qquad BD : DC = KL : CL.$

Consequently $\qquad BK : KL = KL : CL,$

or $\qquad BK \,.\, LC = KL^2.$

But we saw in Lemma II (*b*) that $BK \,.\, LC = AM^2$.

Therefore $\qquad KL = AM$, or $p_1 = d_1$.

For the second circle Lemma III gives us

$$(p_1 + d_1) : d_1 = p_2 : d_2,$$

whence, since $p_1 = d_1$, $p_2 = 2d_2$.

For the third circle

$$(p_2 + d_2) : d_2 = p_3 : d_3,$$

whence $p_3 = 3d_3$.

And so on *ad infinitum*.

The same proposition holds when the successive circles, instead of being placed between the large and one of the small semicircles, come down between the two small semicircles.

Pappus next deals with special cases (1) where the two smaller semicircles become straight lines perpendicular to the diameter of the other semicircle at its extremities, (2) where we replace one of the smaller semicircles by a straight line through D at right angles to BC, and lastly (3) where instead of the semicircle DUC we simply have the straight line DC and make the first circle touch it and the two other semicircles.

Pappus's propositions of course include as particular cases the partial propositions of the same kind included in the 'Book of Lemmas' attributed to Archimedes (Props. 5, 6); cf. p. 102.

Sections (3) and (4). *Methods of squaring the circle, and of trisecting (or dividing in any ratio) any given angle.*

The last sections of Book IV (pp. 234–302) are mainly devoted to the solutions of the problems (1) of squaring or rectifying the circle and (2) of trisecting any given angle or dividing it into two parts in any ratio. To this end Pappus gives a short account of certain curves which were used for the purpose.

(α) *The Archimedean spiral.*

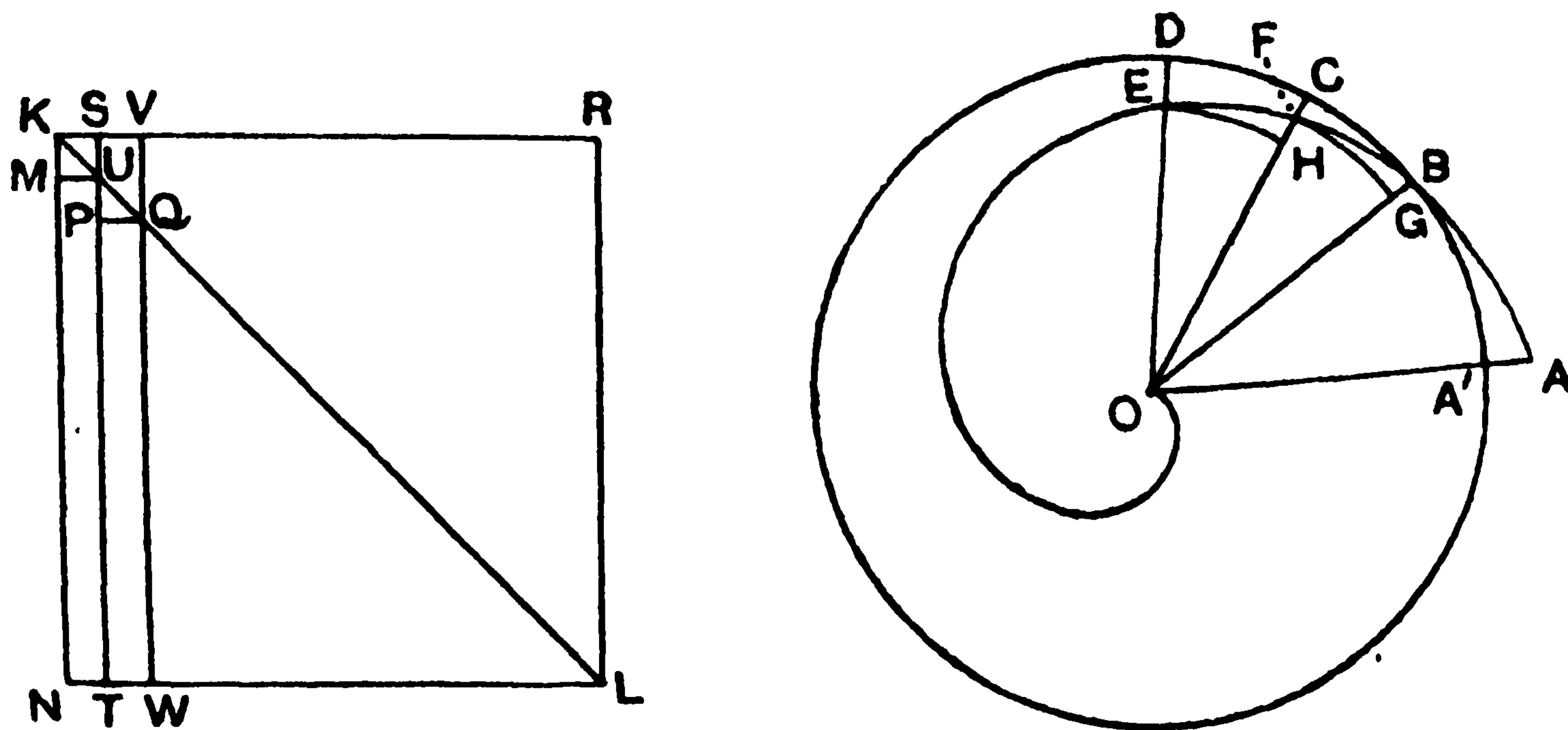
He begins with the spiral of Archimedes, proving some of the fundamental properties. His method of finding the area included (1) between the first turn and the initial line, (2) between any radius vector on the first turn and the curve, is worth giving because it differs from the method of Archimedes. It is the area of the whole first turn which Pappus works out in detail. We will take the area up to the radius vector OB, say.

With centre O and radius OB draw the circle $A'BCD$.

Let BC be a certain fraction, say $1/n$th, of the arc $BCDA'$, and CD the same fraction, OC, OD meeting the spiral in F, E respectively. Let KS, SV be the same fraction of a straight line KR, the side of a square $KNLR$. Draw ST, VW parallel to KN meeting the diagonal KL of the square in U, Q respectively, and draw MU, PQ parallel to KR.

With O as centre and OE, OF as radii draw arcs of circles meeting OF, OB in H, G respectively.

For brevity we will now denote a cylinder in which r is the radius of the base and h the height by (cyl. r, h) and the cone with the same base and height by (cone r, h).



By the property of the spiral,

$$OB : BG = (\text{arc } A'DCB) : (\text{arc } CB)$$
$$= RK : KS$$
$$= NK : KM,$$

whence $$OB : OG = NK : NM.$$

Now

$$(\text{sector } OBC) : (\text{sector } OGF) = OB^2 : OG^2 = NK^2 : MN^2$$
$$= (\text{cyl. } KN, NT) : (\text{cyl. } MN, NT).$$

Similarly

$$(\text{sector } OCD) : (\text{sector } OEH) = (\text{cyl. } ST, TW) : (\text{cyl. } PT, TW),$$

and so on.

The sectors OBC, OCD ... form the sector $OA'DB$, and the sectors OFG, OEH ... form a figure inscribed to the spiral. In like manner the cylinders (KN, TN), (ST, TW) ... form the cylinder (KN, NL), while the cylinders (MN, NT), (PT, TW) ... form a figure inscribed to the cone (KN, NL).

Consequently

$$(\text{sector } OA'DB) : (\text{fig. inscr. in spiral})$$
$$= (\text{cyl. } KN, NL) : (\text{fig. inscr. in cone } KN, NL).$$

We have a similar proportion connecting a figure circumscribed to the spiral and a figure circumscribed to the cone.

By increasing n the inscribed and circumscribed figures can be compressed together, and by the usual method of exhaustion we have ultimately

$$(\text{sector } OA'DB) : (\text{area of spiral}) = (\text{cyl. } KN, NL) : (\text{cone } KN, NL)$$
$$= 3 : 1,$$

or (area of spiral cut off by OB) $= \frac{1}{3}$ (sector $OA'DB$).

The ratio of the sector $OA'DB$ to the complete circle is that of the angle which the radius vector describes in passing from the position OA to the position OB to four right angles, that is, by the property of the spiral, $r : a$, where $r = OB$, $a = OA$.

Therefore (area of spiral cut off by OB) $= \frac{1}{3}\frac{r}{a} \cdot \pi r^2$.

Similarly the area of the spiral cut off by any other radius vector $r' = \frac{1}{3}\frac{r'}{a} \cdot \pi r'^2$.

Therefore (as Pappus proves in his next proposition) the first area is to the second as r^3 to r'^3.

Considering the areas cut off by the radii vectores at the points where the revolving line has passed through angles of $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$ and 2π respectively, we see that the areas are in the ratio of $(\frac{1}{4})^3$, $(\frac{1}{2})^3$, $(\frac{3}{4})^3$, 1 or 1, 8, 27, 64, so that the areas of the spiral included in the four quadrants are in the ratio of 1, 7, 19, 37 (Prop. 22).

(β) *The conchoid of Nicomedes.*

The conchoid of Nicomedes is next described (chaps. 26–7), and it is shown (chaps. 28, 29) how it can be used to find two geometric means between two straight lines, and consequently to find a cube having a given ratio to a given cube (see vol. i, pp. 260–2 and pp. 238–40, where I have also mentioned Pappus's remark that the conchoid which he describes is the *first* conchoid, while there also exist a *second*, a *third* and a *fourth* which are of use for other theorems).

(γ) *The quadratrix.*

The *quadratrix* is taken next (chaps. 30–2), with Sporus's criticism questioning the construction as involving a *petitio*

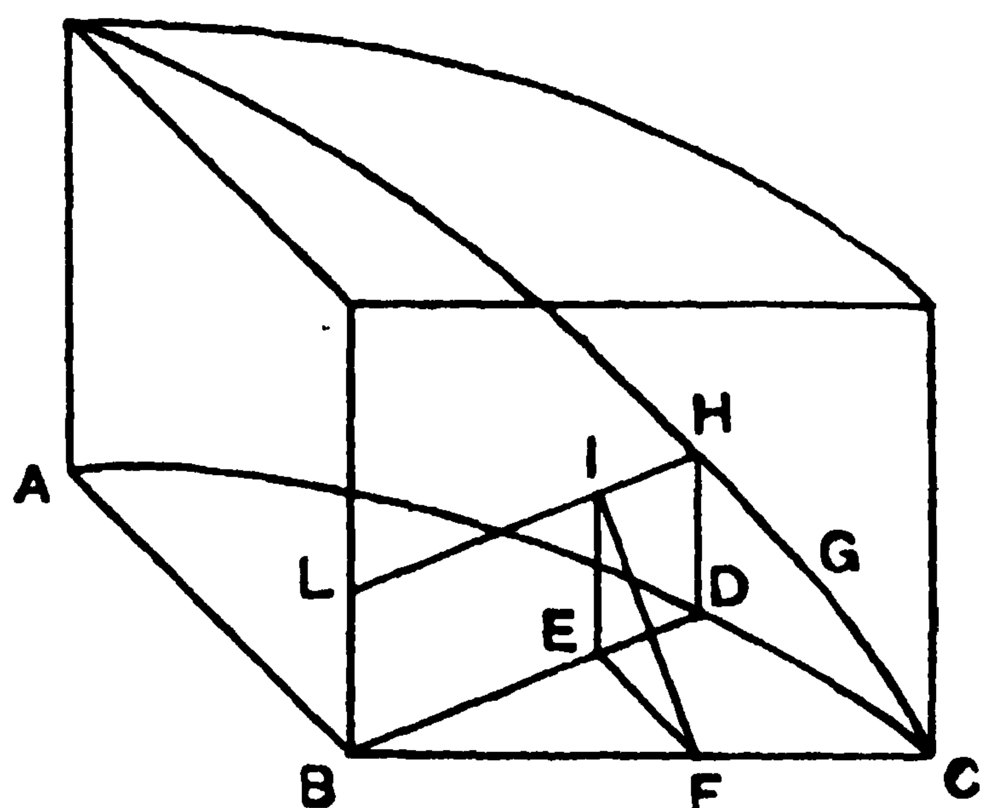
principii. Its use for squaring the circle is attributed to Dinostratus and Nicomedes. The whole substance of this subsection is given above (vol. i, pp. 226–30).

Two constructions for the quadratrix by means of 'surface-loci'.

In the next chapters (chaps. 33, 34, Props. 28, 29) Pappus gives two alternative ways of producing the *quadratrix* 'by means of surface-loci', for which he claims the merit that they are geometrical rather than 'too mechanical' as the traditional method (of Hippias) was.

(1) The first method uses a cylindrical helix thus.

Let ABC be a quadrant of a circle with centre B, and let BD be any radius. Suppose that EF, drawn from a point E on the radius BD perpendicular to BC, is (for all such radii) in a given ratio to the arc DC.



'I say', says Pappus, 'that the locus of E is a certain curve.'

Suppose a right cylinder erected from the quadrant and a cylindrical helix CGH drawn upon its surface. Let DH be the generator of this cylinder through D, meeting the helix in H. Draw BL, EI at right angles to the plane of the quadrant, and draw HIL parallel to BD.

Now, by the property of the helix, $EI\,(= DH)$ is to the arc CD in a given ratio. Also EF : (arc CD) = a given ratio. Therefore the ratio $EF : EI$ is given. And since EF, EI are given in position, FI is given in position. But FI is perpendicular to BC. Therefore FI is in a plane given in position, and so therefore is I.

But I is also on a certain surface described by the line LH, which moves always parallel to the plane ABC, with one extremity L on BL and the other extremity H on the helix. Therefore I lies on the intersection of this surface with the plane through FI.

Hence I lies on a certain curve. Therefore E, its projection on the plane ABC, also lies on a curve.

In the particular case where the given ratio of EF to the arc CD is equal to the ratio of BA to the arc CA, the locus of E is a *quadratrix.*

[The surface described by the straight line LH is a *plectoid.* The shape of it is perhaps best realized as a *continuous* spiral staircase, i.e. a spiral staircase with infinitely small steps. The *quadratrix* is thus produced as the orthogonal projection of the curve in which the plectoid is intersected by a plane through BC inclined at a given angle to the plane ABC. It is not difficult to verify the result analytically.]

(2) The second method uses a right cylinder the base of which is an Archimedean spiral.

Let ABC be a quadrant of a circle, as before, and EF, perpendicular at F to BC, a straight line of such length that EF is to the arc DC as AB is to the arc ADC.

Let a point on AB move uniformly from A to B while, in the same time, AB itself revolves uniformly about B from the position BA to the position BC. The point thus describes the spiral AGB. If the spiral cuts BD in G,

$$BA : BG = (\text{arc } ADC) : (\text{arc } DC),$$

or

$$BG : (\text{arc } DC) = BA : (\text{arc } ADC).$$

Therefore $BG = EF$.

Draw GK at right angles to the plane ABC and equal to BG. Then GK, and therefore K, lies on a right cylinder with the spiral as base.

But BK also lies on a conical surface with vertex B such that its generators all make an angle of $\frac{1}{4}\pi$ with the plane ABC.

Consequently K lies on the intersection of two surfaces, and therefore on a curve.

Through K draw LKI parallel to BD, and let BL, EI be at right angles to the plane ABC.

Then LKI, moving always parallel to the plane ABC, with one extremity on BL and passing through K on a certain

curve, describes a certain plectoid, which therefore contains the point *I*.

Also $IE = EF$, *IF* is perpendicular to *BC*, and hence *IF*, and therefore *I*, lies on a fixed plane through *BC* inclined to *ABC* at an angle of $\frac{1}{4}\pi$.

Therefore *I*, lying on the intersection of the plectoid and the said plane, lies on a certain curve. So therefore does the projection of *I* on *ABC*, i.e. the point *E*.

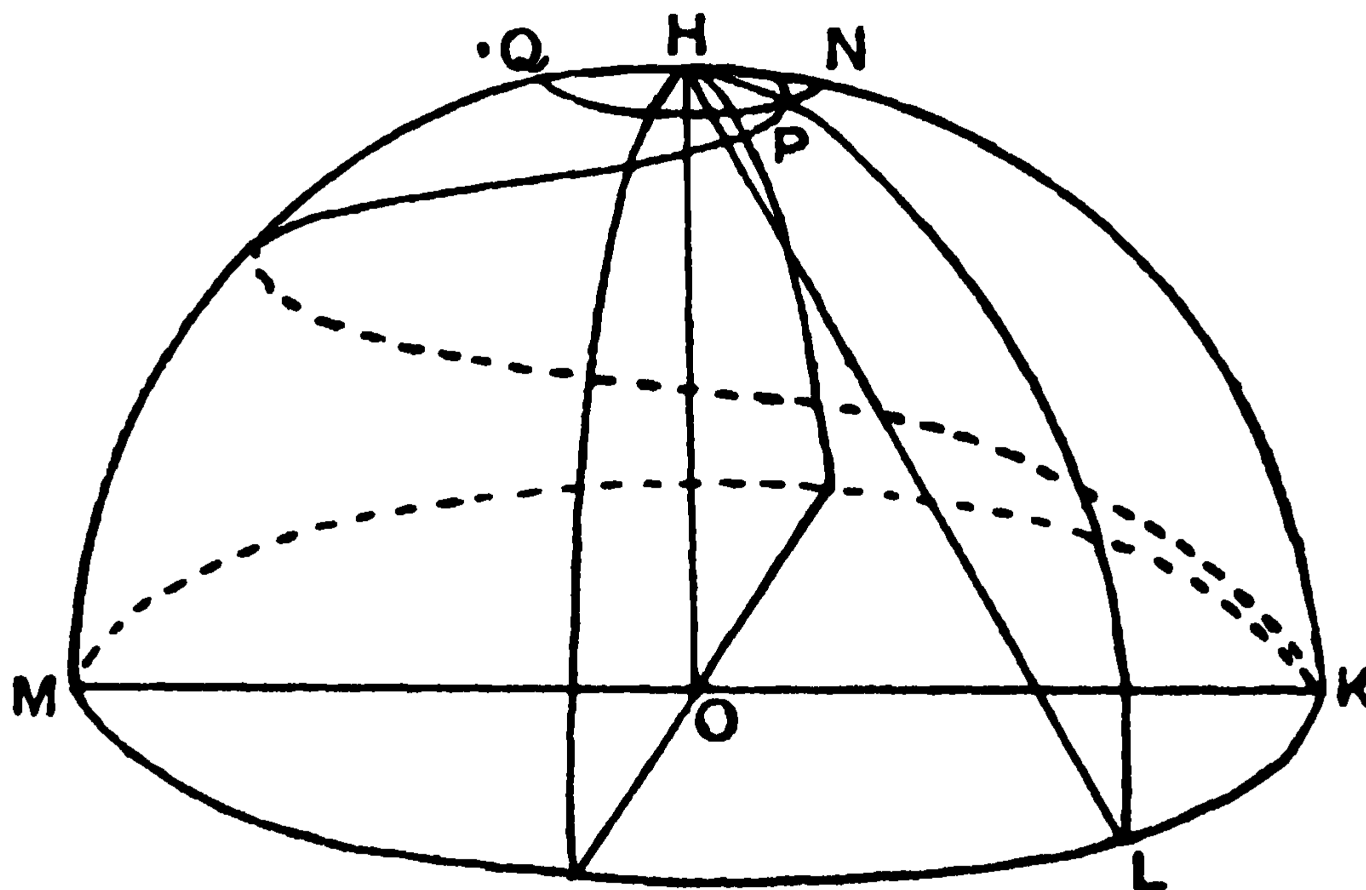
The locus of *E* is clearly the *quadratrix*.

[This result can also be verified analytically.]

(δ) *Digression: a spiral on a sphere.*

Prop. 30 (chap. 35) is a digression on the subject of a certain spiral described on a sphere, suggested by the discussion of a spiral in a plane.

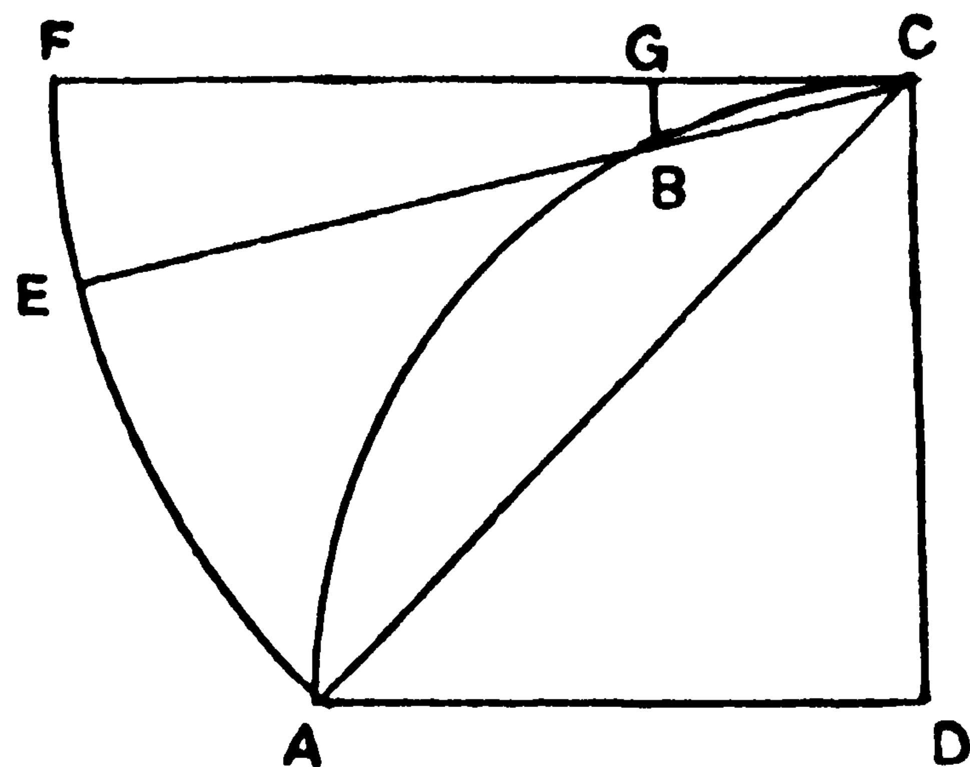
Take a hemisphere bounded by the great circle *KLM*, with *H* as pole. Suppose that the quadrant of a great circle *HNK* revolves uniformly about the radius *HO* so that *K* describes the circle *KLM* and returns to its original position at *K*, and suppose that a point moves uniformly at the same



time from *H* to *K* at such speed that the point arrives at *K* at the same time that *HK* resumes its original position. The point will thus describe a spiral on the surface of the sphere between the points *H* and *K* as shown in the figure.

Pappus then sets himself to prove that the portion of the surface of the sphere cut off towards the pole between the spiral and the arc *HNK* is to the surface of the hemisphere in

a certain ratio shown in the second figure where ABC is a quadrant of a circle equal to a great circle in the sphere, namely the ratio of the segment ABC to the sector $DABC$.



Draw the tangent CF to the quadrant at C. With C as centre and radius CA draw the circle AEF meeting CF in F.

Then the sector CAF is equal to the sector ADC (since $CA^2 = 2AD^2$, while $\angle ACF = \frac{1}{2}\angle ADC$).

It is required, therefore, to prove that, if S be the area cut off by the spiral as above described,

$$S : (\text{surface of hemisphere}) = (\text{segmt. } ABC) : (\text{sector } CAF).$$

Let KL be a (small) fraction, say $1/n$th, of the circumference of the circle KLM, and let HPL be the quadrant of the great circle through H, L meeting the spiral in P. Then, by the property of the spiral,

$$(\text{arc } HP) : (\text{arc } HL) = (\text{arc } KL) : (\text{circumf. of } KLM)$$
$$= 1 : n.$$

Let the small circle NPQ passing through P be described about the pole H.

Next let FE be the same fraction, $1/n$th, of the arc FA that KL is of the circumference of the circle KLM, and join EC meeting the arc ABC in B. With C as centre and CB as radius describe the arc BG meeting CF in G.

Then the arc CB is the same fraction, $1/n$th, of the arc CBA that the arc FE is of FA (for it is easily seen that $\angle FCE = \frac{1}{2}\angle BDC$, while $\angle FCA = \frac{1}{2}\angle CDA$). Therefore, since (arc CBA) = (arc HPL), (arc CB) = (arc HP), and chord CB = chord HP.

Now (sector HPN on sphere) : (sector HKL on sphere)

$$= (\text{chord } HP)^2 : (\text{chord } HL)^2$$

(a consequence of Archimedes, *On Sphere and Cylinder*, I. 42).

And $$HP^2 : HL^2 = CB^2 : CA^2$$
$$= CB^2 : CE^2.$$

Therefore

(sector HPN) : (sector HKL) = (sector CBG) : (sector CEF).

Similarly, if the arc LL' be taken equal to the arc KL and the great circle through H, L' cuts the spiral in P', and a small circle described about H and through P' meets the arc HPL in p; and if likewise the arc BB' is made equal to the arc BC, and CB' is produced to meet AF in E', while again a circular arc with C as centre and CB' as radius meets CE in b,

(sector $HP'p$ on sphere) : (sector HLL' on sphere)

$$= (\text{sector } CB'b) : (\text{sector } CE'E).$$

And so on.

Ultimately then we shall get a figure consisting of sectors on the sphere circumscribed about the area S of the spiral and a figure consisting of sectors of circles circumscribed about the segment CBA; and in like manner we shall have inscribed figures in each case similarly made up.

The method of exhaustion will then give

S : (surface of hemisphere) = (segmt. ABC) : (sector CAF)

$$= (\text{segmt. } ABC) : (\text{sector } DAC).$$

[We may, as an illustration, give the analytical equivalent of this proposition. If ρ, ω be the spherical coordinates of P with reference to H as pole and the arc HNK as polar axis, the equation of Pappus's curve is obviously $\omega = 4\rho$.

If now the radius of the sphere is taken as unity, we have as the element of area

$$dA = d\omega\,(1-\cos\rho) = 4\,d\rho\,(1-\cos\rho).$$

Therefore $$A = \int_0^{\frac{1}{2}\pi} 4\,d\rho\,(1-\cos\rho) = 2\pi - 4.$$

Therefore

$$\frac{A}{(\text{surface of hemisphere})} = \frac{2\pi - 4}{2\pi} = \frac{\frac{1}{4}\pi - \frac{1}{2}}{\frac{1}{4}\pi}$$

$$= \frac{(\text{segment } ABC)}{(\text{sector } DABC)}.]$$

The second part of the last section of Book IV (chaps. 36–41, pp. 270–302) is mainly concerned with the problem of trisecting any given angle or dividing it into parts in any given ratio. Pappus begins with another account of the distinction between *plane, solid* and *linear* problems (cf. Book III, chaps. 20–2) according as they require for their solution (1) the straight line and circle only, (2) conics or their equivalent, (3) higher curves still, 'which have a more complicated and forced (or unnatural) origin, being produced from more irregular surfaces and involved motions. Such are the curves which are discovered in the so-called *loci on surfaces*, as well as others more complicated still and many in number discovered by Demetrius of Alexandria in his *Linear considerations* and by Philon of Tyana by means of the interlacing of plectoids and other surfaces of all sorts, all of which curves possess many remarkable properties peculiar to them. Some of these curves have been thought by the more recent writers to be worthy of considerable discussion; one of them is that which also received from Menelaus the name of the *paradoxical* curve. Others of the same class are spirals, quadratrices, cochloids and cissoids.' He adds the often-quoted reflection on the error committed by geometers when they solve a problem by means of an 'inappropriate class' (of curve or its equivalent), illustrating this by the use in Apollonius, Book V, of a rectangular hyperbola for finding the feet of normals to a *parabola* passing through one point (where a circle would serve the purpose), and by the assumption by Archimedes of a *solid* νεῦσις in his book *On Spirals* (see above, pp. 65–8).

Trisection (or division in any ratio) of any angle.

The method of trisecting any angle based on a certain νεῦσις is next described, with the solution of the νεῦσις itself by

means of a hyperbola which has to be constructed from certain data, namely the asymptotes and a certain point through which the curve must pass (this easy construction is given in Prop. 33, chap. 41–2). Then the problem is directly solved (chaps. 43, 44) by means of a hyperbola in two ways practically equivalent, the hyperbola being determined in the one case by the ordinary Apollonian property, but in the other by means of the *focus-directrix* property. Solutions follow of the problem of dividing any angle in a given ratio by means (1) of the *quadratrix*, (2) of the spiral of Archimedes (chaps. 45, 46). All these solutions have been sufficiently described above (vol. i, pp. 235–7, 241–3, 225–7).

Some problems follow (chaps. 47–51) depending on these results, namely those of constructing an isosceles triangle in which either of the base angles has a given ratio to the vertical angle (Prop. 37), inscribing in a circle a regular polygon of any number of sides (Prop. 38), drawing a circle the circumference of which shall be equal to a given straight line (Prop. 39), constructing on a given straight line *AB* a segment of a circle such that the arc of the segment may have a given ratio to the base (Prop. 40), and constructing an angle incommensurable with a given angle (Prop. 41).

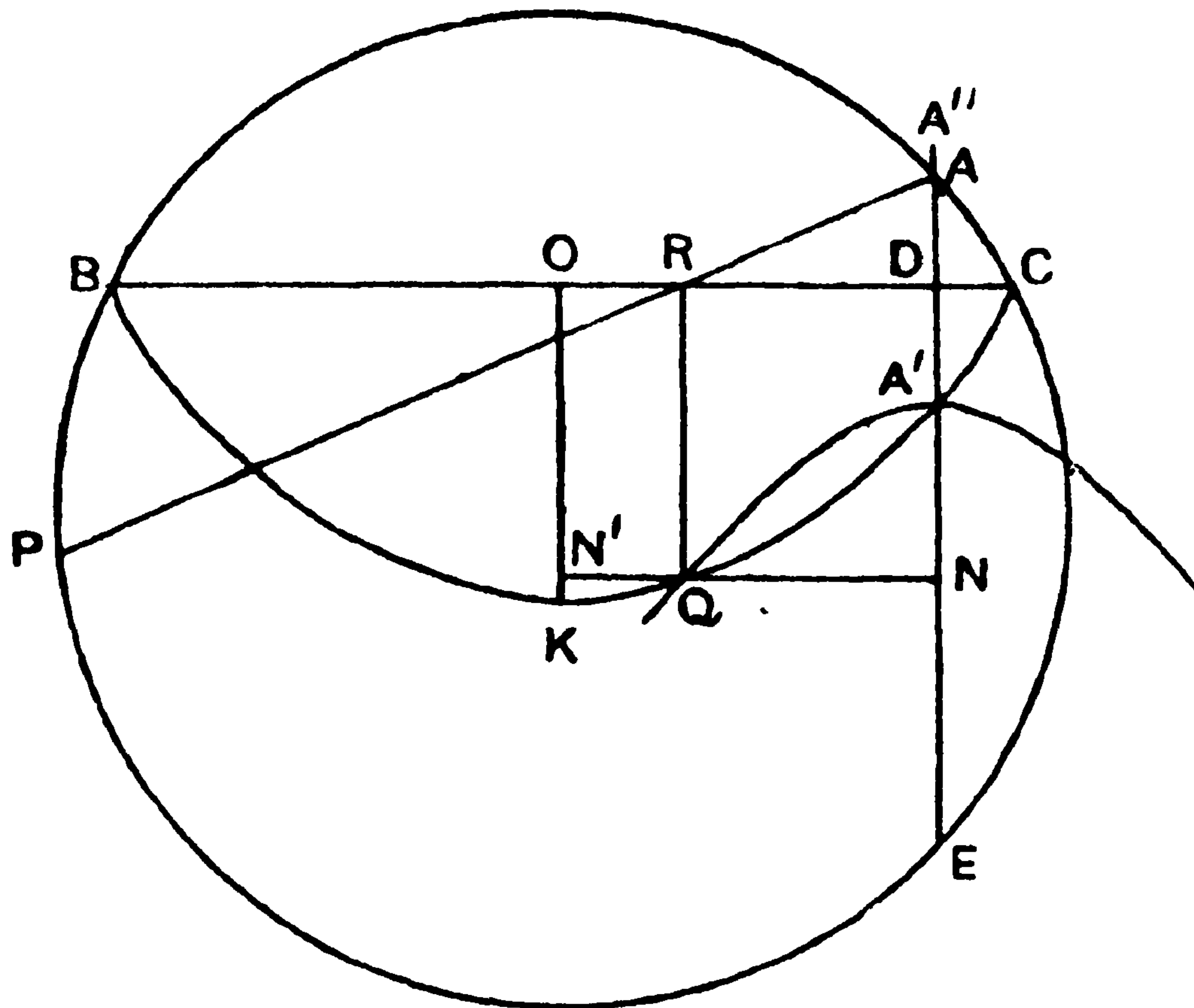
Section (5). *Solution of the νεῦσις of Archimedes, 'On Spirals', Prop. 8, by means of conics.*

Book IV concludes with the solution of the νεῦσις which, according to Pappus, Archimedes unnecessarily assumed in *On Spirals*, Prop. 8. Archimedes's assumption is this. Given a circle, a chord (*BC*) in it less than the diameter, and a point *A* on the circle the perpendicular from which to *BC* cuts *BC* in a point *D* such that $BD > DC$ and meets the circle again in *E*, it is possible to draw through *A* a straight line *ARP* cutting *BC* in *R* and the circle in *P* in such a way that *RP* shall be equal to *DE* (or, in the phraseology of νεύσεις, to place between the straight line *BC* and the circumference of the circle a straight line equal to *DE* and *verging* towards *A*).

Pappus makes the problem rather more general by not requiring *PR* to be equal to *DE*, but making it of any given

length (consistent with a real solution). The problem is best exhibited by means of analytical geometry.

If $BD = a$, $DC = b$, $AD = c$ (so that $DE = ab/c$), we have



to find the point R on BC such that AR produced solves the problem by making PR equal to k, say.

Let $DR = x$. Then, since $BR.RC = PR.RA$, we have

$$(a-x)(b+x) = k\sqrt{(c^2+x^2)}.$$

An obvious expedient is to put y for $\sqrt{(c^2+x^2)}$, when we have

$$(a-x)(b+x) = ky, \qquad (1)$$

and

$$y^2 = c^2+x^2. \qquad (2)$$

These equations represent a parabola and a hyperbola respectively, and Pappus does in fact solve the problem by means of the intersection of a parabola and a hyperbola; one of his preliminary lemmas is, however, again a little more general. In the above figure y is represented by RQ.

The first lemma of Pappus (Prop. 42, p. 298) states that, if from a given point A any straight line be drawn meeting a straight line BC given in position in R, and if RQ be drawn at right angles to BC and of length bearing a given ratio to AR, the locus of Q is a *hyperbola*.

For draw AD perpendicular to BC and produce it to A' so that

$$QR:RA = A'D:DA = \text{the given ratio.}$$

Measure DA'' along DA equal to DA'.

Then, if QN be perpendicular to AD,

$$(AR^2 - AD^2):(QR^2 - A'D^2) = (\text{const.}),$$

that is, $$QN^2 : A'N \cdot A''N = (\text{const.}),$$

and the locus of Q is a hyperbola.

The equation of the hyperbola is clearly

$$x^2 = \mu(y^2 - c^2),$$

where μ is a constant. In the particular case taken by Archimedes $QR = RA$, or $\mu = 1$, and the hyperbola becomes the rectangular hyperbola (2) above.

The second lemma (Prop. 43, p. 300) proves that, if BC is given in length, and Q is such a point that, when QR is drawn perpendicular to BC, $BR \cdot RC = k \cdot QR$, where k is a given length, the locus of Q is a *parabola*.

Let O be the middle point of BC, and let OK be drawn at right angles to BC and of length such that

$$OC^2 = k \cdot KO.$$

Let QN' be drawn perpendicular to OK.

Then
$$\begin{aligned} QN'^2 &= OR^2 \\ &= OC^2 - BR \cdot RC \\ &= k \cdot (KO - QR), \text{ by hypothesis,} \\ &= k \cdot KN'. \end{aligned}$$

Therefore the locus of Q is a parabola.

The equation of the parabola referred to DB, DE as axes of x and y is obviously

$$\{\tfrac{1}{2}(a-b) - x\}^2 = k\left\{\frac{(a+b)^2}{4k} - y\right\},$$

which easily reduces to

$$(a-x)\,(b+x) = ky, \text{ as above (1).}$$

In Archimedes's particular case $k = ab/c$.

To solve the problem then we have only to draw the parabola and hyperbola in question, and their intersection then gives Q, whence R, and therefore ARP, is determined.

Book V. Preface on the Sagacity of Bees.

It is characteristic of the great Greek mathematicians that, whenever they were free from the restraint of the technical language of mathematics, as when for instance they had occasion to write a preface, they were able to write in language of the highest literary quality, comparable with that of the philosophers, historians, and poets. We have only to recall the introductions to Archimedes's treatises and the prefaces to the different Books of Apollonius's *Conics*. Heron, though severely practical, is no exception when he has any general explanation, historical or other, to give. We have now to note a like case in Pappus, namely the preface to Book V of the *Collection*. The editor, Hultsch, draws attention to the elegance and purity of the language and the careful writing; the latter is illustrated by the studied avoidance of hiatus.[1] The subject is one which a writer of taste and imagination would naturally find attractive, namely the practical intelligence shown by bees in selecting the hexagonal form for the cells in the honeycomb. Pappus does not disappoint us; the passage is as attractive as the subject, and deserves to be reproduced.

'It is of course to men that God has given the best and most perfect notion of wisdom in general and of mathematical science in particular, but a partial share in these things he allotted to some of the unreasoning animals as well. To men, as being endowed with reason, he vouchsafed that they should do everything in the light of reason and demonstration, but to the other animals, while denying them reason, he granted that each of them should, by virtue of a certain natural instinct, obtain just so much as is needful to support life. This instinct may be observed to exist in very many other species of living creatures, but most of all in bees. In the first place their orderliness and their submission to the queens who rule in their state are truly admirable, but much more admirable still is their emulation, the cleanliness they observe in the gathering of honey, and the forethought and housewifely care they devote to its custody. Presumably because they know themselves to be entrusted with the task of bringing from the gods to the accomplished portion of mankind a share of

[1] Pappus, vol. iii, p. 1233.

ambrosia in this form, they do not think it proper to pour it carelessly on ground or wood or any other ugly and irregular material; but, first collecting the sweets of the most beautiful flowers which grow on the earth, they make from them, for the reception of the honey, the vessels which we call honeycombs, (with cells) all equal, similar and contiguous to one another, and hexagonal in form. And that they have contrived this by virtue of a certain geometrical forethought we may infer in this way. They would necessarily think that the figures must be such as to be contiguous to one another, that is to say, to have their sides common, in order that no foreign matter could enter the interstices between them and so defile the purity of their produce. Now only three rectilineal figures would satisfy the condition, I mean regular figures which are equilateral and equiangular; for the bees would have none of the figures which are not uniform. . . . There being then three figures capable by themselves of exactly filling up the space about the same point, the bees by reason of their instinctive wisdom chose for the construction of the honeycomb the figure which has the most angles, because they conceived that it would contain more honey than either of the two others.

'Bees, then, know just this fact which is of service to themselves, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material used in constructing the different figures. We, however, claiming as we do a greater share in wisdom than bees, will investigate a problem of still wider extent, namely that, of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always greater, and the greatest plane figure of all those which have a perimeter equal to that of the polygons is the circle.'

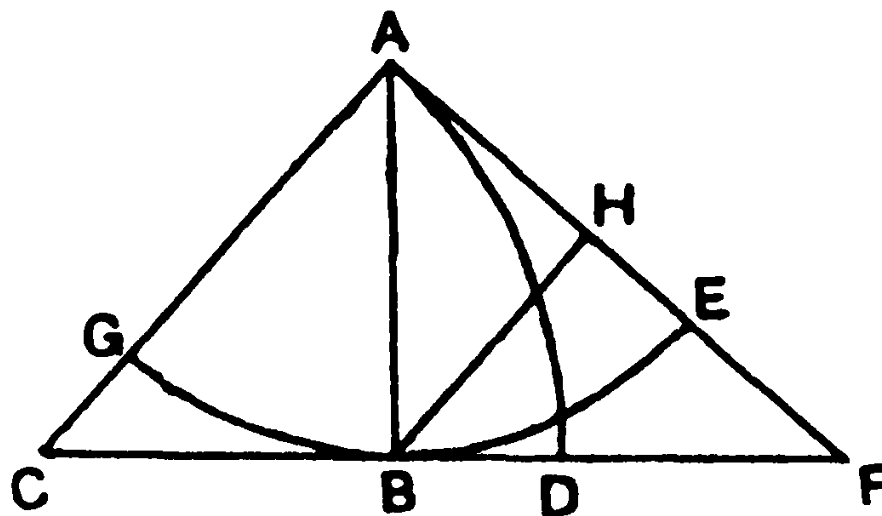
Book V then is devoted to what we may call *isoperimetry*, including in the term not only the comparison of the areas of different plane figures with the same perimeter, but that of the contents of different solid figures with equal surfaces.

Section (1). *Isoperimetry after Zenodorus.*

The first section of the Book relating to plane figures (chaps. 1–10, pp. 308–34) evidently followed very closely the exposition of Zenodorus περὶ ἰσομέτρων σχημάτων (see pp. 207–13, above); but before passing to solid figures Pappus inserts the proposition that *of all circular segments having*

the same circumference the semicircle is the greatest, with some preliminary lemmas which deserve notice (chaps. 15, 16).

(1) ABC is a triangle right-angled at B. With C as centre and radius CA describe the arc AD cutting CB produced in D. To prove that (R denoting a right angle)



$$(\text{sector } CAD) : (\text{area } ABD) > R : \angle BCA.$$

Draw AF at right angles to CA meeting CD produced in F, and draw BH perpendicular to AF. With A as centre and AB as radius describe the arc GBE.

Now $(\text{area } EBF) : (\text{area } EBH) > (\text{area } EBF) : (\text{sector } ABE)$,

and, *componendo*, $\triangle FBH : (EBH) > \triangle ABF : (ABE)$.

But (by an easy lemma which has just preceded)

$$\triangle FBH : (EBH) = \triangle ABF : (ABD),$$

whence $\triangle ABF : (ABD) > \triangle ABF : (ABE)$,

and $(ABE) > (ABD)$.

Therefore $(ABE) : (ABG) > (ABD) : (ABG)$

$> (ABD) : \triangle ABC$, *a fortiori*.

Therefore $\angle BAF : \angle BAC > (ABD) : \triangle ABC$,

whence, inversely, $\triangle ABC : (ABD) > \angle BAC : \angle BAF$.

and, *componendo*, $(\text{sector } ACD) : (ABD) > R : \angle BCA$.

[If α be the circular measure of $\angle BCA$, this gives (if $AC = b$)

$$\tfrac{1}{2}\alpha b^2 : (\tfrac{1}{2}\alpha b^2 - \tfrac{1}{2}\sin\alpha\cos\alpha \, . \, b^2) > \tfrac{1}{2}\pi : \alpha,$$

or $$2\alpha : (2\alpha - \sin 2\alpha) > \pi : 2\alpha;$$

that is, $\theta/(\theta - \sin\theta) > \pi/\theta$, where $0 < \theta < \pi$.]

(2) ABC is again a triangle right-angled at B. With C as centre and CA as radius draw a circle AD meeting BC produced in D. To prove that

$$(\text{sector } CAD) : (\text{area } ABD) > R : \angle ACD.$$

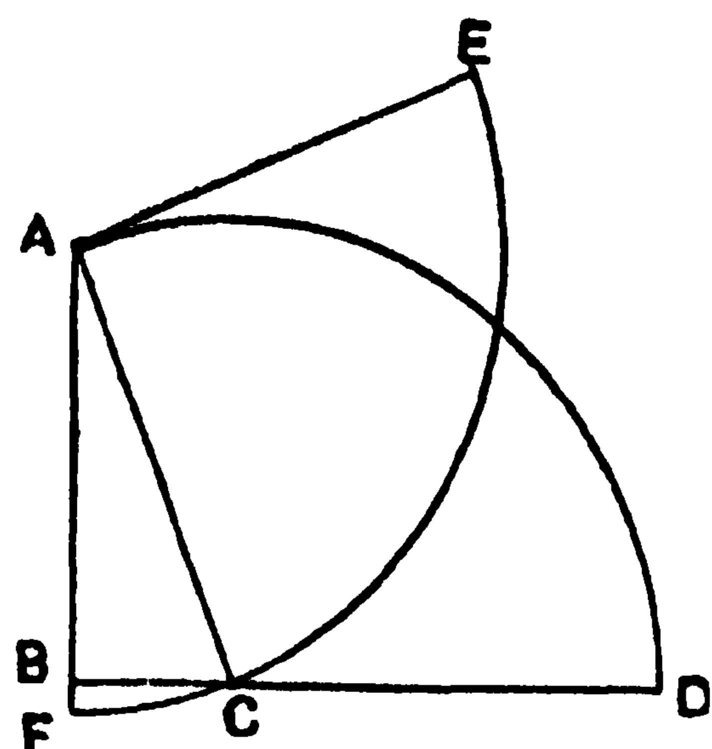
Draw AE at right angles to AC. With A as centre and AC as radius describe the circle FCE meeting AB produced in F and AE in E.

Then, since $\angle ACD > \angle CAE$, (sector ACD) > (sector ACE).

Therefore $(ACD):\Delta ABC > (ACE):\Delta ABC$

$> (ACE):(ACF)$, *a fortiori*,

$> \angle EAC:\angle CAB.$



Inversely,

$\Delta ABC:(ACD) < \angle CAB:\angle EAC,$

and, *componendo*,

$(ABD):(ACD) < \angle EAB:\angle EAC.$

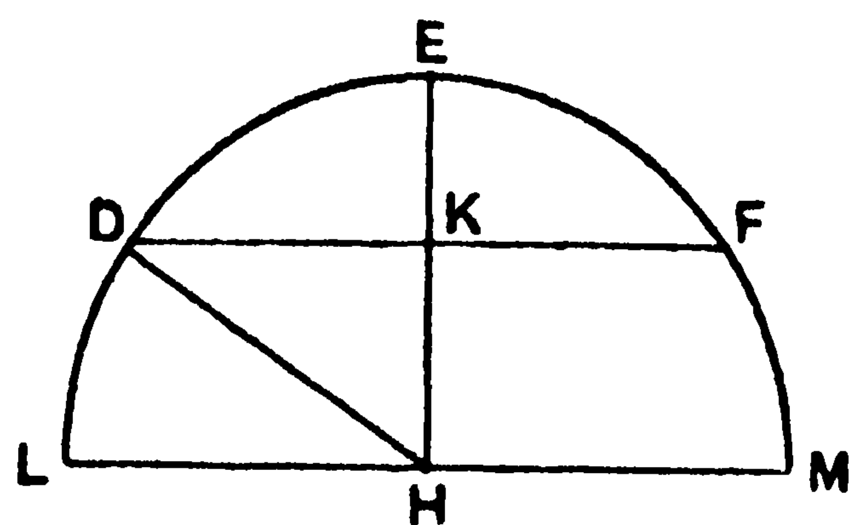
Inversely, $(ACD):(ABD) > \angle EAC:\angle EAB$

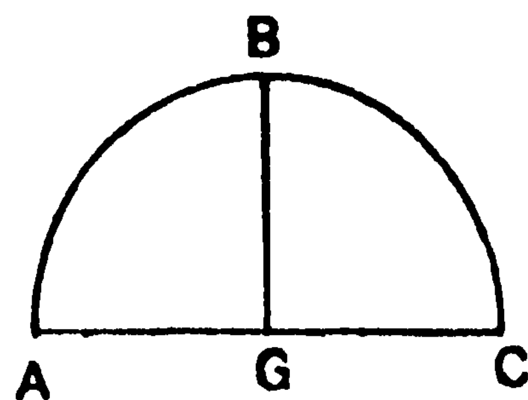
$> R:\angle ACD.$

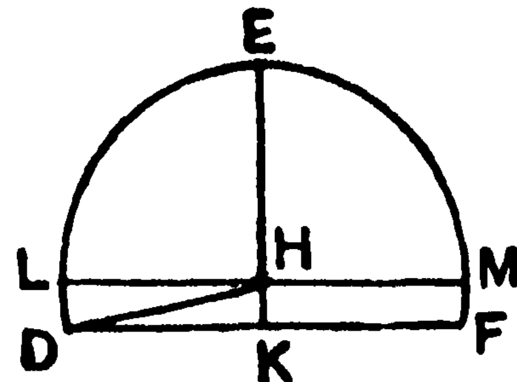
We come now to the application of these lemmas to the proposition comparing the area of a semicircle with that of other segments of equal circumference (chaps. 17, 18).

A semicircle is the greatest of all segments of circles which have the same circumference.

Let ABC be a semicircle with centre G, and DEF another segment of a circle such that the circumference DEF is equal







to the circumference ABC. I say that the area of ABC is greater than the area of DEF.

Let H be the centre of the circle DEF. Draw EHK, BG at right angles to DF, AC respectively. Join DH, and draw LHM parallel to DF.

Then $LH : AG = (\text{arc } LE) : (\text{arc } AB)$

$$= (\text{arc } LE) : (\text{arc } DE)$$

$$= (\text{sector } LHE) : (\text{sector } DHE).$$

Also $LH^2 : AG^2 = (\text{sector } LHE) : (\text{sector } AGB).$

Therefore the sector LHE is to the sector AGB in the ratio duplicate of that which the sector LHE has to the sector DHE.

Therefore

$$(\text{sector } LHE) : (\text{sector } DHE) = (\text{sector } DHE) : (\text{sector } AGB).$$

Now (1) in the case of the segment less than a semicircle and (2) in the case of the segment greater than a semicircle

$$(\text{sector } EDH) : (EDK) > R : \angle DHE,$$

by the lemmas (1) and (2) respectively.

That is,

$$(\text{sector } EDH) : (EDK) > \angle LHE : \angle DHE$$

$$> (\text{sector } LHE) : (\text{sector } DHE)$$

$$> (\text{sector } EDH) : (\text{sector } AGB),$$

from above.

Therefore the half segment EDK is less than the half semicircle AGB, whence the semicircle ABC is greater than the segment DEF.

We have already described the content of Zenodorus's treatise (pp. 207–13, above) to which, so far as plane figures are concerned, Pappus added nothing except the above proposition relating to segments of circles.

Section (2). *Comparison of volumes of solids having their surfaces equal. Case of the sphere.*

The portion of Book V dealing with solid figures begins (p. 350. 20) with the statement that the philosophers who considered that the creator gave the universe the form of a sphere because that was the most beautiful of all shapes also asserted that the sphere is the greatest of all solid figures

which have their surfaces equal; this, however, they had not proved, nor could it be proved without a long investigation. Pappus himself does not attempt to prove that the sphere is greater than *all* solids with the same surface, but only that the sphere is greater than any of the five regular solids having the same surface (chap. 19) and also greater than either a cone or a cylinder of equal surface (chap. 20).

Section (3). *Digression on the semi-regular solids of Archimedes.*

He begins (chap. 19) with an account of the thirteen *semi-regular* solids discovered by Archimedes, which are contained by polygons all equilateral and all equiangular but not all similar (see pp. 98–101, above), and he shows how to determine the number of solid angles and the number of edges which they have respectively; he then gives them the go-by for his present purpose because they are not completely regular; still less does he compare the sphere with any irregular solid having an equal surface.

The sphere is greater than any of the regular solids which has its surface equal to that of the sphere.

The proof that the sphere is greater than any of the regular solids with surface equal to that of the sphere is the same as that given by Zenodorus. Let P be any one of the regular solids, S the sphere with surface equal to that of P. To prove that $S > P$. Inscribe in the solid a sphere s, and suppose that r is its radius. Then the surface of P is greater than the surface of s, and accordingly, if R is the radius of S, $R > r$. But the volume of S is equal to the cone with base equal to the surface of S, and therefore of P, and height equal to R; and the volume of P is equal to the cone with base equal to the surface of P and height equal to r. Therefore, since $R > r$, volume of $S >$ volume of P.

Section (4). *Propositions on the lines of Archimedes, 'On the Sphere and Cylinder'.*

For the fact that the volume of a sphere is equal to the cone with base equal to the surface, and height equal to the radius,

of the sphere, Pappus quotes Archimedes, *On the Sphere and Cylinder*, but thinks proper to add a series of propositions (chaps. 20–43, pp. 362–410) on much the same lines as those of Archimedes and leading to the same results as Archimedes obtains for the surface of a segment of a sphere and of the whole sphere (Prop. 28), and for the volume of a sphere (Prop. 35). Prop. 36 (chap. 42) shows how to divide a sphere into two segments such that their surfaces are in a given ratio and Prop. 37 (chap. 43) proves that the volume as well as the surface of the cylinder circumscribing a sphere is $1\frac{1}{2}$ times that of the sphere itself.

Among the lemmatic propositions in this section of the Book Props. 21, 22 may be mentioned. Prop. 21 proves that, if C, E be two points on the tangent at H to a semicircle such that $CH = HE$, and if CD, EF be drawn perpendicular to the diameter AB, then $(CD+EF)\,CE = AB\,.\,DF$; Prop. 22 proves a like result where C, E are points on the semicircle, CD, EF are as before perpendicular to AB, and EH is the chord of the circle subtending the arc which with CE makes up a semicircle; in this case $(CD+EF)\,CE = EH\,.\,DF$. Both results are easily seen to be the equivalent of the trigonometrical formula

$$\sin(x+y)+\sin(x-y) = 2\sin x\cos y,$$

or, if certain different angles be taken as x, y,

$$\frac{\sin x+\sin y}{\cos y-\cos x} = \cot\tfrac{1}{2}(x-y).$$

Section (5). *Of regular solids with surfaces equal, that is greater which has more faces.*

Returning to the main problem of the Book, Pappus shows that, of the five regular solid figures assumed to have their surfaces equal, that is greater which has the more faces, so that the pyramid, the cube, the octahedron, the dodecahedron and the icosahedron of equal surface are, as regards solid content, in ascending order of magnitude (Props. 38–56). Pappus indicates (p. 410. 27) that 'some of the ancients' had worked out the proofs of these propositions by the analytical method; for himself, he will give a method of his own by

synthetical deduction, for which he claims that it is clearer and shorter. We have first propositions (with auxiliary lemmas) about the perpendiculars from the centre of the circumscribing sphere to a face of (*a*) the octahedron, (*b*) the icosahedron (Props. 39, 43), then the proposition that, if a dodecahedron and an icosahedron be inscribed in the same sphere, the same small circle in the sphere circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron (Prop. 48); this last is the proposition proved by Hypsicles in the so-called 'Book XIV of Euclid', Prop. 2, and Pappus gives two methods of proof, the second of which (chap. 56) corresponds to that of Hypsicles. Prop. 49 proves that twelve of the regular pentagons inscribed in a circle are together greater than twenty of the equilateral triangles inscribed in the same circle. The final propositions proving that the cube is greater than the pyramid with the same surface, the octahedron greater than the cube, and so on, are Props. 52–6 (chaps. 60–4). Of Pappus's auxiliary propositions, Prop. 41 is practically contained in Hypsicles's Prop. 1, and Prop. 44 in Hypsicles's last lemma; but otherwise the exposition is different.

Book VI.

On the contents of Book VI we can be brief. It is mainly astronomical, dealing with the treatises included in the so-called *Little Astronomy*, that is, the smaller astronomical treatises which were studied as an introduction to the great *Syntaxis* of Ptolemy. The preface says that many of those who taught the *Treasury of Astronomy*, through a careless understanding of the propositions, added some things as being necessary and omitted others as unnecessary. Pappus mentions at this point an incorrect addition to Theodosius, *Sphaerica*, III. 6, an omission from Euclid's *Phaenomena*, Prop. 2, an inaccurate representation of Theodosius, *On Days and Nights*, Prop. 4, and the omission later of certain other things as being unnecessary. His object is to put these mistakes right. Allusions are also found in the Book to Menelaus's *Sphaerica*, e.g. the statement (p. 476. 16) that Menelaus in his *Sphaerica* called a spherical triangle τρίπλευρον, *three-side*.

The *Sphaerica* of Theodosius is dealt with at some length (chaps. 1–26, Props. 1–27), and so are the theorems of Autolycus *On the moving Sphere* (chaps. 27–9), Theodosius *On Days and Nights* (chaps. 30–6, Props. 29–38), Aristarchus *On the sizes and distances of the Sun and Moon* (chaps. 37–40, including a proposition, Prop. 39 with two lemmas, which is corrupt at the end and is not really proved), Euclid's *Optics* (chaps. 41–52, Props. 42–54), and Euclid's *Phaenomena* (chaps. 53–60, Props. 55–61).

Problem arising out of Euclid's 'Optics'.

There is little in the Book of general mathematical interest except the following propositions which occur in the section on Euclid's *Optics*.

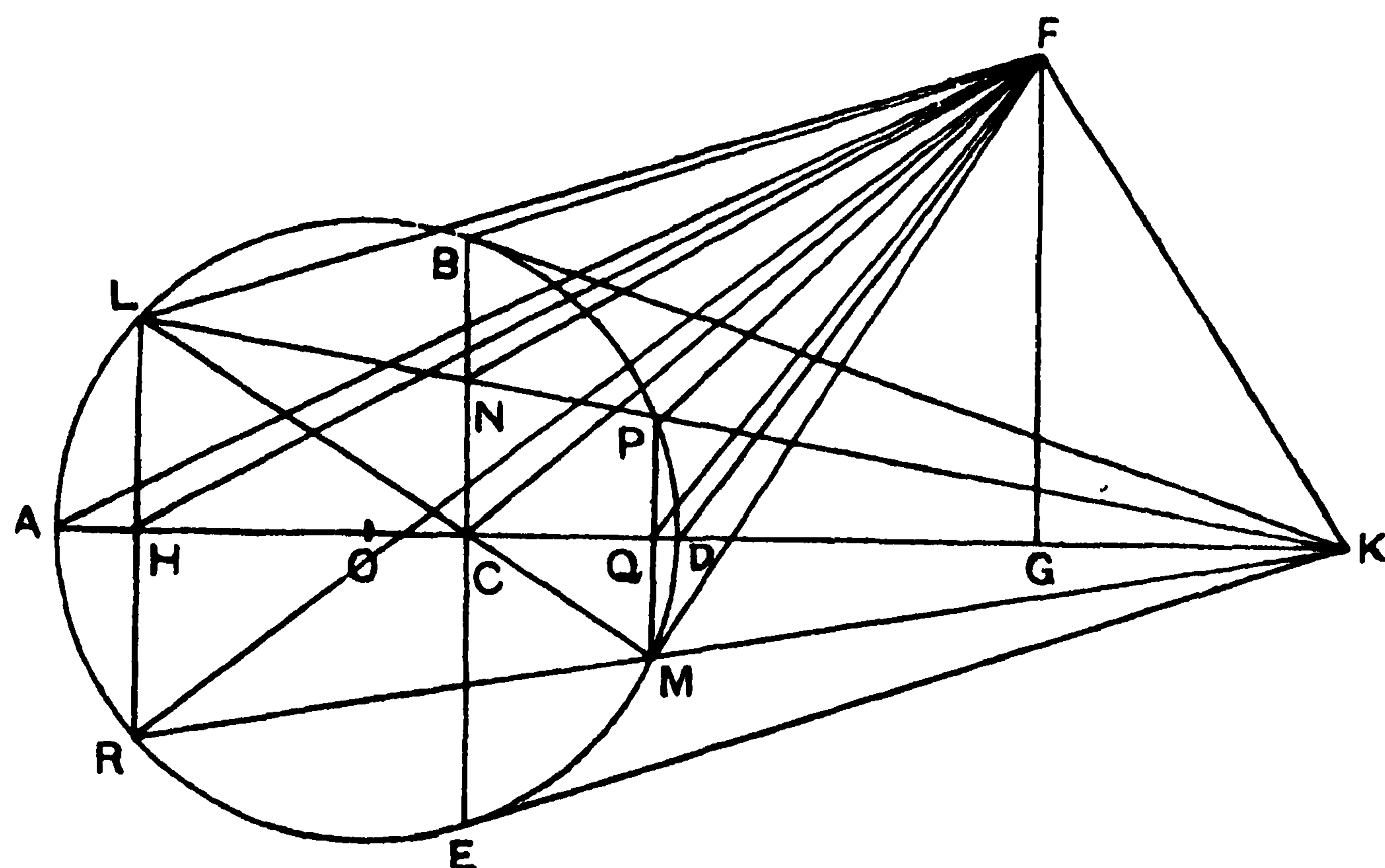
Two propositions are fundamental in solid geometry, namely:

(*a*) If from a point *A* above a plane *AB* be drawn perpendicular to the plane, and if from *B* a straight line *BD* be drawn perpendicular to any straight line *EF* in the plane, then will *AD* also be perpendicular to *EF* (Prop. 43).

(*b*) If from a point *A* above a plane *AB* be drawn to the plane but not at right angles to it, and *AM* be drawn perpendicular to the plane (i.e. if *BM* be the orthogonal projection of *BA* on the plane), the angle *ABM* is the least of all the angles which *AB* makes with any straight lines through *B*, as *BP*, in the plane; the angle *ABP* increases as *BP* moves away from *BM* on either side; and, given any straight line *BP* making a certain angle with *BA*, only one other straight line in the plane will make the same angle with *BA*, namely a straight line *BP′* on the other side of *BM* making the same angle with it that *BP* does (Prop. 44).

These are the first of a series of lemmas leading up to the main problem, the investigation of the apparent form of a circle as seen from a point outside its plane. In Prop. 50 (= Euclid, *Optics*, 34) Pappus proves the fact that all the diameters of the circle will appear equal if the straight line drawn from the point representing the eye to the centre of the circle is either (*a*) at right angles to the plane of the circle or (*b*), if not at right angles to the plane of the circle, is equal

in length to the radius of the circle. In all other cases (Prop. 51 = Eucl. *Optics*, 35) the diameters will appear unequal. Pappus's other propositions carry farther Euclid's remark that the circle seen under these conditions will appear deformed or distorted (παρεσπασμένος), proving (Prop. 53, pp. 588-92) that the apparent form will be an ellipse with its centre not, 'as some think', at the centre of the circle but at another point in it, determined in this way. Given a circle *ABDE* with centre *O*, let the eye be at a point *F* above the plane of the circle such that *FO* is neither perpendicular to that plane nor equal to the radius of the circle. Draw *FG* perpendicular to the plane of the circle and let *ADG* be the diameter through *G*. Join *AF*, *DF*, and bisect the angle *AFD* by the straight line *FC* meeting *AD* in *C*. Through *C* draw *BE* perpendicular to *AD*, and let the tangents at *B*, *E* meet *AG* produced in *K*. Then Pappus proves that *C* (not *O*) is the centre of the apparent ellipse, that *AD*, *BE* are its major and minor axes respectively, that the ordinates to *AD* are parallel to *BE* both really and apparently, and that the ordinates to *BE* will pass through *K* but will appear to be parallel to *AD*. Thus in the figure, *C* being the centre of the apparent ellipse,



it is proved that, if *LCM* is any straight line through *C*, *LC* is apparently equal to *CM* (it is practically assumed—a proposition proved later in Book VII, Prop. 156—that, if *LK* meet the circle again in *P*, and if *PM* be drawn perpendicular to *AD* to meet the circle again in *M*, *LM* passes through *C*).

The test of apparent equality is of course that the two straight lines should subtend equal angles at F.

The main points in the proof are these. The plane through CF, CK is perpendicular to the planes BFE, PFM and LFR; hence CF is perpendicular to BE, QF to PM and HF to LR, whence BC and CE subtend equal angles at F: so do LH, HR, and PQ, QM.

Since FC bisects the angle AFD, and $AC : CD = AK : KD$ (by the polar property), $\angle CFK$ is a right angle. And CF is the intersection of two planes at right angles, namely AFK and BFE, in the former of which FK lies; therefore KF is perpendicular to the plane BFE, and therefore to FN. Since therefore (by the polar property) $LN : NP = LK : KP$, it follows that the angle LFP is bisected by FN; hence LN, NP are apparently equal.

Again $LC : CM = LN : NP = LF : FP = LF : FM$.

Therefore the angles LFC, CFM are equal, and LC, CM are apparently equal.

Lastly $LR : PM = LK : KP = LN : NP = LF : FP$; therefore the isosceles triangles FLR, FPM are equiangular; therefore the angles PFM, LFR, and consequently PFQ, LFH, are equal. Hence LP, RM will appear to be parallel to AD.

We have, based on this proposition, an easy method of solving Pappus's final problem (Prop. 54). 'Given a circle $ABDE$ and any point within it, to find outside the plane of the circle a point from which the circle will have the appearance of an ellipse with centre C.'

We have only to produce the diameter AD through C to the pole K of the chord BE perpendicular to AD and then, in the plane through AK perpendicular to the plane of the circle, to describe a semicircle on CK as diameter. Any point F on this semicircle satisfies the condition.

Book VII. *On the 'Treasury of Analysis'.*

Book VII is of much greater importance, since it gives an account of the books forming what was called the *Treasury of Analysis* (ἀναλυόμενος τόπος) and, as regards those of the books which are now lost, Pappus's account, with the hints derivable from the large collection of lemmas supplied by him to each

book, practically constitutes our only source of information. The Book begins (p. 634) with a definition of *analysis* and *synthesis* which, as being the most elaborate Greek utterance on the subject, deserves to be quoted in full.

'The so-called Ἀναλυόμενος is, to put it shortly, a special body of doctrine provided for the use of those who, after finishing the ordinary Elements, are desirous of acquiring the power of solving problems which may be set them involving (the construction of) lines, and it is useful for this alone. It is the work of three men, Euclid the author of the Elements, Apollonius of Perga and Aristaeus the elder, and proceeds by way of analysis and synthesis.'

Definition of Analysis and Synthesis.

'*Analysis*, then, takes that which is sought as if it were admitted and passes from it through its successive consequences to something which is admitted as the result of synthesis: for in analysis we assume that which is sought as if it were already done (γεγονός), and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracing our steps we come upon something already known or belonging to the class of first principles, and such a method we call analysis as being solution backwards (ἀνάπαλιν λύσιν).

'But in *synthesis*, reversing the process, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what before were antecedents, and successively connecting them one with another, we arrive finally at the construction of what was sought; and this we call synthesis.

'Now analysis is of two kinds, the one directed to searching for the truth and called *theoretical*, the other directed to finding what we are told to find and called *problematical*. (1) In the *theoretical* kind we assume what is sought as if it were existent and true, after which we pass through its successive consequences, as if they too were true and established by virtue of our hypothesis, to something admitted: then (*a*), if that something admitted is true, that which is sought will also be true and the proof will correspond in the reverse order to the analysis, but (*b*), if we come upon something admittedly false, that which is sought will also be false. (2) In the *problematical* kind we assume that which is propounded as if it were known, after which we pass through its

successive consequences, taking them as true, up to something admitted: if then (*a*) what is admitted is possible and obtainable, that is, what mathematicians call *given*, what was originally proposed will also be possible, and the proof will again correspond in the reverse order to the analysis, but if (*b*) we come upon something admittedly impossible, the problem will also be impossible.'

This statement could hardly be improved upon except that it ought to be added that each step in the chain of inference in the analysis must be *unconditionally convertible*; that is, when in the analysis we say that, if *A* is true, *B* is true, we must be sure that each statement is a necessary consequence of the other, so that the truth of *A* equally follows from the truth of *B*. This, however, is almost implied by Pappus when he says that we inquire, not what it is (namely *B*) which follows from *A*, but what it is (*B*) from which *A* follows, and so on.

List of works in the 'Treasury of Analysis'.

Pappus adds a list, in order, of the books forming the Ἀναλυόμενος, namely:

'Euclid's *Data*, one Book, Apollonius's *Cutting-off of a ratio*, two Books, *Cutting-off of an area*, two Books, *Determinate Section*, two Books, *Contacts*, two Books, Euclid's *Porisms*, three Books, Apollonius's *Inclinations* or *Vergings* (νεύσεις), two Books, the same author's *Plane Loci*, two Books, and *Conics*, eight Books, Aristaeus's *Solid Loci*, five Books, Euclid's *Surface-Loci*, two Books, Eratosthenes's *On means*, two Books. There are in all thirty-three Books, the contents of which up to the *Conics* of Apollonius I have set out for your consideration, including not only the number of the propositions, the *diorismi* and the cases dealt with in each Book, but also the lemmas which are required; indeed I have not, to the best of my belief, omitted any question arising in the study of the Books in question.'

Description of the treatises.

Then follows the short description of the contents of the various Books down to Apollonius's *Conics*; no account is given of Aristaeus's *Solid Loci*, Euclid's *Surface-Loci* and

Eratosthenes's *On means*, nor are there any lemmas to these works except two on the *Surface-Loci* at the end of the Book.

The contents of the various works, including those of the lost treatises so far as they can be gathered from Pappus, have been described in the chapters devoted to their authors, and need not be further referred to here, except for an *addendum* to the account of Apollonius's *Conics* which is remarkable. Pappus has been speaking of the 'locus with respect to three or four lines' (which is a conic), and proceeds to say (p. 678. 26) that we may in like manner have loci with reference to five or six or even more lines; these had not up to his time become generally known, though the synthesis of one of them, not by any means the most obvious, had been worked out and its utility shown. Suppose that there are five or six lines, and that p_1, p_2, p_3, p_4, p_5 or $p_1, p_2, p_3, p_4, p_5, p_6$ are the lengths of straight lines drawn from a point to meet the five or six at given angles, then, if in the first case $p_1 p_2 p_3 = \lambda p_4 p_5 a$ (where λ is a constant ratio and a a given length), and in the second case $p_1 p_2 p_3 = \lambda p_4 p_5 p_6$, the locus of the point is in each case a certain curve given in position. The relation could not be expressed in the same form if there were more lines than six, because there are only three dimensions in geometry, although certain recent writers had allowed themselves to speak of a rectangle multiplied by a square or a rectangle without giving any intelligible idea of what they meant by such a thing (is Pappus here alluding to Heron's proof of the formula for the area of a triangle in terms of its sides given on pp. 322–3, above?). But the system of compounded ratios enables it to be expressed for any number of lines thus, $\frac{p_1}{p_2} \cdot \frac{p_3}{p_4} \ldots \frac{p_n}{a} \left(\text{or } \frac{p_{n-1}}{p_n}\right) = \lambda$. Pappus proceeds in language not very clear (p. 680. 30); but the gist seems to be that the investigation of these curves had not attracted men of light and leading, as, for instance, the old geometers and the best writers. Yet there were other important discoveries still remaining to be made. For himself, he noticed that every one in his day was occupied with the elements, the first principles and the natural origin of the subject-matter of investigation; ashamed to pursue such topics, he had himself proved propositions of much more importance and

utility. In justification of this statement and 'in order that he may not appear empty-handed when leaving the subject', he will present his readers with the following.

(*Anticipation of Guldin's Theorem.*)

The enunciations are not very clearly worded, but there is no doubt as to the sense.

'*Figures generated by a complete revolution of a plane figure about an axis are in a ratio compounded (1) of the ratio of the areas of the figures, and (2) of the ratio of the straight lines similarly drawn to (i.e. drawn to meet at the same angles) the axes of rotation from the respective centres of gravity. Figures generated by incomplete revolutions are in the ratio compounded (1) of the ratio of the areas of the figures and (2) of the ratio of the arcs described by the centres of gravity of the respective figures, the latter ratio being itself compounded (a) of the ratio of the straight lines similarly drawn (from the respective centres of gravity to the axes of rotation) and (b) of the ratio of the angles contained (i.e. described) about the axes of revolution by the extremities of the said straight lines (i.e. the centres of gravity).*'

Here, obviously, we have the essence of the celebrated theorem commonly attributed to P. Guldin (1577–1643), 'quantitas rotunda in viam rotationis ducta producit Potestatem Rotundam uno grado altiorem Potestate sive Quantitate Rotata'.[1]

Pappus adds that

'these propositions, which are practically one, include any number of theorems of all sorts about curves, surfaces, and solids, all of which are proved at once by one demonstration, and include propositions both old and new, and in particular those proved in the twelfth Book of these Elements.'

Hultsch attributes the whole passage (pp. 680. 30–682. 20) to an interpolator, I do not know for what reason; but it seems to me that the propositions are quite beyond what could be expected from an interpolator, indeed I know of no Greek mathematician from Pappus's day onward except Pappus himself who was capable of discovering such a proposition.

[1] *Centrobaryca*, Lib. ii, chap. viii, Prop. 3. Viennae 1641.

If the passage is genuine, it seems to indicate, what is not elsewhere confirmed, that the *Collection* originally contained, or was intended to contain, twelve Books.

Lemmas to the different treatises.

After the description of the treatises forming the *Treasury of Analysis* come the collections of lemmas given by Pappus to assist the student of each of the books (except Euclid's *Data*) down to Apollonius's *Conics*, with two isolated lemmas to the *Surface-Loci* of Euclid. It is difficult to give any summary or any general idea of these lemmas, because they are very numerous, extremely various, and often quite difficult, requiring first-rate ability and full command of all the resources of pure geometry. Their number is also greatly increased by the addition of alternative proofs, often requiring lemmas of their own, and by the separate formulation of particular cases where by the use of algebra and conventions with regard to sign we can make one proposition cover all the cases. The style is admirably terse, often so condensed as to make the argument difficult to follow without some little filling-out; the hand is that of a master throughout. The only misfortune is that, the books elucidated being lost (except the *Conics* and the *Cutting-off of a ratio* of Apollonius), it is difficult, often impossible, to see the connexion of the lemmas with one another and the problems of the book to which they relate. In the circumstances, all that I can hope to do is to indicate the types of propositions included in the lemmas and, by way of illustration, now and then to give a proof where it is sufficiently out of the common.

(*a*) Pappus begins with Lemmas to the *Sectio rationis* and *Sectio spatii* of Apollonius (Props. 1–21, pp. 684–704). The first two show how to divide a straight line in a given ratio, and how, given the first, second and fourth terms of a proportion between straight lines, to find the third term. The next section (Props. 3–12 and 16) shows how to manipulate relations between greater and less ratios by transforming them, e.g. *componendo*, *convertendo*, &c., in the same way as Euclid transforms *equal* ratios in Book V; Prop. 16 proves that, according as $a:b >$ or $< c:d$, $ad >$ or $< bc$. Props.

17–20 deal with three straight lines a, b, c in geometrical progression, showing how to mark on a straight line containing a, b, c as segments (including the whole among 'segments'), lengths equal to $a+c \pm 2\sqrt{(ac)}$; the lengths are of course equal to $a+c \pm 2b$ respectively. These lemmas are preliminary to the problem (Prop. 21), Given two straight lines AB, BC (C lying between A and B), to find a point D on BA produced such that $BD:DA = CD:(AB+BC-2\sqrt{AB.BC})$. This is, of course, equivalent to the quadratic equation $(a+x):x = (a-c+x):(a+c-2\sqrt{ac})$, and, after marking off AE along AD equal to the fourth term of this proportion, Pappus solves the equation in the usual way by application of areas.

(β) *Lemmas to the 'Determinate Section' of Apollonius.*

The next set of Lemmas (Props. 22–64, pp. 704–70) belongs to the *Determinate Section* of Apollonius. As we have seen (pp. 180–1, above), this work seems to have amounted to a *Theory of Involution.* Whether the application of certain of Pappus's lemmas corresponded to the conjecture of Zeuthen or not, we have at all events in this set of lemmas some remarkable applications of 'geometrical algebra'. They may be divided into groups as follows

I. Props. 22, 25, 29

If in the figure $AD.DC = BD.DE$, then

$$BD:DE = AB.BC:AE.EC.$$



The proofs by proportions are not difficult. Prop. 29 is an alternative proof by means of Prop. 26 (see below). The algebraic equivalent may be expressed thus: if $ax = by$, then

$$\frac{b}{y} = \frac{(a+b)(b+x)}{(a+y)(x+y)}.$$

II. Props. 30, 32, 34.

If in the same figure $AD.DE = BD.DC$, then

$$BD:DC = AB.BE:EC.CA.$$

Props. 32, 34 are alternative proofs based on other lemmas (Props. 31, 33 respectively). The algebraic equivalent may be stated thus: if $ax = by$, then $\frac{b}{y} = \frac{(a+b)(b-x)}{(x+y)(a-y)}$.

III. Props. 35, 36.

If $AB.BE = CB.BD$, then $AB:BE = DA.AC:CE.ED$, and $CB:BD = AC.CE:AD.DE$, results equivalent to the following: if $ax = by$, then

$$\frac{a}{x} = \frac{(a-y)(a-b)}{(b-x)(y-x)} \text{ and } \frac{b}{y} = \frac{(a-b)(b-x)}{(a-y)(y-x)}.$$

IV. Props. 23, 24, 31, 57, 58.

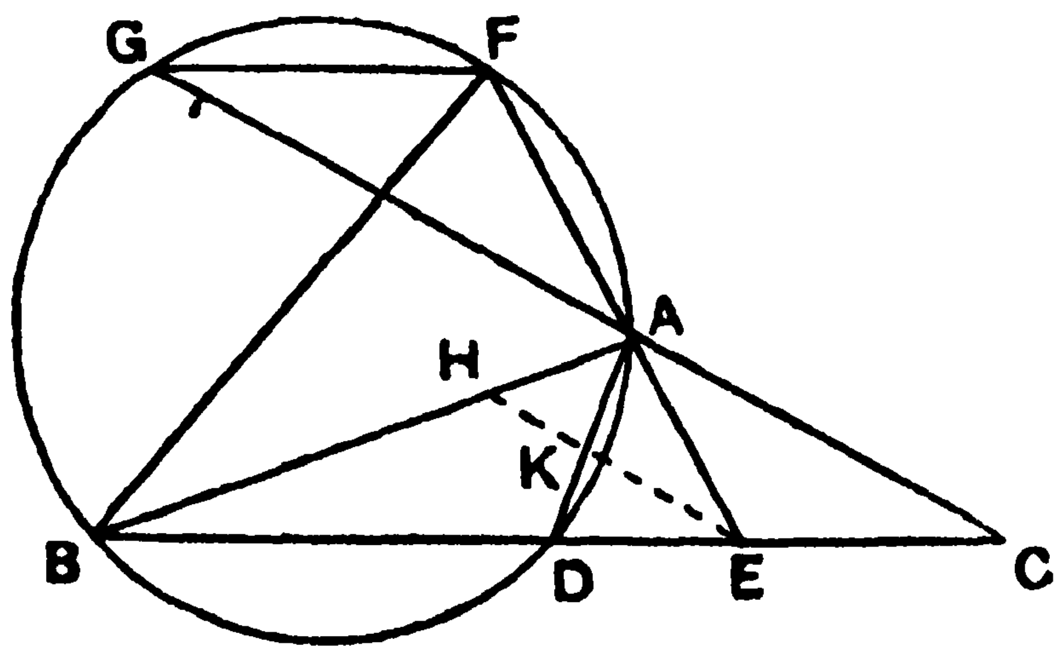


If $AB = CD$, and E is any point in CD,

$$AC.CD = AE.ED + BE.EC,$$

and similar formulae hold for other positions of E. If E is between B and C, $AC.CD = AE.ED - BE.EC$; and if E is on AD produced, $BE.EC = AE.ED + BD.DC$.

V. A small group of propositions relate to a triangle ABC with two straight lines AD, AE drawn from the vertex A to points on the base BC in accordance with one or other of the conditions (a) that the angles BAC, DAE are supplementary, (b) that the angles BAE, DAC are both right angles or, as we



may add from Book VI, Prop. 12, (c) that the angles BAD, EAC are equal. The theorems are:

In case (a) $BC.CD : BE.ED = CA^2 : AE^2$,

(b) $BC.CE : BD.DE = CA^2 : AD^2$,

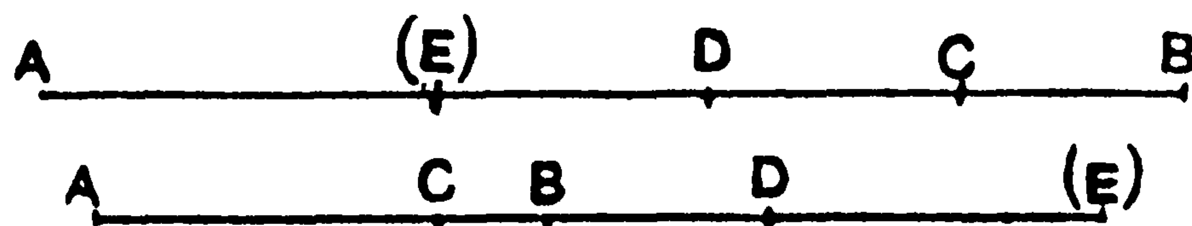
(c) $DC.CE : EB.BD = AC^2 : AB^2$.

VI. Props. 37, 38.

If $AB:BC = AD^2:DC^2$, whether AB be greater or less than AD, then

$$AB.BC = BD^2.$$

[E in the figure is a point such that $ED = CD$.]



The algebraical equivalent is: If $\frac{a}{c} = \frac{(a \pm b)^2}{(b \pm c)^2}$, then $ac = b^2$.

These lemmas are subsidiary to the next (Props. 39, 40), being used in the first proofs of them.

Props. 39, 40 prove the following:

If $ACDEB$ be a straight line, and if

$$BA.AE:BD.DE = AC^2:CD^2,$$

then $$AB.BD:AE.ED = BC^2:CE^2;$$

if, again, $$AC.CB:AE.EB = CD^2:DE^2,$$

then $$EA.AC:CB.BE = AD^2:DB^2.$$

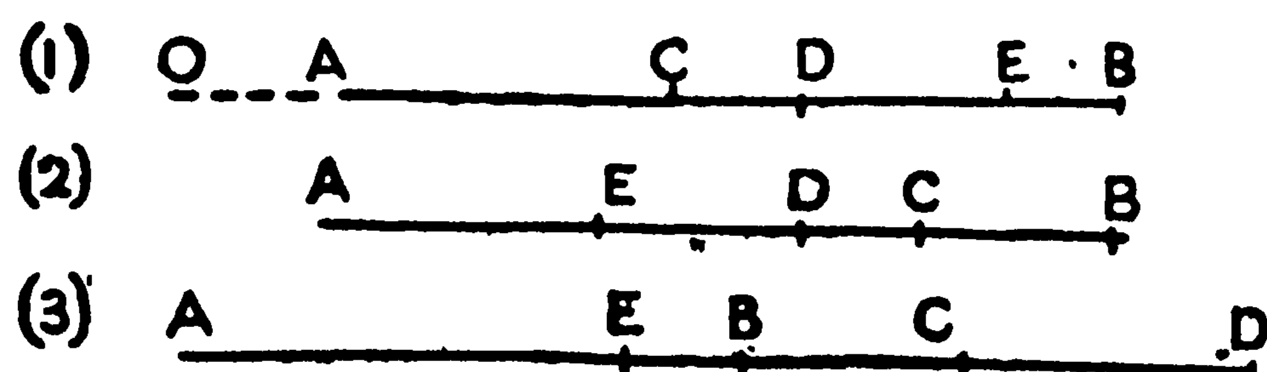
If $AB = a$, $BC = b$, $BD = c$, $BE = d$, the algebraic equivalents are the following.

If $\frac{a(a-d)}{c(c-d)} = \frac{(a-b)^2}{(b-c)^2}$, then $\frac{ac}{(a-d)(c-d)} = \frac{b^2}{(b-d)^2}$;

and if $\frac{(a-b)b}{(a-d)d} = \frac{(b-c)^2}{(c-d)^2}$, then $\frac{(a-d)(a-b)}{bd} = \frac{(a-c)^2}{c^2}$.

VII. Props. 41, 42, 43.

If $AD.DC = BD.DE$, suppose that in Figures (1) and (2)



$k = AE + CB$, and in Figure (3) $k = AE - BC$, then
$k.AD = BA.AE$, $k.CD = BC.CE$, $k.BD = AB.BC$,

$$k.DE = AE.EC.$$

The algebraical equivalents for Figures (1) and (2) respectively may be written (if $a = AD$, $b = DC$, $c = BD$, $d = DE$):

$$\text{If}\quad ab = cd, \text{ then } (a \pm d + c \pm b)\ a = (a + c)(a \pm d),$$
$$(a \pm d + c \pm b)\ b = (c \pm b)(b + d),$$
$$(a \pm d + c \pm b)\ c = (c + a)(c \pm b),$$
$$(a \pm d + c \pm b)\ d = (a \pm d)(d + b).$$

Figure (3) gives other varieties of sign. Troubles about sign can be avoided by measuring all lengths in one direction from an origin O outside the line. Thus, if $OA = a$, $OB = b$, &c., the proposition may be as follows:

If $(d-a)(d-c) = (b-d)(e-d)$ and $k = e-a+b-c$,

then $k(d-a) = (b-a)(e-a)$, $k(d-c) = (b-c)(e-c)$,

$k(b-d) = (b-a)(b-c)$ and $k(e-d) = (e-a)(e-c)$.

VIII. Props. 45–56.

More generally, if $AD.DC = BD.DE$ and $k = AE \pm BC$, then, if F be any point on the line, we have, according to the position of F in relation to A, B, C, D, E,

$$\pm AF.FC \pm EF.FB = k.DF.$$

Algebraically, if $OA = a$, $OB = b \ldots OF = x$, the equivalent is: If $(d-a)(d-c) = (b-d)(e-d)$, and $k = (e-a)+(b-c)$,

then $(x-a)(x-c)+(x-e)(b-x) = k(x-d)$.

By making $x = a, b, c, e$ successively in this equation, we obtain the results of Props. 41–3 above.

IX. Props. 59–64.

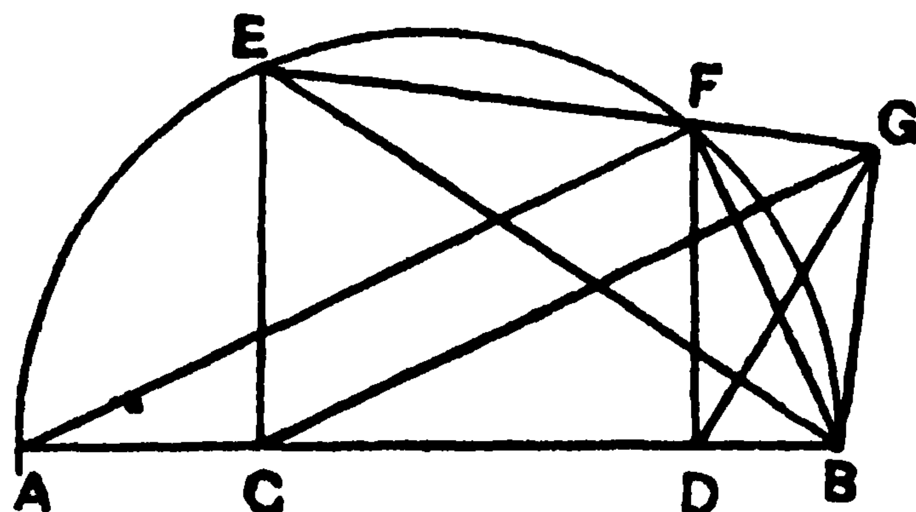
In this group Props. 59, 60, 63 are lemmas required for the remarkable propositions (61, 62, 64) in which Pappus investigates 'singular and minimum' values of the ratio

$$AP.PD : BP.PC,$$

where (A, D), (B, C) are point-pairs on a straight line and P is another point on the straight line. He finds, not only when the ratio has the 'singular and minimum (or maximum)' value,

but also what the value is, for three different positions of P in relation to the four given points.

I will give, as an illustration, the first case, on account of its elegance. It depends on the following *Lemma*. AEB being a semicircle on AB as diameter, C, D any two points on AB, and CE, DF being perpendicular to AB, let EF be joined and



produced, and let BG be drawn perpendicular to EG. To prove that

$$CB \cdot BD = BG^2, \quad (1)$$

$$AC \cdot DB = FG^2, \quad (2)$$

$$AD \cdot BC = EG^2. \quad (3)$$

Join GC, GD, FB, EB, AF.

(1) Since the angles at G, D are right, F, G, B, D are concyclic. Similarly E, G, B, C are concyclic.

Therefore

$\angle BGD = \angle BFD$

$= \angle FAB$

$= \angle FEB$, in the same segment of the semicircle,

$= \angle GCB$, in the same segment of the circle $EGBC$.

And the triangles GCB, DGB also have the angle CBG common; therefore they are similar, and $CB : BG = BG : BD$,

or $$CB \cdot BD = BG^2.$$

(2) We have $AB \cdot BD = BF^2$;

therefore, by subtraction, $AC \cdot DB = BF^2 - BG^2 = FG^2$.

(3) Similarly $AB \cdot BC = BE^2$;

therefore, by subtraction, from the same result (1),

$$AD \cdot BC = BE^2 - BG^2 = EG^2.$$

Thus the lemma gives an extremely elegant construction for squares equal to each of the three rectangles.

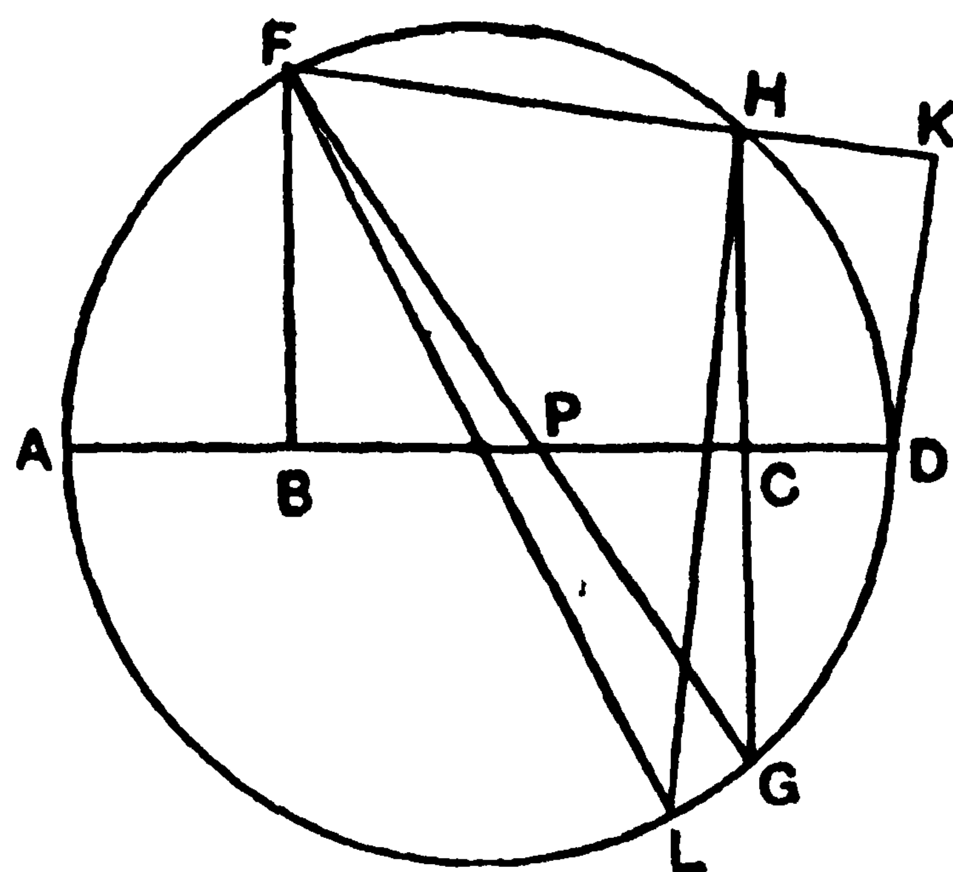
Now suppose (A, D), (B, C) to be two point-pairs on a straight line, and let P, another point on it, be determined by the relation

$$AB . BD : AC . CD = BP^2 : CP^2;$$

then, says Pappus, the ratio $AP . PD : BP . PC$ is singular and a minimum, and is equal to

$$AD^2 : (\sqrt{AC . BD} - \sqrt{AB . CD})^2.$$

On AD as diameter draw a circle, and draw BF, CG perpendicular to AD on opposite sides.



Then, by hypothesis, $AB . BD : AC . CD = BP^2 : CP^2$;

therefore $BF^2 : CG^2 = BP^2 : CP^2$,

or $BF : CG = BP : CP$,

whence the triangles FBP, GCP are similar and therefore equiangular, so that FPG is a straight line.

Produce GC to meet the circle in H, join FH, and draw DK perpendicular to FH produced. Draw the diameter FL and join LH.

Now, by the lemma, $FK^2 = AC . BD$, and $HK^2 = AB . CD$;

therefore $FH = FK - HK = \sqrt{(AC . BD)} - \sqrt{(AB . CD)}$.

Since, in the triangles FHL, PCG, the angles at H, C are right and $\angle FLH = \angle PGC$, the triangles are similar, and

$$GP : PC = FL : FH = AD : FH$$
$$= AD : \{\sqrt{(AC . BD)} - \sqrt{(AB . CD)}\}.$$

But $GP : PC = FP : PB$;

therefore $GP^2 : PC^2 = FP . PG : BP . PC$

$$= AP . PD : BP . PC.$$

Therefore

$$AP . PD : BP . PC = AD^2 : \{ \sqrt{(AC . BD)} - \sqrt{(AB . CD)} \}^2.$$

The proofs of Props. 62 and 64 are different, the former being long and involved. The results are:

Prop. 62. If P is between C and D, and

$$AD . DB : AC . CB = DP^2 : PC^2,$$

then the ratio $AP . PB : CP . PD$ is singular and a minimum and is equal to $\{ \sqrt{(AC . BD)} + \sqrt{(AD . BC)} \}^2 : DC^2$.

Prop. 64. If P is on AD produced, and

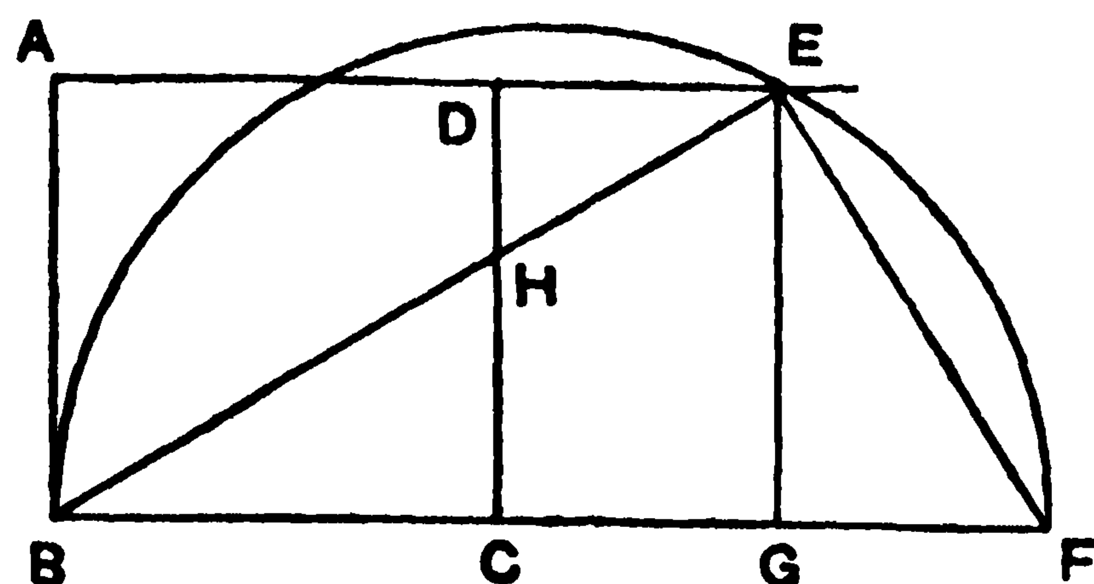
$$AB . BD : AC . CD = BP^2 : CP^2,$$

then the ratio $AP . PD : BP . PC$ is singular and a maximum, and is equal to $AD^2 : \{ \sqrt{(AC . BD)} + \sqrt{(AB . CD)} \}^2$.

(γ) *Lemmas on the Νεύσεις of Apollonius.*

After a few easy propositions (e.g. the equivalent of the proposition that, if $ax + x^2 = by + y^2$, then, according as $a >$ or $< b$, $a + x >$ or $< b + y$), Pappus gives (Prop. 70) the lemma leading to the solution of the νεῦσις with regard to the rhombus (see pp. 190–2, above), and after that the solution by one Heraclitus of the same problem with respect to a square (Props. 71, 72, pp. 780–4). The problem is, *Given a square ABCD, to draw through B a straight line, meeting CD in H and AD produced in E, such that HE is equal to a given length.*

The solution depends on a lemma to the effect that, if any straight line BHE through B meets CD in H and AD pro-



duced in E, and if EF be drawn perpendicular to BE meeting BC produced in F, then

$$CF^2 = BC^2 + HE^2.$$

Draw EG perpendicular to BF.

Then the triangles BCH, EGF are similar and (since $BC = EG$) equal in all respects; therefore $EF = BH$.

Now $$BF^2 = BE^2 + EF^2,$$

or $$BC \,.\, BF + BF \,.\, FC = BH \,.\, BE + BE \,.\, EH + EF^2.$$

But, the angles HCF, HEF being right, H, C, F, E are concyclic, and $$BC \,.\, BF = BH \,.\, BE.$$

Therefore, by subtraction,

$$\begin{aligned} BF \,.\, FC &= BE \,.\, EH + EF^2 \\ &= BE \,.\, EH + BH^2 \\ &= BH \,.\, HE + EH^2 + BH^2 \\ &= EB \,.\, BH + EH^2 \\ &= FB \,.\, BC + EH^2. \end{aligned}$$

Taking away the common part, $BC \,.\, CF$, we have

$$CF^2 = BC^2 + EH^2.$$

Now suppose that we have to draw BHE through B in such a way that $HE = k$. Since BC, EH are both given, we have only to determine a length x such that $x^2 = BC^2 + k^2$, produce BC to F so that $CF = x$, draw a semicircle on BF as diameter, produce AD to meet the semicircle in E, and join BE. BE is thus the straight line required.

Prop. 73 (pp. 784–6) proves that, if D be the middle point of BC, the base of an isosceles triangle ABC, then BC is the shortest of all the straight lines through D terminated by the straight lines AB, AC, and the nearer to BC is shorter than the more remote.

There follows a considerable collection of lemmas mostly showing the equality of certain intercepts made on straight lines through one extremity of the diameter of one of two semicircles having their diameters in a straight line, either one including or partly including the other, or wholly external to one another, on the same or opposite sides of the diameter.

I need only draw two figures by way of illustration.

In the first figure (Prop. 83), *ABC*, *DEF* being the semicircles, *BEKC* is any straight line through *C* cutting both; *FG* is made equal to *AD*; *AB* is joined; *GH* is drawn perpendicular to *BK* produced. It is required to prove that

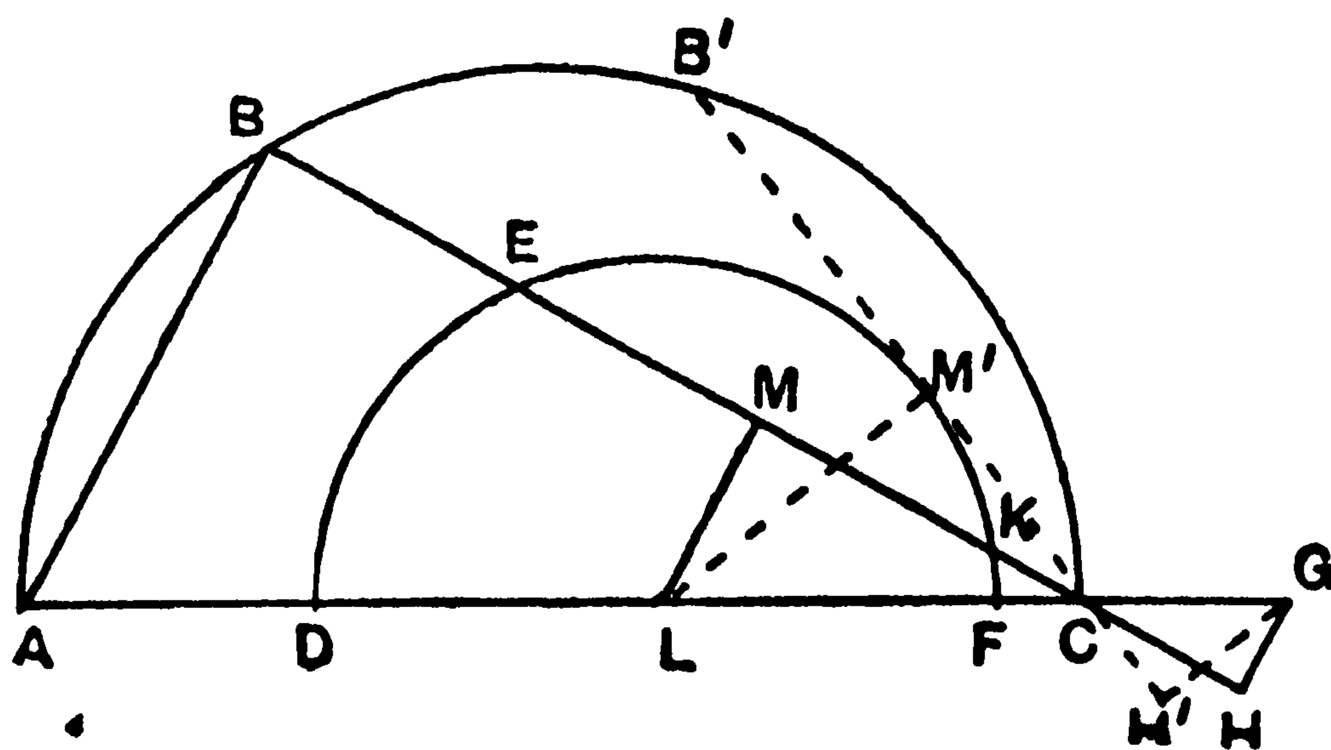


Fig. 1.

BE = *KH*. (This is obvious when from *L*, the centre of the semicircle *DEF*, *LM* is drawn perpendicular to *BK*.) If *E*, *K* coincide in the point *M′* of the semicircle so that *B′CH′* is a tangent, then *B′M′* = *M′H′* (Props. 83, 84).

In the second figure (Prop. 91) *D* is the centre of the semicircle *ABC* and is also the extremity of the diameter of the semicircle *DEF*. If *BEGF* be any straight line through

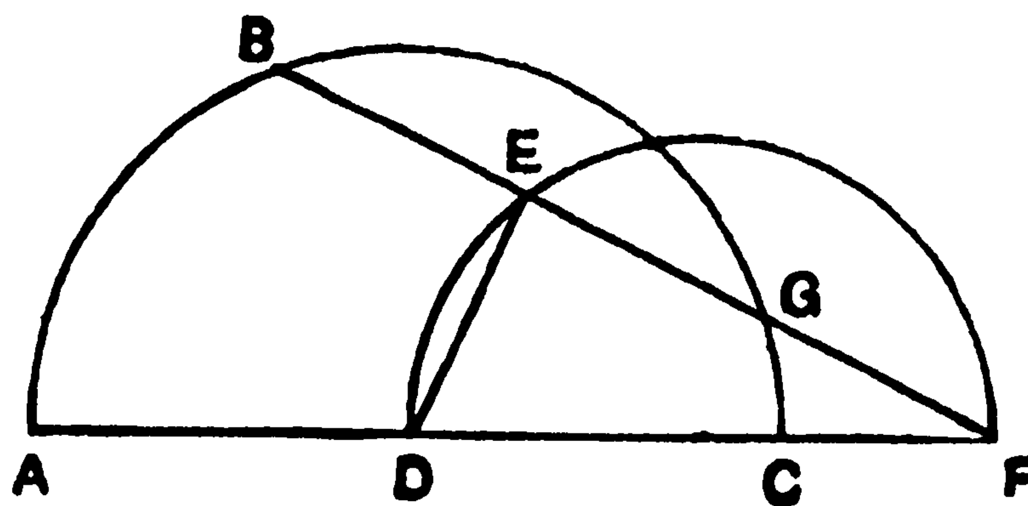


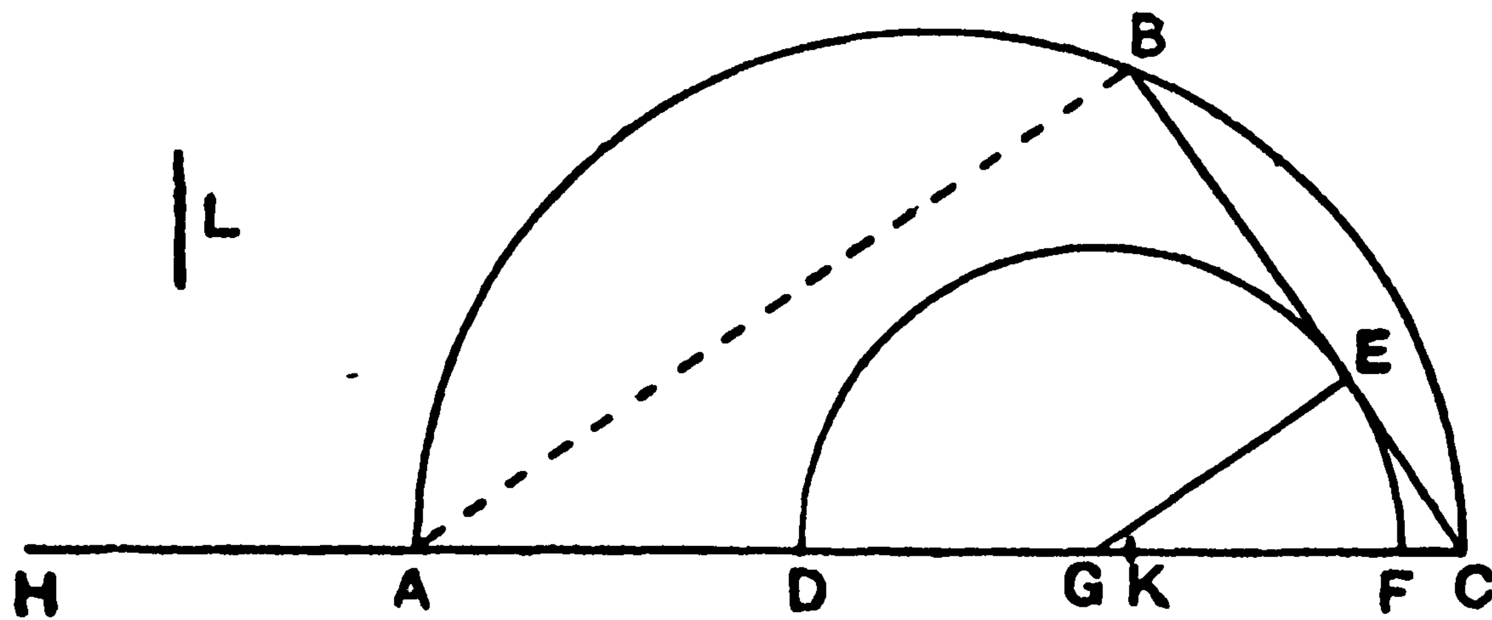
Fig. 2.

F cutting both semicircles, *BE* = *EG*. This is clear, since *DE* is perpendicular to *BG*.

The only problem of any difficulty in this section is Prop. 85 (p. 796). Given a semicircle *ABC* on the diameter *AC* and a point *D* on the diameter, to draw a semicircle passing through *D* and having its diameter along *DC* such that, if *CEB* be drawn touching it at *E* and meeting the semicircle *ABC* in *B*, *BE* shall be equal to *AD*.

The problem is reduced to a problem contained in Apollonius's *Determinate Section* thus.

Suppose the problem solved by the semicircle DEF, BE being equal to AD. Join E to the centre G of the semicircle



DEF. Produce DA to H, making HA equal to AD. Let K be the middle point of DC.

Since the triangles ABC, GEC are similar,

$$AG^2 : GC^2 = BE^2 : EC^2$$
$$= AD^2 : EC^2, \text{ by hypothesis,}$$
$$= AD^2 : GC^2 - DG^2 \text{ (since } DG = GE)$$
$$= AG^2 - AD^2 : DG^2$$
$$= HG \,.\, DG : DG^2$$
$$= HG : DG.$$

Therefore

$$HG : DG = AD^2 : GC^2 - DG^2$$
$$= AD^2 : 2\,DC \,.\, GK.$$

Take a straight line L such that $AD^2 = L \,.\, 2\,DC$;

therefore $$HG : DG = L : GK,$$

or $$HG \,.\, GK = L \,.\, DG.$$

Therefore, given the two straight lines HD, DK (or the three points H, D, K on a straight line), we have to find a point G between D and K such that

$$HG \,.\, GK = L \,.\, DG,$$

which is the second *epitagma* of the third Problem in the *Determinate Section* of Apollonius, and therefore may be taken as solved. (The problem is the equivalent of the

solution of a certain quadratic equation.) Pappus observes that the problem is always possible (requires no διορισμός), and proves that it has only one solution.

(δ) *Lemmas on the treatise 'On contacts' by Apollonius.*

These lemmas are all pretty obvious except two, which are important, one belonging to Book I of the treatise, and the other to Book II. The two lemmas in question have already been set out à propos of the treatise of Apollonius (see pp. 182–5, above). As, however, there are several cases of the first (Props. 105, 107, 108, 109), one case (Prop. 108, pp. 836–8), different from that before given, may be put down here: *Given a circle and two points D, E within it, to draw straight lines through D, E to a point A on the circumference in such a way that, if they meet the circle again in B, C, BC shall be parallel to DE.*

We proceed by analysis. Suppose the problem solved and DA, EA drawn ('inflected') to A in such a way that, if AD, AE meet the circle again in B, C, BC is parallel to DE.

Draw the tangent at B meeting ED produced in F.

Then $\angle FBD = \angle ACB = \angle AED$;

therefore A, E, B, F are concyclic, and consequently

$$FD \,.\, DE = AD \,.\, DB.$$

But the rectangle $AD \,.\, DB$ is given, since it depends only on the position of D in relation to the circle, and the circle is given.

Therefore the rectangle $FD \,.\, DE$ is given.

And DE is given; therefore FD is given, and therefore F.

If follows that the tangent FB is given in position, and therefore B is given. Therefore BDA is given and consequently AE also.

To solve the problem, therefore, we merely take F on ED produced such that $FD \,.\, DE =$ the given rectangle made by the segments of any chord through D, draw the tangent FB, join BD and produce it to A, and lastly draw AE through to C; BC is then parallel to DE.

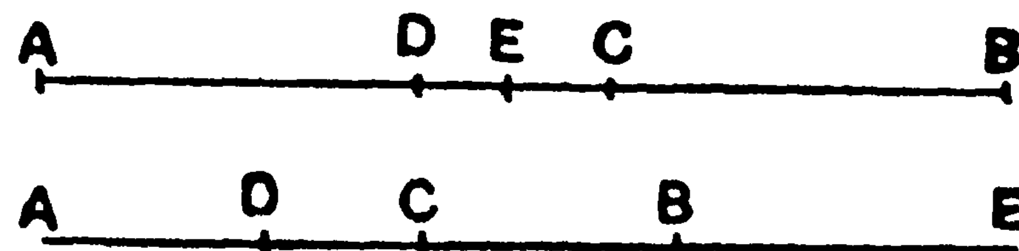
The other problem (Prop. 117, pp. 848–50) is, as we have seen, equivalent to the following: *Given a circle and three points D, E, F in a straight line external to it, to inscribe in the circle a triangle ABC such that its sides pass severally through the three points D, E, F.* For the solution, see pp. 182–4, above.

(ε) The Lemmas to the *Plane Loci* of Apollonius (Props. 119–26, pp. 852–64) are mostly propositions in geometrical algebra worked out by the methods of Eucl., Books II and VI. We may mention the following:

Prop. 122 is the well-known proposition that, if D be the middle point of the side BC in a triangle ABC,

$$BA^2 + AC^2 = 2(AD^2 + DC^2).$$

Props. 123 and 124 are two cases of the same proposition, the enunciation being marked by an expression which is also found in Euclid's *Data*. Let $AB : BC$ be a given ratio, and



let the rectangle $CA \cdot AD$ be given; then, if BE is a mean proportional between DB, BC, 'the square on AE is greater by the rectangle $CA \cdot AD$ than in the ratio of AB to BC to the square on EC', by which is meant that

$$AE^2 = CA \cdot AD + \frac{AB}{BC} \cdot EC^2,$$

or
$$(AE^2 - CA \cdot AD) : EC^2 = AB : BC.$$

The algebraical equivalent may be expressed thus (if $AB = a$, $BC = b$, $AD = c$, $BE = x$):

If $$x = \sqrt{(a-c)b}, \text{ then } \frac{(a \mp x)^2 - (a-b)c}{(x \mp b)^2} = \frac{a}{b}.$$

Prop. 125 is remarkable: If C, D be two points on a straight line AB,

$$AD^2 + \frac{AC}{BC} \cdot DB^2 = AC^2 + AC \cdot CB + \frac{AB}{BC} \cdot CD^2.$$

This is equivalent to the general relation between four points on a straight line discovered by Simson and therefore wrongly known as Stewart's theorem:

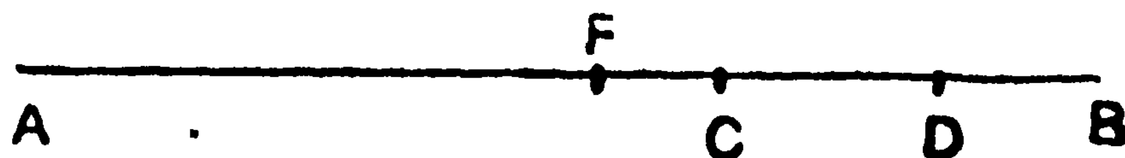
$$AD^2 \cdot BC + BD^2 \cdot CA + CD^2 \cdot AB + BC \cdot CA \cdot AB = 0.$$

(Simson discovered this theorem for the more general case where D is a point outside the line ABC.)

An algebraical equivalent is the identity

$$(d-a)^2(b-c)+(d-b)^2(c-a)+(d-c)^2(a-b)$$
$$+(b-c)(c-a)(a-b) = 0.$$

Pappus's proof of the last-mentioned lemma is perhaps worth giving.



C, D being two points on the straight line AB, take the point F on it such that

$$FD : DB = AC : CB. \quad (1)$$

Then $$FB : BD = AB : BC,$$

and $$(AB - FB) : (BC - BD) = AB : BC,$$

or $$AF : CD = AB : BC,$$

and therefore
$$AF \cdot CD : CD^2 = AB : BC. \quad (2)$$

From (1) we derive

$$\frac{AC}{CB} \cdot DB^2 = FD \cdot DB,$$

and from (2)

$$\frac{AB}{BC} \cdot CD^2 = AF \cdot CD.$$

We have now to prove that

$$AD^2 + BD \cdot DF = AC^2 + AC \cdot CB + AF \cdot CD,$$

or $$AD^2 + BD \cdot DF = CA \cdot AB + AF \cdot CD,$$

i.e. (if $DA \,.\, AC$ be subtracted from each side)

that $$AD \,.\, DC + FD \,.\, DB = AC \,.\, DB + AF \,.\, CD,$$

i.e. (if $AF \,.\, CD$ be subtracted from each side)

that $$FD \,.\, DC + FD \,.\, DB = AC \,.\, DB,$$

or $$FD \,.\, CB = AC \,.\, DB:$$

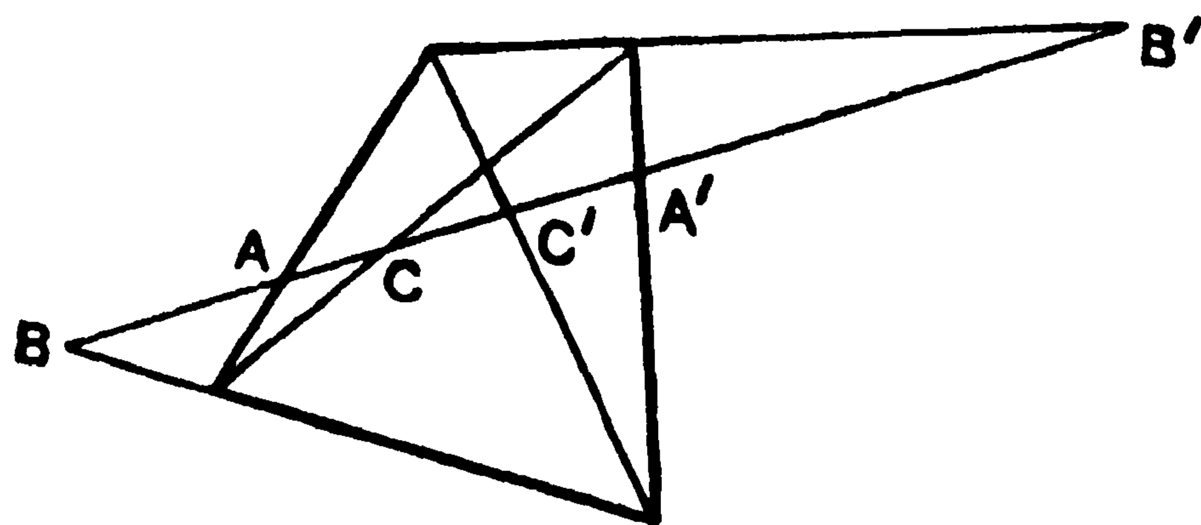
which is true, since, by (1) above, $FD : DB = AC : CB$.

(ζ) *Lemmas to the 'Porisms' of Euclid.*

The 38 Lemmas to the *Porisms* of Euclid form an important collection which, of course, has been included in one form or other in the 'restorations' of the original treatise. Chasles[1] in particular gives a classification of them, and we cannot do better than use it in this place: '23 of the Lemmas relate to rectilineal figures, 7 refer to the harmonic ratio of four points, and 8 have reference to the circle.

'Of the 23 relating to rectilineal figures, 6 deal with the quadrilateral cut by a transversal; 6 with the equality of the anharmonic ratios of two systems of four points arising from the intersections of four straight lines issuing from one point with two other straight lines; 4 may be regarded as expressing a property of the hexagon inscribed in two straight lines; 2 give the relation between the areas of two triangles which have two angles equal or supplementary; 4 others refer to certain systems of straight lines; and the last is a case of the problem of the *Cutting-off of an area*.'

The lemmas relating to the quadrilateral and the transversal are 1, 2, 4, 5, 6 and 7 (Props. 127, 128, 130, 131, 132, 133). Prop. 130 is a general proposition about any transversal



whatever, and is equivalent to one of the equations by which we express the involution of six points. If A, A'; B, B'; C, C' be the points in which the transversal meets the pairs of

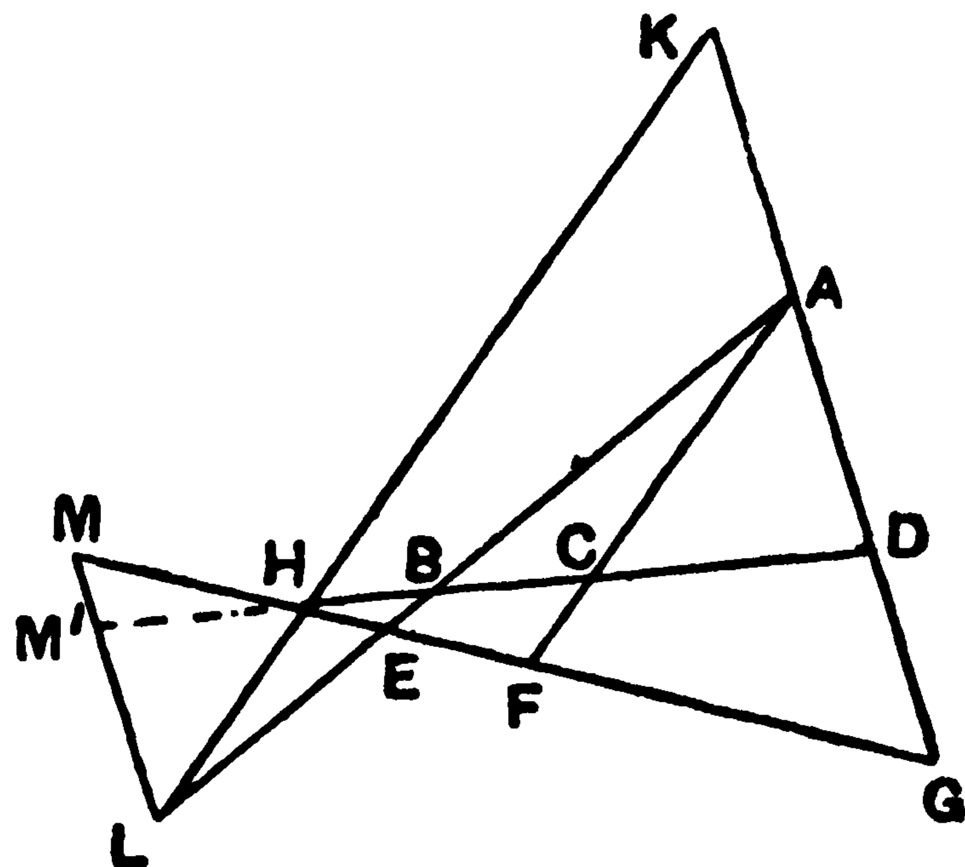
[1] Chasles, *Les trois livres de Porismes d'Euclide*, Paris, 1860, pp. 74 sq.

opposite sides and the two diagonals respectively, Pappus's result is equivalent to

$$\frac{AB\,.\,B'C}{A'B'\,.\,BC'} = \frac{CA}{C'A'}.$$

Props. 127, 128 are particular cases in which the transversal is parallel to a side; in Prop. 131 the transversal passes through the points of concourse of opposite sides, and the result is equivalent to the fact that the two diagonals divide into proportional parts the straight line joining the points of concourse of opposite sides: Prop. 132 is the particular case of Prop. 131 in which the line joining the points of concourse of opposite sides is parallel to a diagonal; in Prop. 133 the transversal passes through one only of the points of concourse of opposite sides and is parallel to a diagonal, the result being $CA^2 = CB\,.\,CB'$.

Props. 129, 136, 137, 140, 142, 145 (Lemmas 3, 10, 11, 14, 16, 19) establish the equality of the anharmonic ratios which four straight lines issuing from a point determine on two transversals; but both transversals are supposed to be drawn from the same point on one of the four straight lines. Let



AB, *AC*, *AD* be cut by transversals *HBCD*, *HEFG*. It is required to prove that

$$\frac{HE\,.\,FG}{HG\,.\,EF} = \frac{HB\,.\,CD}{HD\,.\,BC}.$$

Pappus gives (Prop. 129) two methods of proof which are practically equivalent. The following is the proof 'by compound ratios'.

Draw *HK* parallel to *AF* meeting *DA* and *AE* produced

in K, L; and draw LM parallel to AD meeting GH produced in M.

Then $$\frac{HE.FG}{HG.EF}=\frac{HE}{EF}\cdot\frac{FG}{HG}=\frac{LH}{AF}\cdot\frac{AF}{HK}=\frac{LH}{HK}.$$

In exactly the same way, if DH produced meets LM in M' we prove that

$$\frac{HB.CD}{HD.BC}=\frac{LH}{HK}.$$

Therefore $$\frac{HE.FG}{HG.EF}=\frac{HB.CD}{HD.BC}.$$

(The proposition is proved for $HBCD$ and any other transversal not passing through H by applying our proposition twice, as usual.)

Props. 136, 142 are the reciprocal; Prop. 137 is a particular case in which one of the transversals is parallel to one of the straight lines, Prop. 140 a reciprocal of Prop. 137, Prop. 145 another case of Prop. 129.

The Lemmas 12, 13, 15, 17 (Props. 138, 139, 141, 143) are equivalent to the property of the hexagon inscribed in two straight lines, viz. that, if the vertices of a hexagon are situate, three and three, on two straight lines, the points of concourse of opposite sides are in a straight line; in Props. 138, 141 the straight lines are parallel, in Props. 139, 143 not parallel.

Lemmas 20, 21 (Props. 146, 147) prove that, when one angle of one triangle is equal or supplementary to one angle of another triangle, the areas of the triangles are in the ratios of the rectangles contained by the sides containing the equal or supplementary angles.

The seven Lemmas 22, 23, 24, 25, 26, 27, 34 (Props. 148-53 and 160) are propositions relating to the segments of a straight line on which two intermediate points are marked. Thus:

Props. 148, 150.

If C, D be two points on AB, then

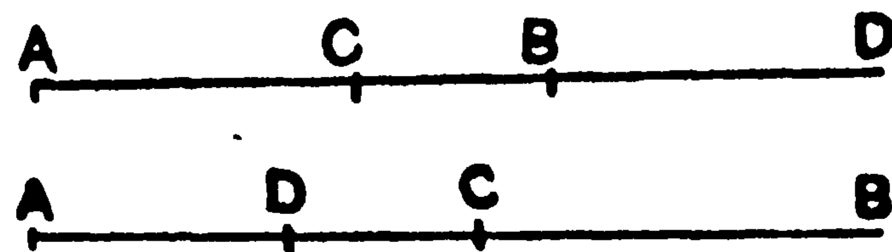
(a) if $2AB.CD=CB^2$, $AD^2=AC^2+DB^2$;



(b) if $2AC.BD=CD^2$, $AB^2=AD^2+CB^2$.

Props. 149, 151.

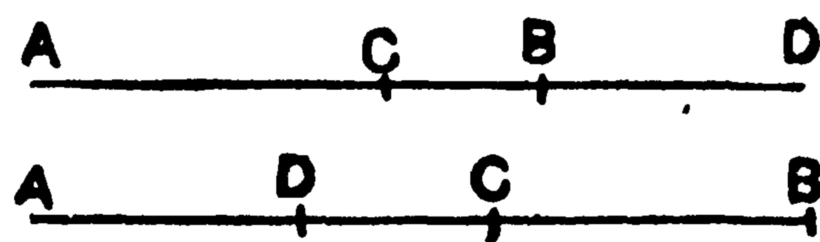
If $$AB \cdot BC = BD^2,$$
then $$(AD \pm DC)\, BD = AD \cdot DC,$$
$$(AD \pm DC)\, BC = DC^2,$$



and $$(AD \pm DC)\, BA = AD^2.$$

Props. 152, 153.

If $AB : BC = AD^2 : DC^2$, then $AB \cdot BC = BD^2$.



Prop. 160.

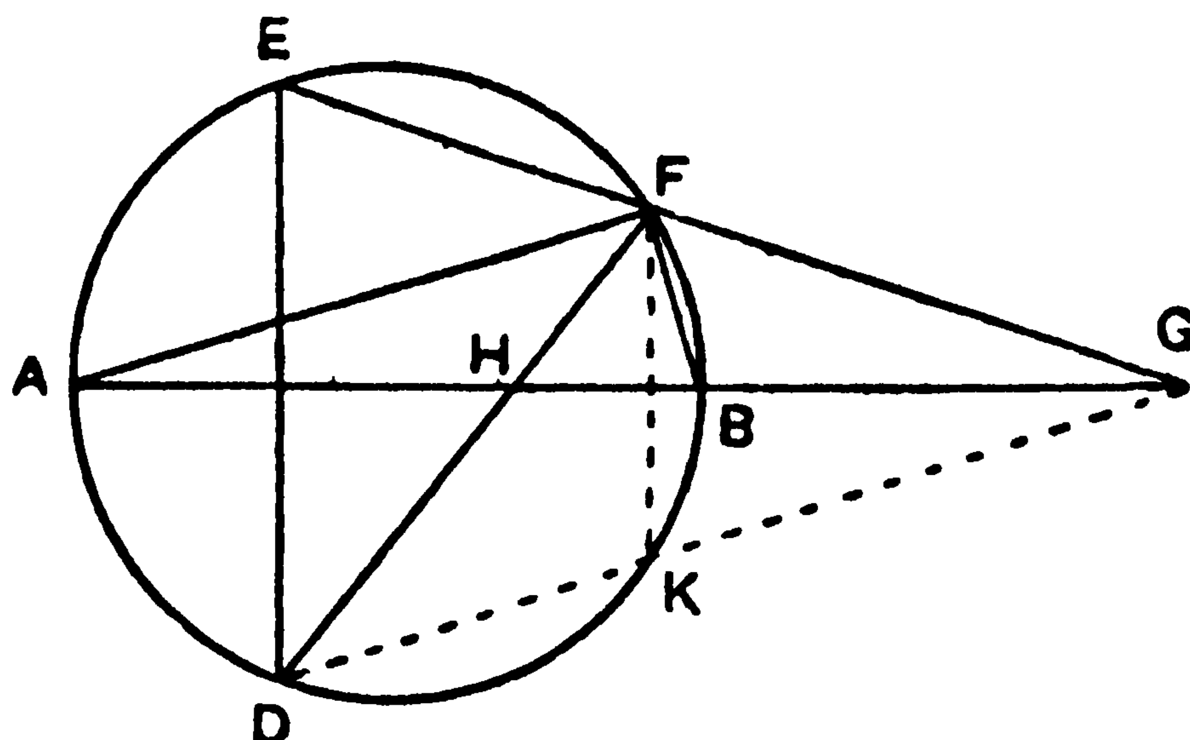
If $AB : BC = AD : DC$, then, if E be the middle point of AC,
$$BE \cdot ED = EC^2,$$
$$BD \cdot DE = AD \cdot DC,$$
$$EB \cdot BD = AB \cdot BC.$$



The Lemmas about the circle include the harmonic properties of the pole and polar, whether the pole is external to the circle (Prop. 154) or internal (Prop. 161). Prop. 155 is a problem, Given a segment of a circle on AB as base, to inflect straight lines AC, BC to the segment in a given ratio to one another.

Prop. 156 is one which Pappus has already used earlier in the *Collection*. It proves that the straight lines drawn from the extremities of a chord (DE) to any point (F) of the circumference divide harmonically the diameter (AB) perpendicular to the chord. Or, if ED, FK be parallel chords, and EF, DK meet in G, and EK, DF in H, then
$$AH : HB = AG : GB.$$

Since AB bisects DE perpendicularly, (arc AE) = (arc AD) and $\angle EFA = \angle AFD$, or AF bisects the angle EFD.

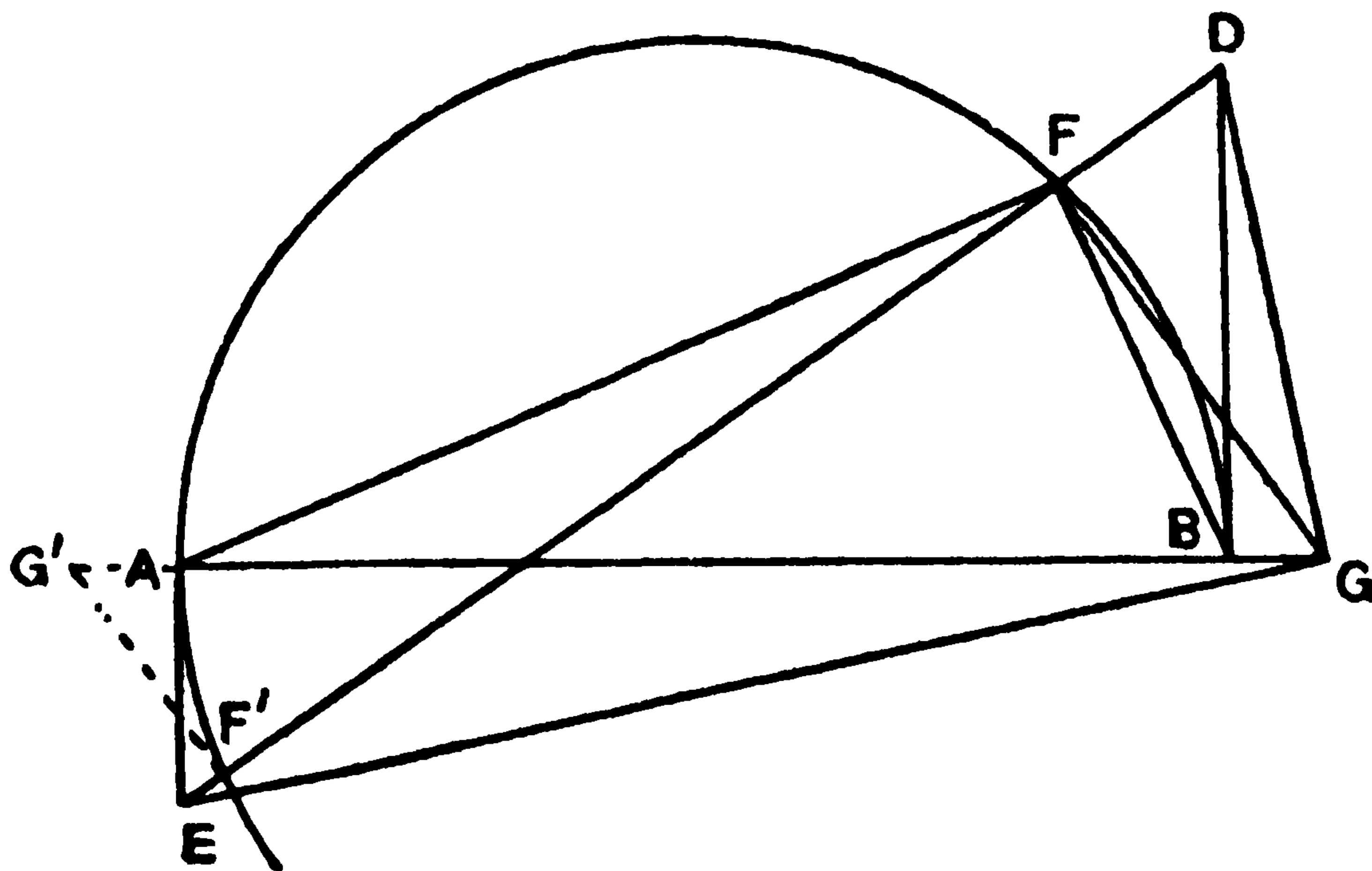


Since the angle AFB is right, FB bisects $\angle HFG$, the supplement of $\angle EFD$.

Therefore (Eucl. VI. 3) $GB : BH = GF : FH = GA : AH$,

and, alternately and inversely, $AH : HB = AG : GB$.

Prop. 157 is remarkable in that (without any mention of a conic) it is practically identical with Apollonius's *Conics* III. 15 about the foci of a central conic. Pappus's theorem is as follows. Let AB be the diameter of a semicircle, and



from A, B let two straight lines AE, BD be drawn at right angles to AB. Let any straight line DE meet the two perpendiculars in D, E and the semicircle in F. Further, let FG be drawn at right angles to DE, meeting AB produced in G.

It is to be proved that

$$AG . GB = AE . BD$$

Since F, D, G, B are concyclic, $\angle BDG = \angle BFG$.

And, since AFB, EFG are both right angles, $\angle BFG = \angle AFE$.

But, since A, E, G, F are concyclic, $\angle AFE = \angle AGE$.

Therefore $\angle BDG = \angle AGE$;

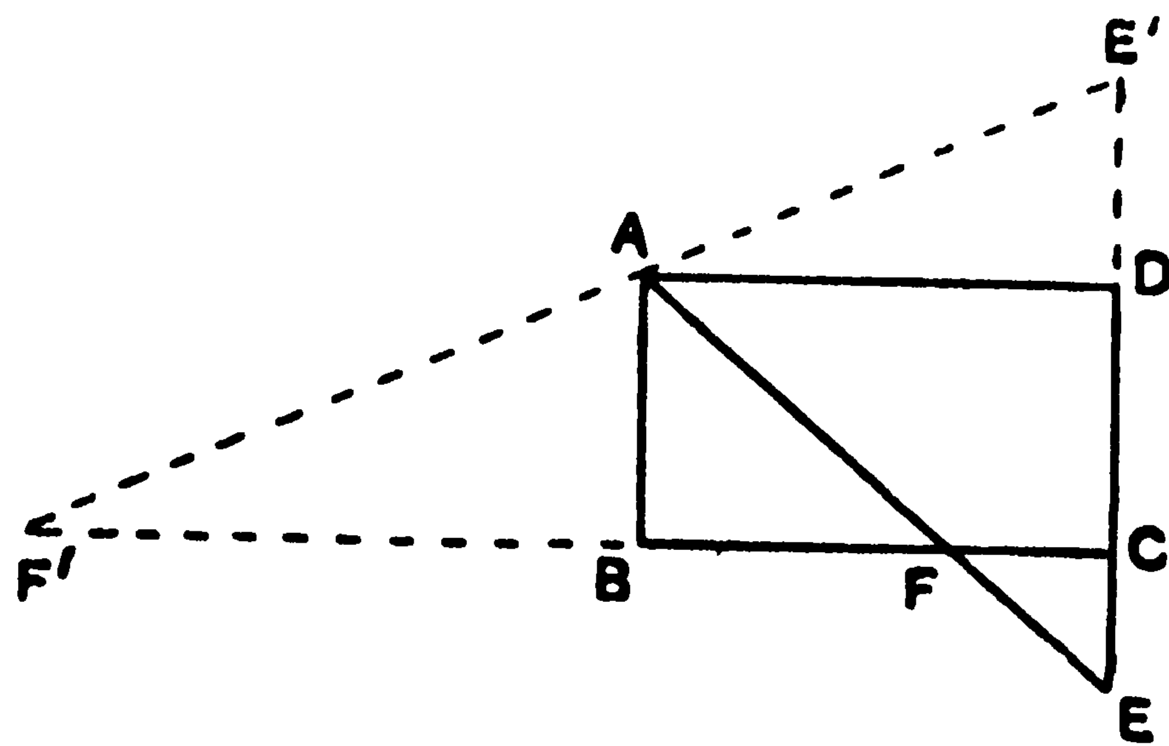
and the right-angled triangles DBG, GAE are similar.

Therefore $AG : AE = BD : GB$,

or $AG \,.\, GB = AE \,.\, DB$.

In Apollonius G and the corresponding point G' on BA produced which is obtained by drawing $F'G'$ perpendicular to ED (where DE meets the circle again in F') are the foci of a central conic (in this case a hyperbola), and DE is any tangent to the conic; the rectangle $AE \,.\, BD$ is of course equal to the square on half the conjugate axis.

(η) The Lemmas to the *Conics* of Apollonius (pp. 918–1004) do not call for any extended notice. There are a large number of propositions in geometrical algebra of the usual kind, relating to the segments of a straight line marked by a number of points on it; propositions about lines divided into proportional segments and about similar figures; two propositions relating to the construction of a hyperbola (Props. 204, 205) and a proposition (208) proving that two hyperbolas with the same asymptotes do not meet one another. There are also two propositions (221, 222) equivalent to an obvious trigono-



metrical formula. Let $ABCD$ be a rectangle, and let any straight line through A meet DC produced in E and BC (produced if necessary) in F.

Then $EA \,.\, AF = ED \,.\, DC + CB \,.\, BF$.

For $EA^2 + AF^2 = ED^2 + DA^2 + AB^2 + BF^2$

$= ED^2 + BC^2 + CD^2 + BF^2.$

Also $EA^2 + AF^2 = EF^2 + 2EA \,.\, AF.$

Therefore

$$\begin{aligned} 2EA \,.\, AF &= EA^2 + AF^2 - EF^2 \\ &= ED^2 + BC^2 + CD^2 + BF^2 - EF^2 \\ &= (ED^2 + CD^2) + (BC^2 + BF^2) - EF^2 \\ &= EC^2 + 2ED \,.\, DC + CF^2 + 2CB \,.\, BF - EF^2 \\ &= 2ED \,.\, DC + 2CB \,.\, BF; \end{aligned}$$

i.e. $EA \,.\, AF = ED \,.\, DC + CB \,.\, BF.$

This is equivalent to $\sec\theta \operatorname{cosec}\theta = \tan\theta + \cot\theta.$

The algebraical equivalents of some of the results obtained by the usual geometrical algebra may be added.

Props. 178, 179, 192–4.

$$(a+2b)a + (b+x)(b-x) = (a+b+x)(a+b-x).$$

Prop. 195. $4a^2 = 2\{(a-x)(a+x) + (a-y)(a+y) + x^2 + y^2\}.$

Prop. 196.

$$(a+b-x)^2 + (a+b+x)^2 = (x-b)^2 + (x+b)^2 + 2(a+2b)a.$$

Props. 197, 199, 198.

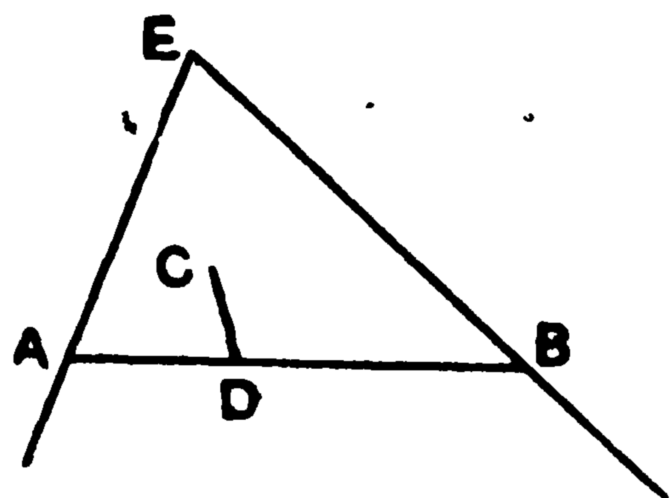
$$\left.\begin{array}{ll} \text{If} & (x+y+a)a + x^2 = (a+x)^2, \\ \text{or if} & (x+y+a)a + x^2 = (a+y)^2, \\ \text{or if} & (x+y-a)a + (x-a)^2 = y^2, \end{array}\right\} \text{ then } x = y.$$

Props. 200, 201. If $(a+b)x = b^2$, then $\dfrac{2b+a}{a} = \dfrac{b+x}{b-x}$ and $(2b+a)a = (a+b)(a+b-x)$.

Prop. 207. If $(a+b)b = 2a^2$, then $a = b$.

(θ) The two Lemmas to the *Surface-Loci* of Euclid have already been mentioned as significant. The first has the appearance of being a general enunciation, such as Pappus

is fond of giving, to cover a class of propositions. The enunciation may be translated as follows: 'If AB be a straight line, and CD a straight line parallel to a straight line given in position, and if the ratio $AD \cdot DB : DC^2$ be given, the point C lies on a conic section. If now AB be no longer given in position, and the points A, B are no longer given but lie (respectively) on straight lines AE, EB given in position, the point C raised above (the plane containing AE, EB) lies on a surface given in position. And this was proved.' Tannery was the first to explain this intelligibly; and his interpretation only requires the very slight change in the text of substituting εὐθείαις for εὐθεῖα in the phrase γένηται δὲ πρὸς θέσει εὐθεῖα ταῖς AE, EB. It is not clear whether, when AB ceases to be given in *position*, it is still given in *length*. If it is given in *length* and A, B move on the lines AE, EB respectively, the surface which is the locus of C is a complicated one such as Euclid would hardly have been in a position to investigate. But two possible cases are indicated which he may have discussed, (1) that in which AB moves always parallel to itself and varies in length accordingly, (2) that in which the two lines on which A, B move are parallel instead of meeting at a point. The loci in these two cases would of course be a cone and a cylinder respectively.



The second Lemma is still more important, since it is the first statement on record of the focus-directrix property of the three conic sections. The proof, after Pappus, has been set out above (pp. 119–21).

(ι) *An unallocated Lemma.*

Book VII ends (pp. 1016–18) with a lemma which is not given under any particular treatise belonging to the *Treasury of Analysis*, but is simply called 'Lemma to the Ἀναλυόμενος'. If ABC be a triangle right-angled at B, and AB, BC be divided at F, G so that $AF : FB = BG : GC = AB : BC$, and if AEG, CEF be joined and BE joined and produced to D, then shall BD be perpendicular to AC.

The text is unsatisfactory, for there is a long interpolation containing an attempt at a proof by *reductio ad absurdum*;

but the genuine proof is indicated, although it breaks off before it is quite complete.

Since $AF : FB = BG : GC$,

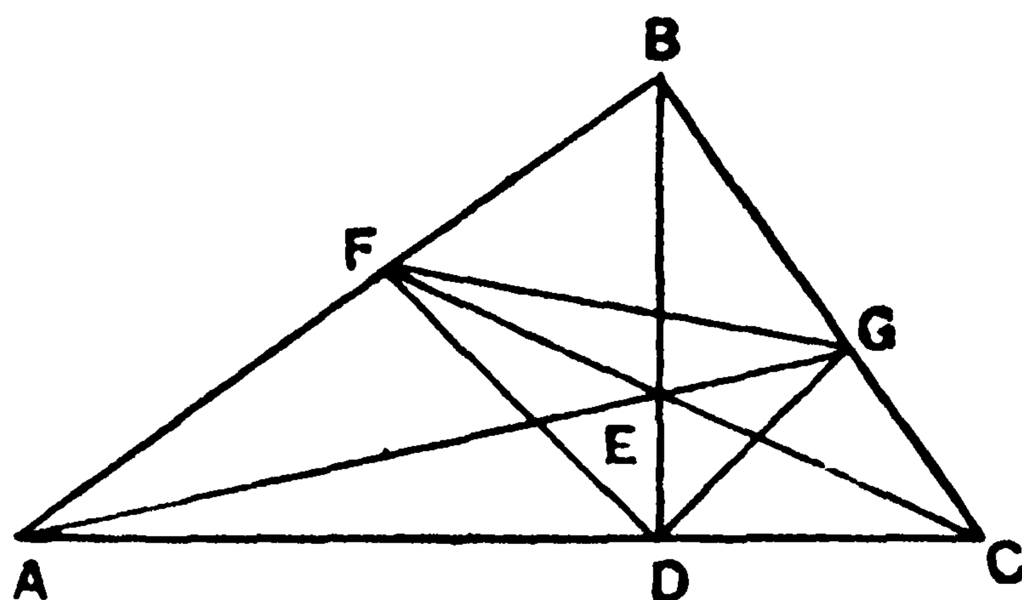
$AB : FB = BC : GC$,

or $AB : BC = FB : GC$.

But, by hypothesis, $AB : BC = BG : GC$;

therefore $BF = BG$.

From this point the proof apparently proceeded by analysis. 'Suppose it done' (γεγονέτω), i.e. suppose the proposition true, and BED perpendicular to AC.



Then, by similarity of triangles, $AD : DB = AB : BC$;

therefore $AF : FB = AD : DB$, and consequently the angle ADB is bisected by DF.

Similarly the angle BDC is bisected by DG.

Therefore each of the angles BDF, BDG is half a right angle, and consequently the angle FDG is a right angle.

Therefore B, G, D, F are concyclic; and, since the angles FDB, BDG are equal, $FB = BG$.

This is of course the result above proved.

Evidently the interpolator tried to clinch the argument by proving that the angle BDA could not be anything but a right angle.

Book VIII.

Book VIII of the *Collection* is mainly on mechanics, although it contains, in addition, some propositions of purely geometrical interest.

Historical preface.

It begins with an interesting preface on the claim of theoretical mechanics, as distinct from the merely practical or industrial, to be regarded as a mathematical subject. Archimedes, Philon, Heron of Alexandria are referred to as the principal exponents of the science, while Carpus of Antioch is also mentioned as having applied geometry to 'certain (practical) arts'.

The date of Carpus is uncertain, though it is probable that he came after Geminus; the most likely date seems to be the first or second century A.D. Simplicius gives the authority of Iamblichus for the statement that Carpus squared the circle by means of a certain curve, which he simply called a curve generated by a double motion.[1] Proclus calls him 'Carpus the writer on mechanics (ὁ μηχανικός)', and quotes from a work of his on Astronomy some remarks about the relation between problems and theorems and the 'priority in order' of the former.[2] Proclus also mentions him as having held that an angle belongs to the category of *quantity* (ποσόν), since it represents a sort of 'distance' between the two lines forming it, this distance being 'extended one way' (ἐφ' ἓν διεστώς) though in a different sense from that in which a line represents extension one way, so that Carpus's view appeared to be 'the greatest possible paradox'[3]; Carpus seems in reality to have been anticipating the modern view of an angle as representing *divergence* rather than distance, and to have meant by ἐφ' ἓν *in one sense* (rotationally), as distinct from one way or in one dimension (linearly).

Pappus tells us that Heron distinguished the logical, i.e. theoretical, part of mechanics from the practical or manual (χειρουργικόν), the former being made up of geometry, arithmetic, astronomy and physics, the latter of work in metal, architecture, carpentering and painting; the man who had been trained from his youth up in the *sciences* aforesaid as well as practised in the said *arts* would naturally prove the best architect and inventor of mechanical devices, but, as it is difficult or impossible for the same person to do both the necessary

[1] Simplicius on Arist. *Categ.*, p. 192, Kalbfleisch.
[2] Proclus on Eucl. I, pp. 241–3. [3] *Ib.*, pp. 125. 25–126. 6.

mathematics and the practical work, he who has not the former must perforce use the resources which practical experience in his particular art or craft gives him. Other varieties of mechanical work included by the ancients under the general term mechanics were (1) the use of the mechanical powers, or devices for moving or lifting great weights by means of a small force, (2) the construction of engines of war for throwing projectiles a long distance, (3) the pumping of water from great depths, (4) the devices of 'wonder-workers' (*θαυμασιουργοί*), some depending on pneumatics (like Heron in the *Pneumatica*), some using strings, &c., to produce movements like those of living things (like Heron in 'Automata and Balancings'), some employing floating bodies (like Archimedes in 'Floating Bodies'), others using water to measure time (like Heron in his 'Water-clocks'), and lastly 'sphere-making', or the construction of mechanical imitations of the movements of the heavenly bodies with the uniform circular motion of water as the motive power. Archimedes, says Pappus, was held to be the one person who had understood the cause and the reason of all these various devices, and had applied his extraordinarily versatile genius and inventiveness to all the purposes of daily life, and yet, although this brought him unexampled fame the world over, so that his name was on every one's lips, he disdained (according to Carpus) to write any mechanical work save a tract on sphere-making, but diligently wrote all that he could in a small compass of the most advanced parts of geometry and of subjects connected with arithmetic. Carpus himself, says Pappus, as well as others applied geometry to practical arts, and with reason: 'for geometry is in no wise injured, nay it is by nature capable of giving substance to many arts by being associated with them, and, so far from being injured, it may be said, while itself advancing those arts, to be honoured and adorned by them in return.'

The object of the Book.

Pappus then describes the object of the Book, namely to set out the propositions which the ancients established by geometrical methods, besides certain useful theorems discovered by himself, but in a shorter and clearer form and

in better logical sequence than his predecessors had attained. The sort of questions to be dealt with are (1) a comparison between the force required to move a given weight along a horizontal plane and that required to move the same weight upwards on an inclined plane, (2) the finding of two mean proportionals between two unequal straight lines, (3) given a toothed wheel with a certain number of teeth, to find the diameter of, and to construct, another wheel with a given number of teeth to work on the former. Each of these things, he says, will be clearly understood in its proper place if the principles on which the 'centrobaric doctrine' is built up are first set out. It is not necessary, he adds, to define what is meant by 'heavy' and 'light' or upward and downward motion, since these matters are discussed by Ptolemy in his *Mathematica*; but the notion of the centre of gravity is so fundamental in the whole theory of mechanics that it is essential in the first place to explain what is meant by the 'centre of gravity' of any body.

On the centre of gravity.

Pappus then defines the centre of gravity as 'the point within a body which is such that, if the weight be conceived to be suspended from the point, it will remain at rest in any position in which it is put'.[1] The method of determining the point by means of the intersection, first of planes, and then of straight lines, is next explained (chaps. 1, 2), and Pappus then proves (Prop. 2) a proposition of some difficulty, namely that, if D, E, F be points on the sides BC, CA, AB of a triangle ABC such that

$$BD : DC = CE : EA = AF : FB,$$

then the centre of gravity of the triangle ABC is also the centre of gravity of the triangle DEF.

Let H, K be the middle points of BC, CA respectively; join AH, BK. Join HK meeting DE in L.

Then AH, BK meet in G, the centre of gravity of the triangle ABC, and $AG = 2\,GH$, $BG = 2\,GK$, so that

$$CA : AK = AB : HK = BG : GK = AG : GH.$$

[1] Pappus, viii, p. 1030. 11–13.

Now, by hypothesis,

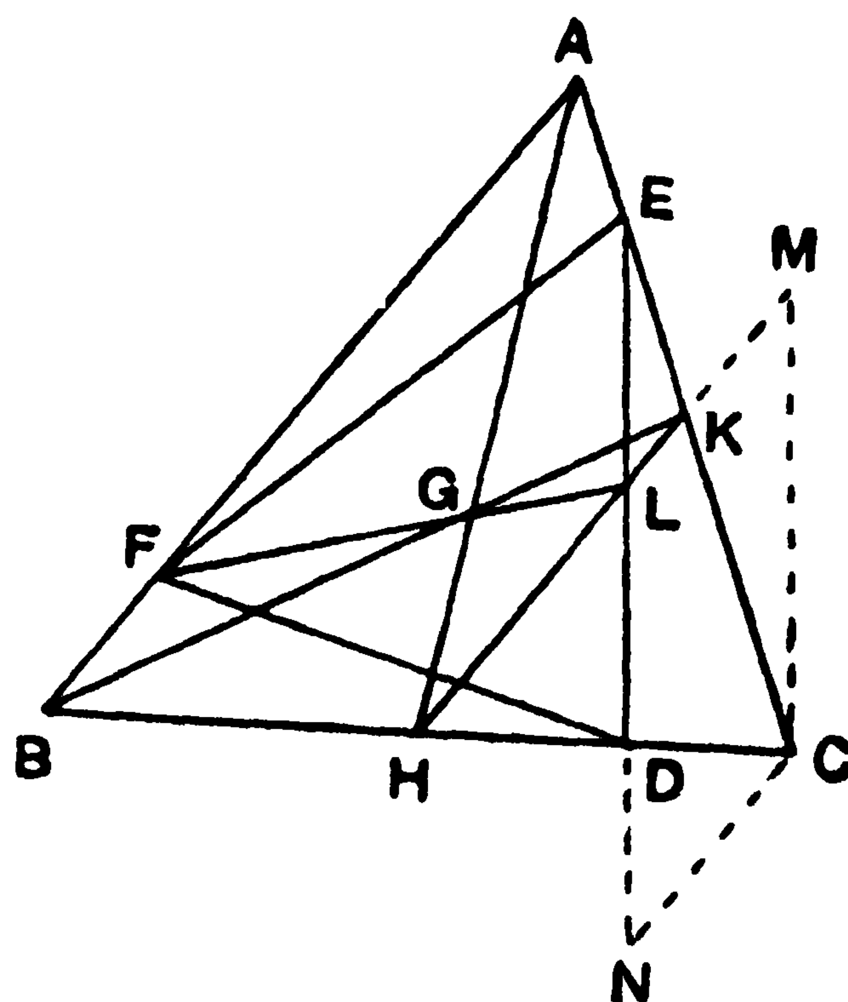
$$CE : EA = BD : DC,$$

whence $$CA : AE = BC : CD,$$

and, if we halve the antecedents,

$$AK : AE = HC : CD;$$

therefore $$AK : EK = HC : HD \text{ or } BH : HD,$$



whence, *componendo*, $CE : EK = BD : DH$. (1)

But $AF : FB = BD : DC = (BD : DH) . (DH : DC)$

$$= (CE : EK) . (DH : DC). \quad (2)$$

Now, ELD being a transversal cutting the sides of the triangle KHC, we have

$$HL : KL = (CE : EK) . (DH : DC). \quad (3)$$

[This is 'Menelaus's theorem'; Pappus does not, however, quote it, but proves the relation *ad hoc* in an added lemma by drawing CM parallel to DE to meet HK produced in M. The proof is easy, for $HL : LK = (HL : LM) . (LM : LK)$

$$= (HD : DC) . (CE : EK).]$$

It follows from (2) and (3) that

$$AF : FB = HL : LK,$$

and, since AB is parallel to HK, and AH, BK are straight lines meeting in G, FGL is a straight line.

[This is proved in another easy lemma by *reductio ad absurdum*.]

We have next to prove that $EL = LD$.

Now [again by 'Menelaus's theorem', proved *ad hoc* by drawing CN parallel to HK to meet ED produced in N]

$$EL : LD = (EK : KC) . (CH : HD). \qquad (4)$$

But, by (1) above, $CE : EK = BD : DH$;

therefore $\quad CK : KE = BH : HD = CH : HD,$

so that $(EK : KC) . (CH : HD) = 1$, and therefore, from (4),

$$EL = LD.$$

It remains to prove that $FG = 2\,GL$, which is obvious by parallels, since $\quad FG : GL = AG : GH = 2 : 1.$

Two more propositions follow with reference to the centre of gravity. The first is, Given a rectangle with AB, BC as adjacent sides, to draw from C a straight line meeting the side opposite BC in a point D such that, if the trapezium $ADCB$ is hung from the point D, it will rest with AD, BC horizontal.

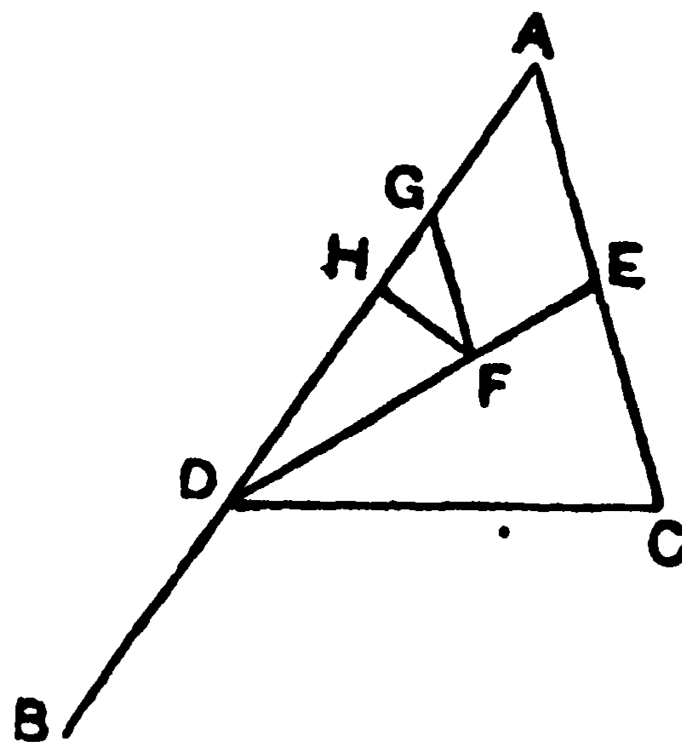
In other words, the centre of gravity must be in DL drawn perpendicular to BC. Pappus proves by analysis that $CL^2 = 3\,BL^2$, so that the problem is reduced to that of dividing BC into parts BL, LC such that this relation holds. The latter problem is solved (Prop. 6) by taking a point, say X, in CB such that $CX = 3\,XB$, describing a semicircle on BC as diameter and drawing XY at right angles to BC to meet the semicircle in Y, so that $XY^2 = \frac{3}{16}\,BC^2$, and then dividing CB at L so that

$$CL : LB = CX : XY \left(= \tfrac{3}{4} : \tfrac{1}{4}\sqrt{3} = \sqrt{3} : 1\right).$$

The second proposition is this (Prop. 7). Given two straight lines AB, AC, and B a fixed point on AB, if CD be drawn

with its extremities on AC, AB and so that $AC : BD$ is a given ratio, then the centre of gravity of the triangle ADC will lie on a straight line.

Take E, the middle point of AC, and F a point on DE such that $DF = 2\,FE$. Also let H be a point on BA such that $BH = 2\,HA$. Draw FG parallel to AC. Then $AG = \frac{1}{3}AD$, and $AH = \frac{1}{3}AB$; therefore $HG = \frac{1}{3}BD$.



Also $FG = \frac{2}{3}AE = \frac{1}{3}AC$. Therefore, since the ratio $AC:BD$ is given, the ratio $GH : GF$ is given.

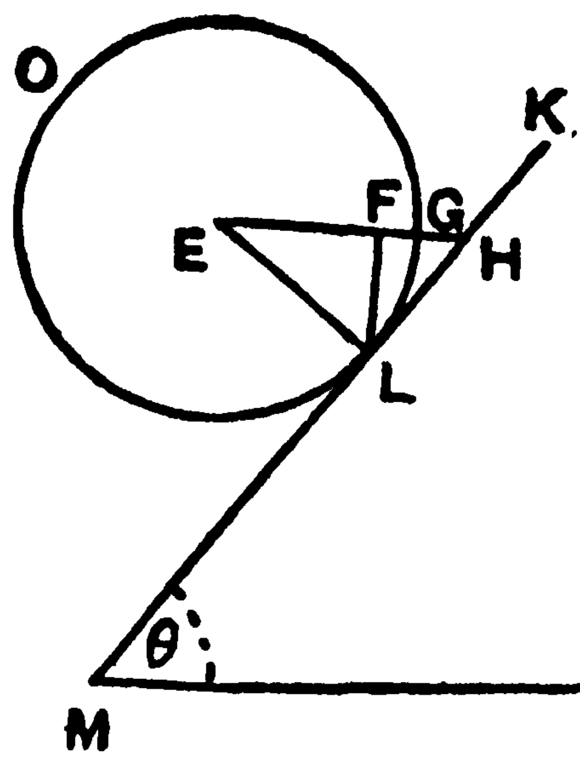
And the angle FGH $(= A)$ is given; therefore the triangle FGH is given in species, and consequently the angle GHF is given. And H is a given point. Therefore HF is a given straight line, and it contains the centre of gravity of the triangle ADC.

The inclined plane.

Prop. 8 is on the construction of a plane at a given inclination to another plane parallel to the horizon, and with this Pappus leaves theory and proceeds to the practical part. Prop. 9 (p. 1054. 4 sq.) investigates the problem 'Given a weight which can be drawn along a plane parallel to the horizon by a given force, and a plane inclined to the horizon at a given angle, to find the force required to draw the weight upwards on the inclined plane'. This seems to be the first or only attempt in ancient times to investigate motion on an inclined plane, and as such it is curious, though of no value.

Let A be the weight which can be moved by a force C along a horizontal plane. Conceive a sphere with weight equal to A placed in contact at L with the given inclined plane; the circle OGL represents a section of the sphere by a vertical plane passing through E its centre and LK the line of greatest slope drawn through the point L. Draw EGH horizontal and therefore parallel to MN in the plane of section, and draw LF perpendicular to EH. Pappus seems to regard the plane as rough, since he proceeds to make a system in equilibrium

about FL as if L were the fulcrum of a lever. Now the weight A acts vertically downwards along a straight line through E. To balance it, Pappus supposes a weight B attached with its centre of gravity at G.



Then $A : B = GF : EF$

$= (EL - EF) : EF$

$[= (1 - \sin\theta) : \sin\theta,$

where $\angle KMN = \theta]$;

and, since $\angle KMN$ is given, the ratio $EF : EL$, and therefore the ratio $(EL - EF) : EF$, is given; thus B is found.

Now, says Pappus, if D is the force which will move B along a horizontal plane, as C is the force which will move A along a horizontal plane, the sum of C and D will be the force required to move the sphere upwards on the inclined plane. He takes the particular case where $\theta = 60°$. Then $\sin\theta$ is approximately $\frac{104}{120}$ (he evidently uses $\frac{1}{2} \cdot \frac{26}{15}$ for $\frac{1}{2}\sqrt{3}$),

and $A : B = 16 : 104$.

Suppose, for example, that $A = 200$ talents; then B is 1300 talents. Suppose further that C is 40 man-power; then, since $D : C = B : A$, $D = 260$ man-power; and it will take $D + C$, or 300 man-power, to move the weight up the plane!

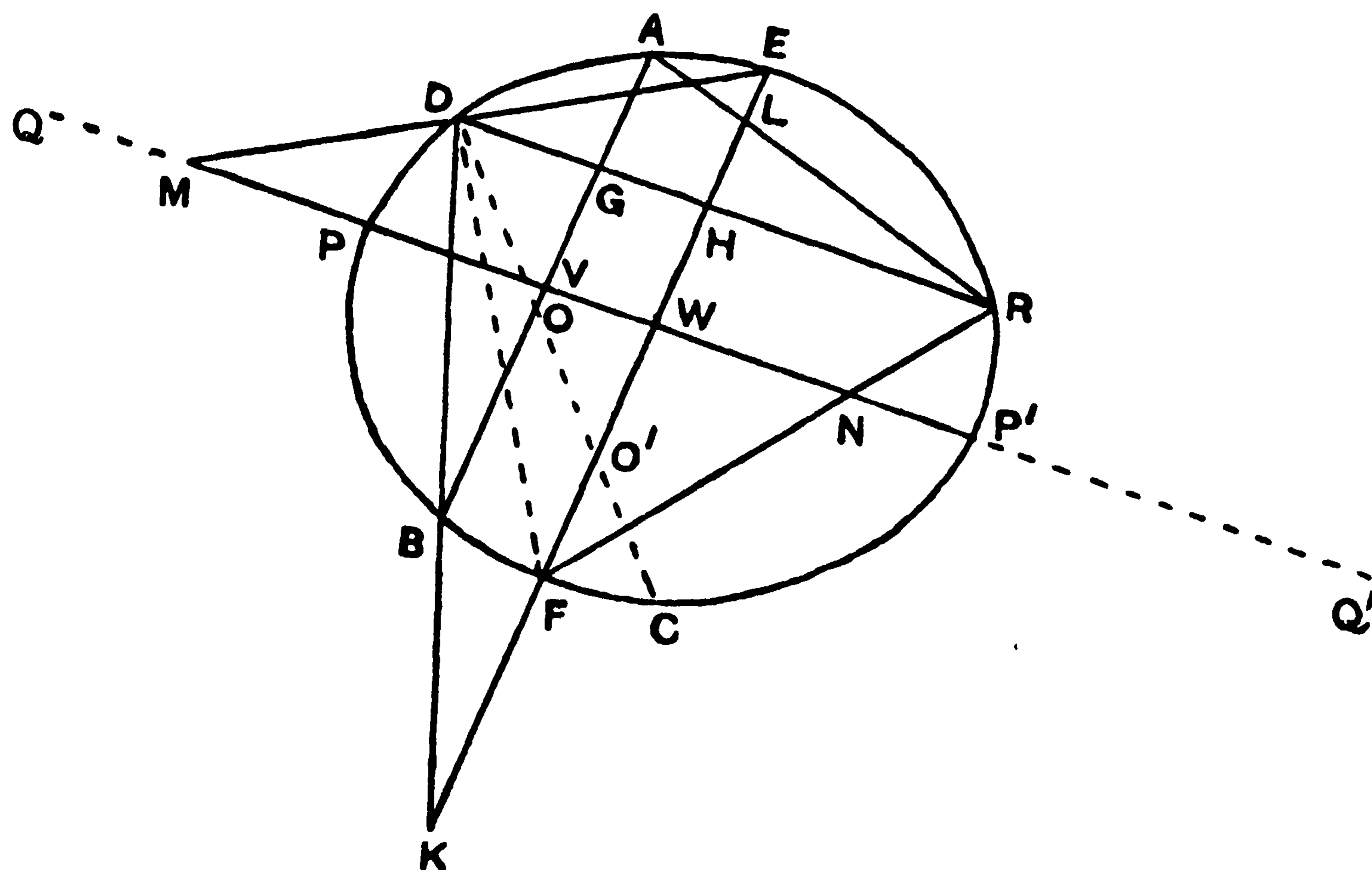
Prop. 10 gives, from Heron's *Barulcus*, the machine consisting of a pulley, interacting toothed wheels, and a spiral screw working on the last wheel and turned by a handle; Pappus merely alters the proportions of the weight to the force, and of the diameter of the wheels. At the end of the chapter (pp. 1070–2) he repeats his construction for the finding of two mean proportionals.

Construction of a conic through five points.

Chaps. 13–17 are more interesting, for they contain the solution of the problem of *constructing a conic through five given points.* The problem arises in this way. Suppose we are given a broken piece of the surface of a cylindrical column such that no portion of the circumference of either of its base

is left intact, and let it be required to find the diameter of a circular section of the cylinder. We take any two points A, B on the surface of the fragment and by means of these we find five points on the surface all lying in one plane section, in general oblique. This is done by taking five different radii and drawing pairs of circles with A, B as centres and with each of the five radii successively. These pairs of circles with equal radii, intersecting at points on the surface, determine five points on the plane bisecting AB at right angles. The five points are then represented on any plane by triangulation.

Suppose the points are A, B, C, D, E and are such that no two of the lines connecting the different pairs are parallel.



This case can be reduced to the construction of a conic through the five points A, B, D, E, F where EF is parallel to AB. This is shown in a subsequent lemma (chap. 16).

For, if EF be drawn through E parallel to AB, and if CD meet AB in O and EF in O', we have, by the well-known proposition about intersecting chords,

$$CO \cdot OD : AO \cdot OB = CO' \cdot O'D : EO' \cdot O'F,$$

whence $O'F$ is known, and F is determined.

We have then (Prop. 13) to construct a conic through A, B, D, E, F, where EF is parallel to AB.

Bisect AB, EF at V, W; then VW produced both ways is a diameter. Draw DR, the chord through D parallel

to this diameter. Then R is determined by means of the relation

$$RG.GD:BG.GA = RH.HD:FH.HE \qquad (1)$$

in this way.

Join DB, RA, meeting EF in K, L respectively.

Then, by similar triangles,

$$RG.GD:BG.GA = (RH:HL).(DH:HK)$$
$$= RH.HD:KH.HL.$$

Therefore, by (1), $FH.HE = KH.HL$,

whence HL is determined, and therefore L. The intersection of AL, DH determines R.

Next, in order to find the extremities P, P' of the diameter through V, W, we draw ED, RF meeting PP' in M, N respectively.

Then, as before,

$FW.WE:P'W.WP = FH.HE:RH.HD$, by the ellipse,

$= FW.WE:NW.WM$, by similar triangles.

Therefore $P'W.WP = NW.WM$;

and similarly we can find the value of $P'V.VP$.

Now, says Pappus, since $P'W.WP$ and $P'V.VP$ are given areas and the points V, W are given, P, P' are given. His determination of P, P' amounts (Prop. 14 following) to an elimination of one of the points and the finding of the other by means of an equation of the second degree.

Take two points Q, Q' on the diameter such that

$$P'V.VP = WV.VQ, \qquad (\alpha)$$
$$P'W.WP = VW.WQ'; \qquad (\beta)$$

Q, Q' are thus known, while P, P' remain to be found.

By (α) $P'V:VW = QV:VP$,

whence $P'W:VW = PQ:PV$.

Therefore, by means of (β),

$$PQ:PV = Q'W:WP,$$

so that $$PQ : QV = Q'W : PQ',$$

or $$PQ . PQ' = QV . Q'W.$$

Thus P can be found, and similarly P'.

The conjugate diameter is found by virtue of the relation

$$(\text{conjugate diam.})^2 : PP'^2 = p : PP'.$$

where p is the latus rectum to PP' determined by the property of the curve

$$p : PP' = AV^2 : PV . VP'.$$

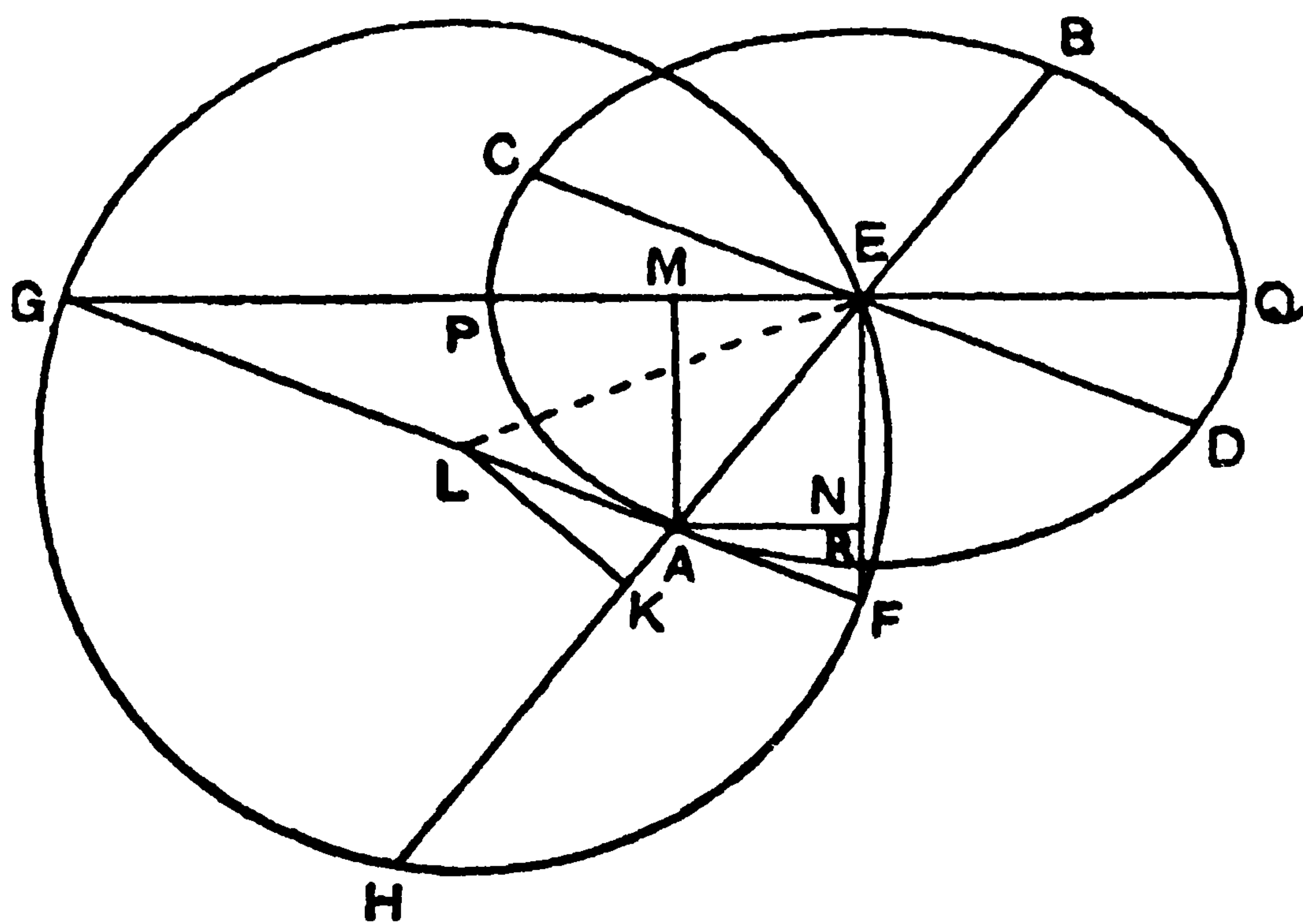
Problem, Given two conjugate diameters of an ellipse, to find the axes.

Lastly, Pappus shows (Prop. 14, chap. 17) how, when we are given two conjugate diameters, we can find the axes. The construction is as follows. Let AB, CD be conjugate diameters (CD being the greater), E the centre.

Produce EA to H so that

$$EA . AH = DE^2.$$

Through A draw FG parallel to CD. Bisect EH in K, and draw KL at right angles to EH meeting FG in L.



With L as centre, and LE as radius, describe a circle cutting GF in G, F.

Join EF, EG, and from A draw AM, AN parallel to EF, EG respectively.

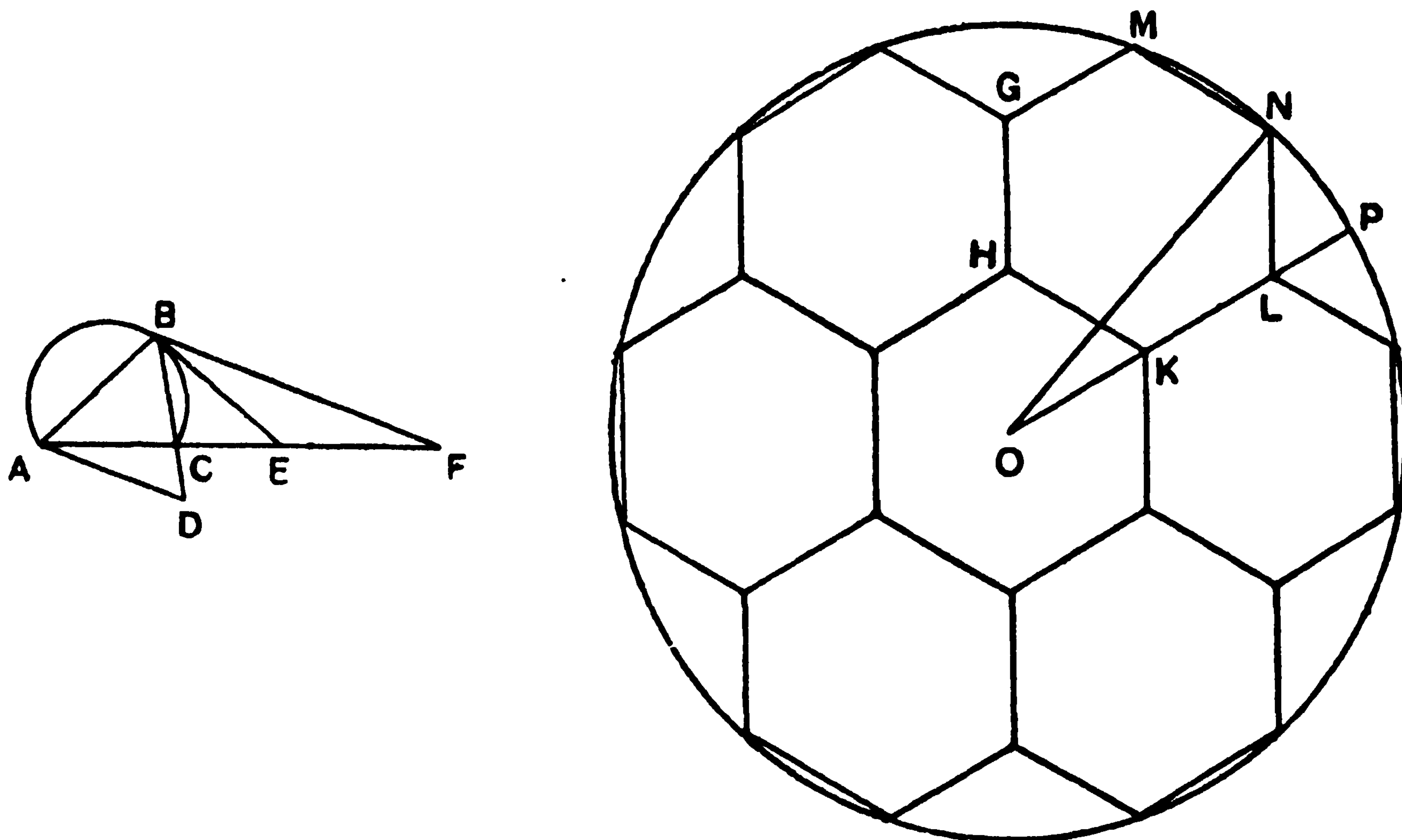
Take points P, R on EG, EF such that

$$EP^2 = GE.EM, \text{ and } ER^2 = FE.EN.$$

Then EP is half the major axis, and ER half the minor axis. Pappus omits the proof.

Problem of seven hexagons in a circle.

Prop. 19 (chap. 23) is a curious problem. To inscribe seven equal regular hexagons in a circle in such a way that one



is about the centre of the circle, while six others stand on its sides and have the opposite sides in each case placed as chords in the circle.

Suppose $GHKLNM$ to be the hexagon so described on HK, a side of the inner hexagon; OKL will then be a straight line. Produce OL to meet the circle in P.

Then $OK = KL = LN$. Therefore, in the triangle OLN, $OL = 2LN$, while the included angle OLN ($= 120°$) is also given. Therefore the triangle is given in species; therefore the ratio $ON : NL$ is given, and, since ON is given, the side NL of each of the hexagons is given.

Pappus gives the auxiliary construction thus. Let AF be taken equal to the radius OP. Let $AC = \frac{1}{3}AF$, and on AC as base describe a segment of a circle containing an angle of 60°. Take CE equal to $\frac{4}{5}AC$, and draw EB to touch the circle at B.

Then he proves that, if we join AB, AB is equal to the length of the side of the hexagon required.

Produce BC to D so that $BD = BA$, and join DA. ABD is then equilateral.

Since EB is a tangent to the segment, $AE.EC = EB^2$ or $AE:EB = EB:EC$, and the triangles EAB, EBC are similar.

Therefore $BA^2:BC^2 = AE^2:EB^2 = AE:EC = 9:4$;

and $BC = \frac{2}{3}BA = \frac{2}{3}BD$, so that $BC = 2CD$.

But $CF = 2CA$; therefore $AC:CF = DC:CB$, and AD, BF are parallel.

Therefore $BF:AD = BC:CD = 2:1$, so that

$$BF = 2AD = 2AB.$$

Also $\angle FBC = \angle BDA = 60°$, so that $\angle ABF = 120°$, and the triangle ABF is therefore equal and similar to the required triangle NLO.

Construction of toothed wheels and indented screws.

The rest of the Book is devoted to the construction (1) of toothed wheels with a given number of teeth equal to those of a given wheel, (2) of a cylindrical helix, the *cochlias*, indented so as to work on a toothed wheel. The text is evidently defective, and at the end an interpolator has inserted extracts about the mechanical powers from Heron's *Mechanics*.

XX

ALGEBRA: DIOPHANTUS OF ALEXANDRIA

Beginnings learnt from Egypt.

In algebra, as in geometry, the Greeks learnt the beginnings from the Egyptians. Familiarity on the part of the Greeks with Egyptian methods of calculation is well attested. (1) These methods are found in operation in the Heronian writings and collections. (2) Psellus in the letter published by Tannery in his edition of Diophantus speaks of 'the method of arithmetical calculations used by the Egyptians, by which problems in analysis are handled'; he adds details, doubtless taken from Anatolius, of the technical terms used for different kinds of numbers, including the powers of the unknown quantity. (3) The scholiast to Plato's *Charmides* 165 E says that 'parts of λογιστική, the science of calculation, are the so-called Greek and Egyptian methods in multiplications and divisions, and the additions and subtractions of fractions'. (4) Plato himself in the *Laws* 819 A–C says that free-born boys should, as is the practice in Egypt, learn, side by side with reading, simple mathematical calculations adapted to their age, which should be put into a form such as to combine amusement with instruction: problems about the distribution of, say, apples or garlands, the calculation of mixtures, and other questions arising in military or civil life.

'Hau'-calculations.

The Egyptian calculations here in point (apart from their method of writing and calculating in fractions, which, with the exception of $\frac{2}{3}$, were always decomposed and written as the sum of a diminishing series of aliquot parts or submultiples) are the *hau*-calculations. *Hau*, meaning a *heap*, is the term denoting the unknown quantity, and the calculations

in terms of it are equivalent to the solutions of simple equations with one unknown quantity. Examples from the Papyrus Rhind correspond to the following equations:

$$\tfrac{1}{7}x + x = 19,$$

$$\tfrac{2}{3}x + \tfrac{1}{2}x + \tfrac{1}{7}x + x = 33,$$

$$(x + \tfrac{2}{3}x) - \tfrac{1}{3}(x + \tfrac{2}{3}x) = 10.$$

The Egyptians anticipated, though only in an elementary form, a favourite method of Diophantus, that of the 'false supposition' or 'regula falsi'. An arbitrary assumption is made as to the value of the unknown, and the true value is afterwards found by a comparison of the result of substituting the wrong value in the original expression with the actual data. Two examples may be given. The first, from the Papyrus Rhind, is the problem of dividing 100 loaves among five persons in such a way that the shares are in arithmetical progression, and one-seventh of the sum of the first three shares is equal to the sum of the other two. If $a+4d$, $a+3d$, $a+2d$, $a+d$, a be the shares, then

$$3a + 9d = 7(2a + d),$$

or

$$d = 5\tfrac{1}{2}a.$$

Ahmes says, without any explanation, 'make the difference, as it is, $5\frac{1}{2}$', and then, assuming $a = 1$, writes the series 23, $17\frac{1}{2}$, 12, $6\frac{1}{2}$, 1. The addition of these gives 60, and 100 is $1\frac{2}{3}$ times 60. Ahmes says simply 'multiply $1\frac{2}{3}$ times' and thus gets the correct values $38\frac{1}{3}$, $29\frac{1}{6}$, 20, $10\frac{2}{3}\frac{1}{6}$, $1\frac{2}{3}$.

The second example (taken from the Berlin Papyrus 6619) is the solution of the equations

$$x^2 + y^2 = 100,$$

$$x : y = 1 : \tfrac{3}{4}, \text{ or } y = \tfrac{3}{4}x.$$

x is first assumed to be 1, and $x^2 + y^2$ is thus found to be $\frac{25}{16}$. In order to make 100, $\frac{25}{16}$ has to be multiplied by 64 or 8^2. The true value of x is therefore 8 times 1, or 8.

Arithmetical epigrams in the Greek Anthology.

The simple equations solved in the Papyrus Rhind are just the kind of equations of which we find many examples in the

arithmetical epigrams contained in the Greek Anthology. Most of these appear under the name of Metrodorus, a grammarian, probably of the time of the Emperors Anastasius I (A.D. 491–518) and Justin I (A.D. 518–27). They were obviously only collected by Metrodorus, from ancient as well as more recent sources. Many of the epigrams (46 in number) lead to simple equations, and several of them are problems of dividing a number of apples or nuts among a certain number of persons, that is to say, the very type of problem mentioned by Plato. For example, a number of apples has to be determined such that, if four persons out of six receive one-third, one-eighth, one-fourth and one-fifth respectively of the whole number, while the fifth person receives 10 apples, there is one apple left over for the sixth person, i.e.

$$\tfrac{1}{3}x+\tfrac{1}{8}x+\tfrac{1}{4}x+\tfrac{1}{5}x+10+1 = x.$$

Just as Plato alludes to bowls (φιάλαι) of different metals, there are problems in which the weights of bowls have to be found. We are thus enabled to understand the allusions of Proclus and the scholiast on *Charmides* 165 E to μηλῖται and φιαλῖται ἀριθμοί, 'numbers of apples or of bowls'. It is evident from Plato's allusions that the origin of such simple algebraical problems dates back, at least, to the fifth century B.C.

The following is a classification of the problems in the *Anthology*. (1) Twenty-three are simple equations in one unknown and of the type shown above; one of these is an epigram on the age of Diophantus and certain incidents of his life (xiv. 126). (2) Twelve are easy simultaneous equations with two unknowns, like Dioph. I. 6; they can of course be reduced to a simple equation with one unknown by means of an easy elimination. One other (xiv. 51) gives simultaneous equations in three unknowns

$$x = y+\tfrac{1}{3}z,\quad y = z+\tfrac{1}{3}x,\quad z = 10+\tfrac{1}{3}y,$$

and one (xiv. 49) gives four equations in four unknowns,

$$x+y = 40,\quad x+z = 45,\quad x+u = 36,\quad x+y+z+u = 60.$$

With these may be compared Dioph. I. 16–21, as well as the general solution of any number of simultaneous linear equa-

tions of this type with the same number of unknown quantities which was given by Thymaridas, an early Pythagorean, and was called the ἐπάνθημα, 'flower' or 'bloom' of Thymaridas (see vol. i, pp. 94–6). (3) Six more are problems of the usual type about the filling and emptying of vessels by pipes; e.g. (xiv. 130) one pipe fills the vessel in one day, a second in two and a third in three; how long will all three running together take to fill it? Another about brickmakers (xiv. 136) is of the same sort.

Indeterminate equations of the first degree.

The Anthology contains (4) two *indeterminate* equations of the first degree which can be solved in positive integers in an infinite number of ways (xiv. 48, 144); the first is a distribution of apples, $3x$ in number, into parts satisfying the equation $x-3y = y$, where y is not less than 2; the second leads to three equations connecting four unknown quantities:

$$x+y = x_1+y_1,$$
$$x = 2y_1,$$
$$x_1 = 3y,$$

the general solution of which is $x = 4k$, $y = k$, $x_1 = 3k$, $y_1 = 2k$. These very equations, which, however, are made determinate by assuming that $x+y = x_1+y_1 = 100$, are solved in Dioph. I. 12.

Enough has been said to show that Diophantus was not the inventor of Algebra. Nor was he the first to solve indeterminate problems of the second degree.

Indeterminate equations of second degree before Diophantus.

Take first the problem (Dioph. II. 8) of dividing a square number into two squares, or of finding a right-angled triangle with sides in rational numbers. We have already seen that Pythagoras is credited with the discovery of a general formula for finding such triangles, namely,

$$n^2+\{\tfrac{1}{2}(n^2-1)\}^2 = \{\tfrac{1}{2}(n^2+1)\}^2,$$

where n is any odd number, and Plato with another formula of the same sort, namely $(2n)^2+(n^2-1)^2 = (n^2+1)^2$. Euclid (Lemma following X. 28) finds the following more general formula

$$m^2n^2p^2q^2 = \{\tfrac{1}{2}(mnp^2+mnq^2)\}^2 - \{\tfrac{1}{2}(mnp^2-mnq^2)\}^2.$$

The Pythagoreans too, as we have seen (vol. i, pp. 91–3), solved another indeterminate problem, discovering, by means of the series of 'side-' and 'diameter-numbers', any number of successive integral solutions of the equations

$$2x^2-y^2 = \pm 1.$$

Diophantus does not particularly mention this equation, but from the Lemma to VI. 15 it is clear that he knew how to find any number of solutions when one is known. Thus, seeing that $2x^2-1 = y^2$ is satisfied by $x = 1$, $y = 1$, he would put

$$2(1+x)^2-1 = \text{a square}$$
$$= (px-1)^2, \text{ say};$$

whence $$x = (4+2p)/(p^2-2).$$

Take the value $p = 2$, and we have $x = 4$, and $x+1 = 5$; in this case $2 \cdot 5^2-1 = 49 = 7^2$. Putting $x+5$ in place of x, we can find a still higher value, and so on.

Indeterminate equations in the Heronian collections.

Some further Greek examples of indeterminate analysis are now available. They come from the Constantinople manuscript (probably of the twelfth century) from which Schöne edited the *Metrica* of Heron; they have been published and translated by Heiberg, with comments by Zeuthen.[1] Two of the problems (thirteen in number) had been published in a less complete form in Hultsch's Heron (*Geëponicus*, 78, 79); the others are new.

I. The first problem is to find two rectangles such that the perimeter of the second is three times that of the first, and the area of the first is three times that of the second. The

[1] *Bibliotheca mathematica*, viii$_3$, 1907-8, pp. 118-34. See now *Geom.* 24. 1–13 in Heron, vol. iv (ed. Heiberg), pp. 414-26.

number 3 is of course only an illustration, and the problem is equivalent to the solution of the equations

$$\left.\begin{aligned}(1)\quad u+v &= n(x+y)\\ (2)\quad xy &= n.uv\end{aligned}\right\}.$$

The solution given in the text is equivalent to

$$\left.\begin{aligned}x &= 2n^3-1, & y &= 2n^3\\ u &= n(4n^3-2), & v &= n\end{aligned}\right\}.$$

Zeuthen suggests that the solution may have been obtained thus. As the problem is indeterminate, it would be natural to start with some hypothesis, e.g. to put $v = n$. It would follow from equation (1) that u is a multiple of n, say nz. We have then

$$x+y = 1+z,$$

while, by (2), $$xy = n^3z,$$

whence $$xy = n^3(x+y)-n^3,$$

or $$(x-n^3)(y-n^3) = n^3(n^3-1).$$

An obvious solution is

$$x-n^3 = n^3-1, \quad y-n^3 = n^3,$$

which gives $z = 2n^3-1+2n^3-1 = 4n^3-2$, so that

$$u = nz = n(4n^3-2).$$

II. The second is a similar problem about two rectangles, equivalent to the solution of the equations

$$\left.\begin{aligned}(1)\quad x+y &= u+v\\ (2)\quad xy &= n.uv\end{aligned}\right\},$$

and the solution given in the text is

$$x+y = u+v = n^3-1, \tag{3}$$

$$\left.\begin{aligned}u &= n-1, & v &= n(n^2-1)\\ x &= n^2-1, & y &= n^2(n-1)\end{aligned}\right\}. \tag{4}$$

In this case trial may have been made of the assumptions

$$v = nx, \quad y = n^2u,$$

when equation (1) would give

$$(n-1)x = (n^2-1)u,$$

a solution of which is $x = n^2-1$, $u = n-1$.

III. The fifth problem is interesting in one respect. We are asked to find a right-angled triangle (in rational numbers) with area of 5 feet. We are told to multiply 5 by some square *containing 6 as a factor*, e.g. 36. This makes 180, and this is the area of the triangle (9, 40, 41). Dividing each side by 6, we have the triangle required. The author, then, is aware that the area of a right-angled triangle with sides in whole numbers is divisible by 6. If we take the Euclidean formula for a right-angled triangle, making the sides $a \cdot mn$, $a \cdot \frac{1}{2}(m^2-n^2)$, $a \cdot \frac{1}{2}(m^2+n^2)$, where a is any number, and m, n are numbers which are both odd or both even, the area is

$$\tfrac{1}{4}mn(m-n)(m+n)a^2,$$

and, as a matter of fact, the number $mn(m-n)(m+n)$ is divisible by 24, as was proved later (for another purpose) by Leonardo of Pisa.

IV. The last four problems (10 to 13) are of great interest. They are different particular cases of one problem, that of finding a rational right-angled triangle such that the numerical sum of its area and its perimeter is a given number. The author's solution depends on the following formulae, where a, b are the perpendiculars, and c the hypotenuse, of a right-angled triangle, S its area, r the radius of the inscribed circle, and $s = \frac{1}{2}(a+b+c)$;

$$S = rs = \tfrac{1}{2}ab, \quad r+s = a+b, \quad c = s-r.$$

(The proof of these formulae by means of the usual figure, namely that used by Heron to prove the formula

$$S = \sqrt{\{s(s-a)(s-b)(s-c)\}},$$

is easy.)

Solving the first two equations, in order to find a and b, we have

$$\left.\begin{matrix} a \\ b \end{matrix}\right\} = \tfrac{1}{2}[r+s \mp \sqrt{\{(r+s)^2-8rs\}}],$$

which formula is actually used by the author for finding a

and b. The method employed is to take the sum of the area and the perimeter $S+2s$, separated into its two obvious factors $s(r+2)$, to put $s(r+2) = A$ (the given number), and then to separate A into suitable factors to which s and $r+2$ may be equated. They must obviously be such that sr, the area, is divisible by 6. To take the first example where $A = 280$: the possible factors are 2×140, 4×70, 5×56, 7×40, 8×35, 10×28, 14×20. The suitable factors in this case are $r+2 = 8$, $s = 35$, because r is then equal to 6, and rs is a multiple of 6.

The author then says that

$$a = \tfrac{1}{2}[6+35-\sqrt{\{(6+35)^2-8\,.\,6\,.\,35\}}] = \tfrac{1}{2}(41-1) = 20,$$

$$b = \tfrac{1}{2}(41+1) = 21,$$

$$c = 35-6 = 29.$$

The triangle is therefore (20, 21, 29) in this case. The triangles found in the other three cases, by the same method, are (9, 40, 41), (8, 15, 17) and (9, 12, 15).

Unfortunately there is no guide to the date of the problems just given. The probability is that the original formulation of the most important of the problems belongs to the period between Euclid and Diophantus. This supposition best agrees with the fact that the problems include nothing taken from the great collection in the *Arithmetica*. On the other hand, it is strange that none of the seven problems above mentioned is found in Diophantus. The five relating to rational right-angled triangles might well have been included by him; thus he finds rational right-angled triangles such that the area *plus* or *minus* one of the perpendiculars is a given number, but not the rational triangle which has a given area; and he finds rational right-angled triangles such that the area *plus* or *minus* the sum of *two* sides is a given number, but not the rational triangle such that the sum of the area and the *three* sides is a given number. The omitted problems might, it is true, have come in the lost Books; but, on the other hand, Book VI would have been the appropriate place for them.

The crowning example of a difficult indeterminate problem propounded before Diophantus's time is the Cattle-Problem attributed to Archimedes, described above (pp. 97-8).

Numerical solution of quadratic equations.

The *geometrical* algebra of the Greeks has been in evidence all through our history from the Pythagoreans downwards, and no more need be said of it here except that its arithmetical application was no new thing in Diophantus. It is probable, for example, that the solution of the quadratic equation, discovered first by geometry, was applied for the purpose of finding *numerical* values for the unknown as early as Euclid, if not earlier still. In Heron the numerical solution of equations is well established, so that Diophantus was not the first to treat equations algebraically. What he did was to take a step forward towards an algebraic *notation*.

The date of DIOPHANTUS can now be fixed with fair certainty. He was later than Hypsicles, from whom he quotes a definition of a polygonal number, and earlier than Theon of Alexandria, who has a quotation from Diophantus's definitions. The possible limits of date are therefore, say, 150 B.C. to A.D. 350. But the letter of Psellus already mentioned says that Anatolius (Bishop of Laodicea about A.D. 280) dedicated to Diophantus a concise treatise on the Egyptian method of reckoning; hence Diophantus must have been a contemporary, so that he probably flourished A.D. 250 or not much later.

An epigram in the Anthology gives some personal particulars: his boyhood lasted $\frac{1}{6}$th of his life; his beard grew after $\frac{1}{12}$th more; he married after $\frac{1}{7}$th more, and his son was born 5 years later; the son lived to half his father's age, and the father died 4 years after his son. Thus, if x was his age when he died,

$$\tfrac{1}{6}x + \tfrac{1}{12}x + \tfrac{1}{7}x + 5 + \tfrac{1}{2}x + 4 = x,$$

which gives $x = 84$.

Works of Diophantus.

The works on which the fame of Diophantus rests are:

(1) the *Arithmetica* (originally in thirteen Books),

(2) a tract *On Polygonal Numbers*.

Six Books only of the former and a fragment of the latter survive.

Allusions in the *Arithmetica* imply the existence of

(3) A collection of propositions under the title of *Porisms*; in three propositions (3, 5, 16) of Book V, Diophantus quotes as known certain propositions in the Theory of Numbers, prefixing to the statement of them the words 'We have it in the *Porisms* that...'

A scholium on a passage of Iamblichus, where Iamblichus cites a dictum of certain Pythagoreans about the unit being the dividing line (μεθόριον) between number and aliquot parts, says 'thus Diophantus in the *Moriastica*.... for he describes as "parts" the progression without limit in the direction of less than the unit'. The *Moriastica* may be a separate work by Diophantus giving rules for reckoning with fractions; but I do not feel sure that the reference may not simply be to the definitions at the beginning of the *Arithmetica*.

The *Arithmetica*.

The seven lost Books and their place.

None of the manuscripts which we possess contain more than six Books of the *Arithmetica*, the only variations being that some few divide the six Books into seven, while one or two give the fragment on Polygonal Numbers as VIII. The missing Books were evidently lost at a very early date. Tannery suggests that Hypatia's commentary extended only to the first six Books, and that she left untouched the remaining seven, which, partly as a consequence, were first forgotten and then lost (cf. the case of Apollonius's *Conics*, where the only Books which have survived in Greek, I–IV, are those on which Eutocius commented). There is no sign that even the Arabians ever possessed the missing Books. The *Fakhrī*, an algebraical treatise by Abū Bekr Muḥ. b. al-Ḥasan al-Karkhī (d. about 1029), contains a collection of problems in determinate and indeterminate analysis which not only show that their author had deeply studied Diophantus but in many cases are taken direct from the *Arithmetica*, sometimes with a change in constants; in the fourth section of the work,

between problems corresponding to problems in Dioph. II and III, are 25 problems not found in Diophantus, but internal evidence, and especially the admission of irrational results (which are always avoided by Diophantus), exclude the hypothesis that we have here one of the lost Books. Nor is there any sign that more of the work than we possess was known to Abū'l Wafā al-Būzjānī (A.D. 940–98) who wrote a 'commentary on the algebra of Diophantus', as well as a 'Book of proofs of propositions used by Diophantus in his work'. These facts again point to the conclusion that the lost Books were lost before the tenth century.

The old view of the place originally occupied by the lost seven Books is that of Nesselmann, who argued it with great ability.[1] According to him (1) much less of Diophantus is wanting than would naturally be supposed on the basis of the numerical proportion of 7 lost to 6 extant Books, (2) the missing portion came, not at the end, but in the middle of the work, and indeed mostly between the first and second Books. Nesselmann's general argument is that, if we carefully read the last four Books, from the third to the sixth, we shall find that Diophantus moves in a rigidly defined and limited circle of methods and artifices, and seems in fact to be at the end of his resources. As regards the possible contents of the lost portion on this hypothesis, Nesselmann can only point to (1) topics which we should expect to find treated, either because foreshadowed by the author himself or as necessary for the elucidation or completion of the whole subject, (2) the *Porisms*; under head (1) come, (*a*) determinate equations of the second degree, and (*b*) indeterminate equations of the first degree. Diophantus does indeed promise to show how to solve the general quadratic $ax^2 \pm bx \pm c = 0$ so far as it has rational and positive solutions; the suitable place for this would have been between Books I and II. But there is nothing whatever to show that indeterminate equations of the first degree formed part of the writer's plan. Hence Nesselmann is far from accounting for the contents of seven whole Books; and he is forced to the conjecture that the six Books may originally have been divided into even more than seven Books; there is, however, no evidence to support this.

[1] Nesselmann, *Algebra der Griechen*, pp. 264–73.

Relation of the 'Porisms' to the Arithmetica.

Did the *Porisms* form part of the *Arithmetica* in its original form? The phrase in which they are alluded to, and which occurs three times, 'We have it in the *Porisms* that...' suggests that they were a distinct collection of propositions concerning the properties of certain numbers, their divisibility into a certain number of squares, and so on; and it is possible that it was from the same collection that Diophantus took the numerous other propositions which he assumes, explicitly or implicitly. If the collection was part of the *Arithmetica*, it would be strange to quote the propositions under a separate title 'The Porisms' when it would have been more natural to refer to particular propositions of particular Books, and more natural still to say *τοῦτο γὰρ προδέδεικται*, or some such phrase, 'for this has been proved', without any reference to the particular place where the proof occurred. The expression 'We have it in the *Porisms*' (in the plural) would be still more inappropriate if the *Porisms* had been, as Tannery supposed, not collected together as one or more Books of the *Arithmetica*, but scattered about in the work as *corollaries* to particular propositions. Hence I agree with the view of Hultsch that the *Porisms* were not included in the *Arithmetica* at all, but formed a separate work.

If this is right, we cannot any longer hold to the view of Nesselmann that the lost Books were in the middle and not at the end of the treatise; indeed Tannery produces strong arguments in favour of the contrary view, that it is the last and most difficult Books which are lost. He replies first to the assumption that Diophantus could not have proceeded to problems more difficult than those of Book V. 'If the fifth or the sixth Book of the *Arithmetica* had been lost, who, pray, among us would have believed that such problems had ever been attempted by the Greeks? It would be the greatest error, in any case in which a thing cannot clearly be proved to have been unknown to all the ancients, to maintain that it could not have been known to some Greek mathematician. If we do not know to what lengths Archimedes brought the theory of numbers (to say nothing of other things), let us admit our ignorance. But, between the famous problem of the

cattle and the most difficult of Diophantus's problems, is there not a sufficient gap to require seven Books to fill it? And, without attributing to the ancients what modern mathematicians have discovered, may not a number of the things attributed to the Indians and Arabs have been drawn from Greek sources? May not the same be said of a problem solved by Leonardo of Pisa, which is very similar to those of Diophantus but is not now to be found in the *Arithmetica*? In fact, it may fairly be said that, when Chasles made his reasonably probable restitution of the *Porisms* of Euclid, he, notwithstanding that he had Pappus's lemmas to help him, undertook a more difficult task than he would have undertaken if he had attempted to fill up seven Diophantine Books with numerical problems which the Greeks may reasonably be supposed to have solved.'[1]

It is not so easy to agree with Tannery's view of the relation of the treatise *On Polygonal Numbers* to the *Arithmetica*. According to him, just as Serenus's treatise on the sections of cones and cylinders was added to the mutilated *Conics* of Apollonius consisting of four Books only, in order to make up a convenient volume, so the tract on Polygonal Numbers was added to the remains of the *Arithmetica*, though forming no part of the larger work.[2] Thus Tannery would seem to deny the genuineness of the whole tract on Polygonal Numbers, though in his text he only signalizes the portion beginning with the enunciation of the problem 'Given a number, to find in how many ways it can be a polygonal number' as 'a vain attempt by a commentator' to solve this problem. Hultsch, on the other hand, thinks that we may conclude that Diophantus really solved the problem. The tract begins, like Book I of the *Arithmetica*, with definitions and preliminary propositions; then comes the difficult problem quoted, the discussion of which breaks off in our text after a few pages, and to these it would be easy to tack on a great variety of other problems.

The name of Diophantus was used, as were the names of Euclid, Archimedes and Heron in their turn, for the purpose of palming off the compilations of much later authors.

[1] Diophantus, ed. Tannery, vol. ii, p. xx.
[2] *Ib.*, p. xviii.

Tannery includes in his edition three fragments under the heading 'Diophantus Pseudepigraphus'. The first, which is not 'from the Arithmetic of Diophantus' as its heading states, is worth notice as containing some particulars of one of 'two methods of finding the square root of any square number'; we are told to begin by writing the number 'according to the arrangement of the Indian method', i.e. in the Indian numerical notation which reached us through the Arabs. The second fragment is the work edited by C. Henry in 1879 as *Opusculum de multiplicatione et divisione sexagesimalibus Diophanto vel Pappo attribuendum.* The third, beginning with *Διοφάντου ἐπιπεδομετρικά* is a Byzantine compilation from later reproductions of the *γεωμετρούμενα* and *στερεομετρούμενα* of Heron. Not one of the three fragments has anything to do with Diophantus.

Commentators from Hypatia downwards.

The first commentator on Diophantus of whom we hear is Hypatia, the daughter of Theon of Alexandria; she was murdered by Christian fanatics in A.D. 415. I have already mentioned the attractive hypothesis of Tannery that Hypatia's commentary extended only to our six Books, and that this accounts for their survival when the rest were lost. It is possible that the remarks of Psellus (eleventh century) at the beginning of his letter about Diophantus, Anatolius and the Egyptian method of arithmetical reckoning were taken from Hypatia's commentary.

Georgius Pachymeres (1240 to about 1310) wrote in Greek a paraphrase of at least a portion of Diophantus. Sections 25–44 of this commentary relating to Book I, Def. 1 to Prop 11, survive. Maximus Planudes (about 1260–1310) also wrote a systematic commentary on Books I, II. Arabian commentators were Abū'l Wafā al-Būzjānī (940–98), Qusṭā b. Lūqā al-Ba'labakkī (d. about 912) and probably Ibn al-Haitham (about 965–1039).

Translations and editions.

To Regiomontanus belongs the credit of being the first to call attention to the work of Diophantus as being extant in

Greek. In an *Oratio* delivered at the end of 1463 as an introduction to a course of lectures on astronomy which he gave at Padua in 1463–4 he observed: 'No one has yet translated from the Greek into Latin the fine thirteen Books of Diophantus, in which the very flower of the whole of arithmetic lies hid, the *ars rei et census* which to-day they call by the Arabic name of Algebra.' Again, in a letter dated February 5, 1464, to Bianchini, he writes that he has found at Venice 'Diofantus, a Greek arithmetician who has not yet been translated into Latin'. Rafael Bombelli was the first to find a manuscript in the Vatican and to conceive the idea of publishing the work; this was towards 1570, and, with Antonio Maria Pazzi, he translated five Books out of the seven into which the manuscript was divided. The translation was not published, but Bombelli took all the problems of the first four Books and some of those of the fifth and embodied them in his *Algebra* (1572), interspersing them with his own problems.

The next writer on Diophantus was Wilhelm Holzmann, who called himself Xylander, and who with extraordinary industry and care produced a very meritorious Latin translation with commentary (1575). Xylander was an enthusiast for Diophantus, and his preface and notes are often delightful reading. Unfortunately the book is now very rare. The standard edition of Diophantus till recent years was that of Bachet, who in 1621 published for the first time the Greek text with Latin translation and notes. A second edition (1670) was carelessly printed and is untrustworthy as regards the text; on the other hand it contained the epoch-making notes of Fermat; the editor was S. Fermat, his son. The great blot on the work of Bachet is his attitude to Xylander, to whose translation he owed more than he was willing to avow. Unfortunately neither Bachet nor Xylander was able to use the best manuscripts; that used by Bachet was Parisinus 2379 (of the middle of the sixteenth century), with the help of a transcription of part of a Vatican MS. (Vat. gr. 304 of the sixteenth century), while Xylander's manuscript was the Wolfenbüttel MS. Guelferbytanus Gudianus 1 (fifteenth century). The best and most ancient manuscript is that of Madrid (Matritensis 48 of the thirteenth century) which was

unfortunately spoiled by corrections made, especially in Books I, II, from some manuscript of the 'Planudean' class; where this is the case recourse must be had to Vat. gr. 191 which was copied from it before it had suffered the general alteration referred to: these are the first two of the manuscripts used by Tannery in his definitive edition of the Greek text (Teubner, 1893, 1895).

Other editors can only be shortly enumerated. In 1585 Simon Stevin published a French version of the first four Books, based on Xylander. Albert Girard added the fifth and sixth Books, the complete edition appearing in 1625. German translations were brought out by Otto Schulz in 1822 and by G. Wertheim in 1890. Poselger translated the fragment on Polygonal Numbers in 1810. All these translations depended on the text of Bachet.

A reproduction of Diophantus in modern notation with introduction and notes by the present writer (second edition 1910) is based on the text of Tannery and may claim to be the most complete and up-to-date edition.

My account of the *Arithmetica* of Diophantus will be most conveniently arranged under three main headings (1) the notation and definitions, (2) the principal methods employed, so far as they can be generally stated, (3) the nature of the contents, including the assumed Porisms, with indications of the devices by which the problems are solved.

Notation and definitions.

In his work *Die Algebra der Griechen* Nesselmann distinguishes three stages in the evolution of algebra. (1) The first stage he calls 'Rhetorical Algebra' or reckoning by means of complete words. The characteristic of this stage is the absolute want of all symbols, the whole of the calculation being carried on by means of complete words and forming in fact continuous prose. This first stage is represented by such writers as Iamblichus, all Arabian and Persian algebraists, and the oldest Italian algebraists and their followers, including Regiomontanus. (2) The second stage Nesselmann calls the 'Syncopated Algebra', essentially like the first as regards

literary style, but marked by the use of certain abbreviational symbols for constantly recurring quantities and operations. To this stage belong Diophantus and, after him, all the later Europeans until about the middle of the seventeenth century (with the exception of Vieta, who was the first to establish, under the name of *Logistica speciosa*, as distinct from *Logistica numerosa*, a regular system of reckoning with letters denoting magnitudes as well as numbers). (3) To the third stage Nesselmann gives the name of 'Symbolic Algebra', which uses a complete system of notation by signs having no visible connexion with the words or things which they represent, a complete language of symbols, which entirely supplants the 'rhetorical' system, it being possible to work out a solution without using a single word of ordinary language with the exception of a connecting word or two here and there used for clearness' sake.

Sign for the unknown ($= x$), *and its origin.*

Diophantus's system of notation then is merely abbreviational. We will consider first the representation of the unknown quantity (our x). Diophantus defines the unknown quantity as '*containing an indeterminate or undefined multitude of units*' (πλῆθος μονάδων ἀόριστον), adding that it is called ἀριθμός, i.e. *number* simply, and is denoted by a certain sign. This sign is then used all through the book. In the earliest (the Madrid) MS. the sign takes the form ꟸ, in Marcianus 308 it appears as S. In the printed editions of Diophantus before Tannery's it was represented by the final sigma with an accent, ς′, which is sufficiently like the second of the two forms. Where the symbol takes the place of inflected forms ἀριθμόν, ἀριθμοῦ, &c., the termination was put above and to the right of the sign like an exponent, e.g. $ς^{ν}$ for ἀριθμόν as $τ^{ν}$ for τόν, $ς^{οῦ}$ for ἀριθμοῦ; the symbol was, in addition, doubled in the plural cases, thus $ςς^{οί}$, $ςς^{ούς}$, &c. The coefficient is expressed by putting the required Greek numeral immediately after it; thus $ςς^{οἱ}$ ια = 11 ἀριθμοί, equivalent to $11x$, ς′α $= x$, and so on. Tannery gives reasons for thinking that in the archetype the case-endings did not appear, and

that the sign was not duplicated for the plural, although such duplication was the practice of the Byzantines. That the sign was merely an abbreviation for the word *ἀριθμός* and no algebraical symbol is shown by the fact that it occurs in the manuscripts for *ἀριθμός* in the ordinary sense as well as for *ἀριθμός* in the technical sense of the unknown quantity. Nor is it confined to Diophantus. It appears in more or less similar forms in the manuscripts of other Greek mathematicians, e.g. in the Bodleian MS. of Euclid (D'Orville 301) of the ninth century (in the forms ϛ̊ ϛϛ, or as a curved line similar to the abbreviation for *καί*), in the manuscripts of the *Sand-reckoner* of Archimedes (in a form approximating to ϛ), where again there is confusion caused by the similarity of the signs for *ἀριθμός* and *καί*, in a manuscript of the *Geodaesia* included in the Heronian collections edited by Hultsch (where it appears in various forms resembling sometimes ζ, sometimes ρ, sometimes ο, and once ξ, with case-endings superposed) and in a manuscript of Theon of Smyrna.

What is the origin of the sign? It is certainly not the final sigma, as is proved by several of the forms which it takes. I found that in the Bodleian manuscript of Diophantus it is written in the form ϛ, larger than and quite unlike the final sigma. This form, combined with the fact that in one place Xylander's manuscript read *αρ* for the full word, suggested to me that the sign might be a simple contraction of the first two letters of *ἀριθμός*. This seemed to be confirmed by Gardthausen's mention of a contraction for *αρ*, in the form ϙ occurring in a papyrus of A.D. 154, since the transition to the form found in the manuscripts of Diophantus might easily have been made through an intermediate form ϙ. The loss of the downward stroke, or of the loop, would give a close approximation to the forms which we know. This hypothesis as to the origin of the sign has not, so far as I know, been improved upon. It has the immense advantage that it makes the sign for *ἀριθμός* similar to the signs for the powers of the unknown, e.g. Δ^{Υ} for *δύναμις*, K^{Υ} for *κύβος*, and to the sign $\overset{o}{M}$ for the unit, the sole difference being that the two letters coalesce into one instead of being separate.

Signs for the powers of the unknown and their reciprocals.

The powers of the unknown, corresponding to our $x^2, x^3 \dots x^6$, are defined and denoted as follows:

x^2 is δύναμις and is denoted by Δ^Y,

x^3 „ κύβος „ „ „ K^Y,

x^4 „ δυναμοδύναμις „ „ $\Delta^Y\Delta$,

x^5 „ δυναμόκυβος „ „ ΔK^Y,

x^6 „ κυβόκυβος „ „ $K^Y K$.

Beyond the sixth power Diophantus does not go. It should be noted that, while the terms from κύβος onwards may be used for the powers of any ordinary known number as well as for the powers of the unknown, δύναμις is restricted to the square of the unknown; wherever a particular square number is spoken of, the term is τετράγωνος ἀριθμός. The term δυναμοδύναμις occurs once in another author, namely in the *Metrica* of Heron,[1] where it is used for the fourth power of the side of a triangle.

Diophantus has also terms and signs for the reciprocals of the various powers of the unknown, i.e. for $1/x$, $1/x^2$.... As an aliquot part was ordinarily denoted by the corresponding numeral sign with an accent, e.g. $\gamma' = \frac{1}{3}$, $\iota\alpha' = \frac{1}{11}$, Diophantus has a mark appended to the symbols for $x, x^2 \dots$ to denote the reciprocals; this, which is used for aliquot parts as well, is printed by Tannery thus, $^{\chi}$. With Diophantus then

ἀριθμοστόν, denoted by s^{χ}, is equivalent to $1/x$,

δυναμοστόν, „ $\Delta^{Y\chi}$ „ „ $1/x^2$,

and so on.

The coefficient of the term in $x, x^2 \dots$ or $1/x, 1/x^2 \dots$ is expressed by the ordinary numeral immediately following, e.g. $\Delta K^Y \kappa\varsigma = 26x^5$, $\Delta^{Y\chi}\,\sigma\nu = 250/x^2$.

Diophantus does not need any signs for the operations of multiplication and division. Addition is indicated by mere juxtaposition; thus $K^Y \alpha\ \Delta^Y \iota\gamma\ s\ \epsilon$ corresponds to $x^3 + 13x^2 + 5x$.

[1] Heron, *Metrica*, p. 48. 11, 19, Schöne.

When there are units in addition, the units are indicated by the abbreviation $\overset{o}{M}$; thus $K^{Y}\alpha\ \Delta^{Y}\iota\gamma\ \varsigma\ \epsilon\ \overset{o}{M}\ \beta$ corresponds to $x^3 + 13x^2 + 5x + 2$.

The sign (⋔) *for minus and its meaning.*

For subtraction alone is a sign used. The full term for *wanting* is λεῖψις, as opposed to ὕπαρξις, a *forthcoming*, which denotes a *positive* term. The symbol used to indicate a *wanting*, corresponding to our sign for *minus*, is ⋔, which is described in the text as a 'ψ turned downwards and truncated' (Ψ ἐλλιπὲς κάτω νεῦον). The description is evidently interpolated, and it is now certain that the sign has nothing to do with ψ. Nor is it confined to Diophantus, for it appears in practically the same form in Heron's *Metrica*,[1] where in one place the reading of the manuscript is μονάδων οδ ⋔ ιʹδʹ, $74 - \frac{1}{14}$. In the manuscripts of Diophantus, when the sign is resolved by writing the full word instead of it, it is generally resolved into λείψει, the dative of λεῖψις, but in other places the symbol is used instead of parts of the verb λείπειν, namely λιπών or λείψας and once even λίπωσι; sometimes λείψει in the manuscripts is followed by the *accusative*, which shows that in these cases the sign was wrongly resolved. It is therefore a question whether Diophantus himself ever used the dative λείψει for *minus* at all. The use is certainly foreign to classical Greek. Ptolemy has in two places λεῖψαν and λείπουσαν respectively followed, properly, by the accusative, and in one case he has τὸ ἀπὸ τῆς ΓΛ λειφθὲν ὑπὸ τοῦ ἀπὸ τῆς ΖΓ (where the meaning is $Z\Gamma^2 - \Gamma\Lambda^2$). Hence Heron would probably have written a participle where the ⋔ occurs in the expression quoted above, say μονάδων οδ λειψασῶν τεσσαρακαιδέκατον. On the whole, therefore, it is probable that in Diophantus, and wherever else it occurred, ⋔ is a compendium for the root of the verb λείπειν, in fact a Λ with I placed in the middle (cf. Χ, an abbreviation for τάλαντον). This is the hypothesis which I put forward in 1885, and it seems to be confirmed by the fresh evidence now available as shown above.

[1] Heron, *Metrica*, p. 156. 8, 10.

Attached to the definition of *minus* is the statement that 'a *wanting* (i.e. a *minus*) multiplied by a *wanting* makes a *forthcoming* (i.e. a *plus*); and a *wanting* (a *minus*) multiplied by a *forthcoming* (a *plus*) makes a *wanting* (a *minus*)'.

Since Diophantus uses no sign for *plus*, he has to put all the positive terms in an expression together and write all the negative terms together after the sign for *minus*; e.g. for x^3-5x^2+8x-1 he necessarily writes Κ^Υ α ς η ⋔ Δ^Υ ε M̊ α.

The Diophantine notation for fractions as well as for large numbers has been fully explained with many illustrations in Chapter II above. It is only necessary to add here that, when the numerator and denominator consist of composite expressions in terms of the unknown and its powers, he puts the numerator first followed by ἐν μορίῳ or μορίου and the denominator.

Thus Δ^Υ ξ M̊ ͵βφκ ἐν μορίῳ Δ^ΥΔ α M̊ ϡ ⋔ Δ^Υ ξ

$= (60x^2+2520)/(x^4+900-60x^2)$, [VI. 12]

and Δ^Υ ιε ⋔ M̊ λς ἐν μορίῳ Δ^ΥΔ α M̊ λς ⋔ Δ^Υ ιβ

$= (15x^2-36)/(x^4+36-12x^2)$ [VI. 14].

For a *term* in an algebraical expression, i.e. a power of x with a certain coefficient, and the term containing a certain number of units, Diophantus uses the word εἶδος, 'species', which primarily means the particular power of the variable without the coefficient. At the end of the definitions he gives directions for simplifying equations until each side contains positive terms only, by the addition or subtraction of coefficients, and by getting rid of the negative terms (which is done by adding the necessary quantities to both sides); the object, he says, is to reduce the equation until one term only is left on each side; 'but', he adds, 'I will show you later how, in the case also where two terms are left equal to one term, such a problem is solved'. We find in fact that, when he has to solve a quadratic equation, he endeavours by means of suitable assumptions to reduce it either to a simple equation or a *pure* quadratic. The solution of the mixed quadratic

in three terms is clearly assumed in several places of the *Arithmetica*, but Diophantus never gives the necessary explanation of this case as promised in the preface.

Before leaving the notation of Diophantus, we may observe that the form of it limits him to the use of one unknown at a time. The disadvantage is obvious. For example, where we can begin with any number of unknown quantities and gradually eliminate all but one, Diophantus has practically to perform his eliminations beforehand so as to express every quantity occurring in the problem in terms of only one unknown. When he handles problems which are by nature indeterminate and would lead in our notation to an indeterminate equation containing two or three unknowns, he has to assume for one or other of these some particular number arbitrarily chosen, the effect being to make the problem determinate. However, in doing so, Diophantus is careful to say that we may for such and such a quantity put any number whatever, say such and such a number; there is therefore (as a rule) no real loss of generality. The particular devices by which he contrives to express all his unknowns in terms of one unknown are extraordinarily various and clever. He can, of course, use the same variable ϛ in the same problem with different significations *successively*, as when it is necessary in the course of the problem to solve a subsidiary problem in order to enable him to make the coefficients of the different terms of expressions in x such as will answer his purpose and enable the original problem to be solved. There are, however, two cases, II. 28, 29, where for the proper working-out of the problem two unknowns are imperatively necessary. We should of course use x and y; Diophantus calls the first ϛ as usual; the second, for want of a term, he agrees to call in the first instance '*one unit*', i.e. 1. Then later, having completed the part of the solution necessary to find x, he substitutes its value and uses ϛ over again for what he had originally called 1. That is, he has to put his finger on the place to which the 1 has passed, so as to substitute ϛ for it. This is a *tour de force* in the particular cases, and would be difficult or impossible in more complicated problems.

The methods of Diophantus.

It should be premised that Diophantus will have in his solutions no numbers whatever except 'rational' numbers; he admits fractional solutions as well as integral, but he excludes not only surds and imaginary quantities but also negative quantities. Of a negative quantity *per se*, i.e. without some greater positive quantity to subtract it from, he had apparently no conception. Such equations then as lead to imaginary or negative roots he regards as useless for his purpose; the solution is in these cases ἀδύνατος, impossible. So we find him (V. 2) describing the equation $4 = 4x + 20$ as ἄτοπος, absurd, because it would give $x = -4$. He does, it is true, make occasional use of a quadratic which would give a root which is positive but a surd, but only for the purpose of obtaining limits to the root which are integers or numerical fractions; he never uses or tries to express the actual root of such an equation. When therefore he arrives in the course of solution at an equation which would give an 'irrational' result, he retraces his steps, finds out how his equation has arisen, and how he may, by altering the previous work, substitute for it another which shall give a rational result. This gives rise in general to a subsidiary problem the solution of which ensures a rational result for the problem itself.

It is difficult to give a complete account of Diophantus's methods without setting out the whole book, so great is the variety of devices and artifices employed in the different problems. There are, however, a few general methods which do admit of differentiation and description, and these we proceed to set out under subjects.

I. Diophantus's treatment of equations.

(A) *Determinate equations.*

Diophantus solved without difficulty determinate equations of the first and second degrees; of a cubic we find only one example in the *Arithmetica*, and that is a very special case.

(1) *Pure determinate equations.*

Diophantus gives a general rule for this case without regard to degree. We have to take like from like on both sides of an

equation and neutralize negative terms by adding to both sides, then take like from like again, until we have one term left equal to one term. After these operations have been performed, the equation (after dividing out, if both sides contain a power of x, by the lesser power) reduces to $Ax^m = B$, and is considered solved. Diophantus regards this as giving one root only, excluding any negative value as 'impossible'. No equation of the kind is admitted which does not give a 'rational' value, integral or fractional. The value $x = 0$ is ignored in the case where the degree of the equation is reduced by dividing out by any power of x.

(2) *Mixed quadratic equations.*

Diophantus never gives the explanation of the method of solution which he promises in the preface. That he had a definite method like that used in the Geometry of Heron is proved by clear verbal explanations in different propositions. As he requires the equation to be in the form of two positive terms being equal to one positive term, the possible forms for Diophantus are

$$(a)\ mx^2+px = q,\quad (b)\ mx^2 = px+q,\quad (c)\ mx^2+q = px.$$

It does not appear that Diophantus divided by m in order to make the first term a square; rather he multiplied by m for this purpose. It is clear that he stated the roots in the above cases in a form equivalent to

$$(a)\ \frac{-\frac{1}{2}p+\sqrt{(\frac{1}{4}p^2+mq)}}{m},\quad (b)\ \frac{\frac{1}{2}p+\sqrt{(\frac{1}{4}p^2+mq)}}{m},$$

$$(c)\ \frac{\frac{1}{2}p+\sqrt{(\frac{1}{4}p^2-mq)}}{m}.$$

The explanations which show this are to be found in VI. 6, in IV. 39 and 31, and in V. 10 and VI. 22 respectively. For example in V. 10 he has the equation $17x^2+17 < 72x$, and he says 'Multiply half the coefficient of x into itself and we have 1296; subtract the product of the coefficient of x^2 and the term in units, or 289. The remainder is 1007, the square root of which is not greater than 31. Add half the coefficient of x and the result is not greater than 67. Divide by the coefficient of x^2, and x is not greater than $\frac{67}{17}$.' In IV. 39 he has the

equation $2x^2 > 6x + 18$ and says, 'To solve this, take the square of half the coefficient of x, i.e. 9, and the product of the unit-term and the coefficient of x^2, i.e. 36. Adding, we have 45, the square root of which is not less than 7. Add half the coefficient of x [and divide by the coefficient of x^2]; whence x is not less than 5.' In these cases it will be observed that 31 and 7 are not accurate limits, but are the nearest integral limits which will serve his purpose.

Diophantus always uses the positive sign with the radical, and there has been much discussion as to whether he knew that a quadratic equation has *two* roots. The evidence of the text is inconclusive because his only object, in every case, is to get one solution; in some cases the other root would be negative, and would therefore naturally be ignored as 'absurd' or 'impossible'. In yet other cases where the second root is possible it can be shown to be useless from Diophantus's point of view. For my part, I find it difficult or impossible to believe that Diophantus was unaware of the existence of two real roots in such cases. It is so obvious from the geometrical form of solution based on Eucl. II. 5, 6 and that contained in Eucl. VI. 27–9; the construction of VI. 28, too, corresponds in fact to the *negative* sign before the radical in the case of the particular equation there solved, while a quite obvious and slight variation of the construction would give the solution corresponding to the *positive* sign.

The following particular cases of quadratics occurring in the *Arithmetica* may be quoted, with the results stated by Diophantus.

$x^2 = 4x - 4$; therefore $x = 2$. (IV. 22)

$325x^2 = 3x + 18$; $x = \frac{78}{325}$ or $\frac{6}{25}$. (IV. 31)

$84x^2 + 7x = 7$; $x = \frac{1}{4}$. (VI. 6)

$84x^2 - 7x = 7$; $x = \frac{1}{3}$. (VI. 7)

$630x^2 - 73x = 6$; $x = \frac{6}{35}$. (VI. 9)

$630x^2 + 73x = 6$; x is rational. (VI. 8)

$5x < x^2 - 60 < 8x$; x not < 11 and not > 12. (V. 30)

$17x^2 + 17 < 72x < 19x^2 + 19$; x not $> \frac{67}{17}$ and not $< \frac{66}{19}$. (V. 10)

$22x < x^2 + 60 < 24x$; x not < 19 but < 21. (V. 30)

In the first and third of the last three cases the limits are not accurate, but are *integral* limits which are *a fortiori* safe. In the second $\frac{66}{19}$ should have been $\frac{67}{19}$, and it would have been more correct to say that, if x is not greater than $\frac{67}{17}$ and not less than $\frac{67}{19}$, the given conditions are *a fortiori* satisfied.

For comparison with Diophantus's solutions of quadratic equations we may refer to a few of his solutions of

(3) *Simultaneous equations involving quadratics.*

In I. 27, 28, and 30 we have the following pairs of equations.

$$(\alpha)\ \left.\begin{aligned}\xi+\eta &= 2a\\ \xi\eta &= B\end{aligned}\right\},\qquad (\beta)\ \left.\begin{aligned}\xi+\eta &= 2a\\ \xi^2+\eta^2 &= B\end{aligned}\right\},\qquad (\gamma)\ \left.\begin{aligned}\xi-\eta &= 2a\\ \xi\eta &= B\end{aligned}\right\}.$$

I use the Greek letters for the numbers required to be found as distinct from the one unknown which Diophantus uses, and which I shall call x.

In (α), he says, let $\xi-\eta = 2x$ ($\xi > \eta$).

It follows, by addition and subtraction, that $\xi = a+x$, $\eta = a-x$;

therefore $\qquad \xi\eta = (a+x)(a-x) = a^2-x^2 = B,$

and x is found from the pure quadratic equation.

In (β) similarly he assumes $\xi-\eta = 2x$, and the resulting equation is $\quad \xi^2+\eta^2 = (a+x)^2+(a-x)^2 = 2(a^2+x^2) = B.$

In (γ) he puts $\xi+\eta = 2x$ and solves as in the case of (α).

(4) *Cubic equation.*

Only one very particular case occurs. In VI. 17 the problem leads to the equation

$$x^2+2x+3 = x^3+3x-3x^2-1.$$

Diophantus says simply 'whence x is found to be 4'. In fact the equation reduces to

$$x^3+x = 4x^2+4.$$

Diophantus no doubt detected, and divided out by, the common factor x^2+1, leaving $x = 4$.

(B) *Indeterminate equations.*

Diophantus says nothing of indeterminate equations of the first degree. The reason is perhaps that it is a principle with him to admit rational *fractional* as well as integral solutions, whereas the whole point of indeterminate equations of the first degree is to obtain a solution in *integral* numbers. Without this limitation (foreign to Diophantus) such equations have no significance.

(a) *Indeterminate equations of the second degree.*

The form in which these equations occur is invariably this: one or two (but never more) functions of x of the form Ax^2+Bx+C or simpler forms are to be made rational square numbers by finding a suitable value for x. That is, we have to solve, in the most general case, one or two equations of the form $Ax^2+Bx+C=y^2$.

(1) *Single equation.*

The solutions take different forms according to the particular values of the coefficients. Special cases arise when one or more of them vanish or they satisfy certain conditions.

1. When A or C or both vanish, the equation can always be solved rationally.

Form $Bx=y^2$.

Form $Bx+C=y^2$.

Diophantus puts for y^2 any determinate square m^2, and x is immediately found.

Form $Ax^2+Bx=y^2$.

Diophantus puts for y any multiple of x, as $\frac{m}{n}x$.

2. The equation $Ax^2+C=y^2$ can be rationally solved according to Diophantus:

(α) when A is positive and a square, say a^2;

in this case we put $a^2x^2+C=(ax\pm m)^2$, whence

$$x=\pm\frac{C-m^2}{2ma}$$

(m and the sign being so chosen as to give x a positive value);

(β) when C is positive and a square, say c^2;
in this case Diophantus puts $Ax^2+c^2 = (mx \pm c)^2$, and obtains

$$x = \pm \frac{2mc}{A-m^2}.$$

(γ) When one solution is known, any number of other solutions can be found. This is stated in the Lemma to VI. 15. It would be true not only of the cases $\pm Ax^2 \mp C = y^2$, but of the general case $Ax^2+Bx+C = y^2$. Diophantus, however, only states it of the case $Ax^2 - C = y^2$.

His method of finding other (greater) values of x satisfying the equation when one (x_0) is known is as follows. If $Ax_0^2 - C = q^2$, he substitutes in the original equation (x_0+x) for x and $(q-kx)$ for y, where k is some integer.

Then, since $A(x_0+x)^2 - C = (q-kx)^2$, while $Ax_0^2 - C = q^2$, it follows by subtraction that

$$2x(Ax_0+kq) = x^2(k^2-A),$$

whence $$x = 2(Ax_0+kq)/(k^2-A),$$

and the new value of x is $x_0 + \dfrac{2(Ax_0+kq)}{k^2-A}$.

Form $Ax^2 - c^2 = y^2$.

Diophantus says (VI. 14) that a rational solution of this case is only possible when A is the sum of two squares.

[In fact, if $x = p/q$ satisfies the equation, and $Ax^2 - c^2 = k^2$,

we have $$Ap^2 = c^2q^2 + k^2q^2,$$

or $$A = \left(\frac{cq}{p}\right)^2 + \left(\frac{kq}{p}\right)^2.]$$

Form $Ax^2 + C = y^2$.

Diophantus proves in the Lemma to VI. 12 that this equation has an infinite number of solutions when $A+C$ is a square, i.e. in the particular case where $x = 1$ is a solution. (He does not, however, always bear this in mind, for in III. 10 he regards the equation $52x^2+12 = y^2$ as impossible though $52+12 = 64$ is a square, just as, in III. 11, $266x^2 - 10 = y^2$ is regarded as impossible.)

Suppose that $A+C = q^2$; the equation is then solved by

substituting in the original equation $1+x$ for x and $(q-kx)$ for y, where k is some integer.

3. Form $Ax^2+Bx+C=y^2$.

This can be reduced to the form in which the second term is wanting by replacing x by $z-\frac{B}{2A}$.

Diophantus, however, treats this case separately and less fully. According to him, a rational solution of the equation $Ax^2+Bx+C=y^2$ is only possible

(α) when A is positive and a square, say a^2;

(β) when C is positive and a square, say c^2;

(γ) when $\frac{1}{4}B^2-AC$ is positive and a square.

In case (α) y is put equal to $(ax-m)$, and in case (β) y is put equal to $(mx-c)$.

Case (γ) is not expressly enunciated, but occurs, as it were, accidentally (IV. 31). The equation to be solved is $3x+18-x^2=y^2$. Diophantus first assumes $3x+18-x^2=4x^2$, which gives the quadratic $3x+18=5x^2$; but this 'is not rational'. Therefore the assumption of $4x^2$ for y^2 will not do, 'and we must find a square [to replace 4] such that 18 times (this square + 1) + $(\frac{3}{2})^2$ may be a square'. The auxiliary equation is therefore $18(m^2+1)+\frac{9}{4}=y^2$, or $72m^2+81=$ a square, and Diophantus assumes $72m^2+81=(8m+9)^2$, whence $m=18$. Then, assuming $3x+18-x^2=(18)^2x^2$, he obtains the equation $325x^2-3x-18=0$, whence $x=\frac{78}{325}$, that is, $\frac{6}{25}$.

(2) *Double equation.*

The Greek term is διπλοϊσότης, διπλῆ ἰσότης or διπλῆ ἴσωσις. Two different functions of the unknown have to be made simultaneously squares. The general case is to solve in rational numbers the equations

$$\left.\begin{aligned} mx^2+\alpha x+a &= u^2 \\ nx^2+\beta x+b &= w^2 \end{aligned}\right\}.$$

The necessary preliminary condition is that each of the two expressions can be made a square. This is always possible when the first term (in x^2) is wanting. We take this simplest case first.

1. *Double equation of the first degree.*

The equations are

$$\alpha x + a = u^2,$$

$$\beta x + b = w^2.$$

Diophantus has one general method taking slightly different forms according to the nature of the coefficients.

(*a*) First method of solution.

This depends upon the identity

$$\{\tfrac{1}{2}(p+q)\}^2 - \{\tfrac{1}{2}(p-q)\}^2 = pq.$$

If the difference between the two expressions in x can be separated into two factors p, q, the expressions themselves are equated to $\{\frac{1}{2}(p+q)\}^2$ and $\{\frac{1}{2}(p-q)\}^2$ respectively. As Diophantus himself says in II. 11, we 'equate either the square of half the difference of the two factors to the lesser of the expressions, or the square of half the sum to the greater'.

We will consider the general case and investigate to what particular classes of cases the method is applicable from Diophantus's point of view, remembering that the final quadratic in x must always reduce to a single equation.

Subtracting, we have $(\alpha - \beta)x + (a - b) = u^2 - w^2$.

Separate $(\alpha - \beta)x + (a - b)$ into the factors

$$p,\ \{(\alpha - \beta)x + (a - b)\}/p.$$

We write accordingly

$$u \pm w = \frac{(\alpha - \beta)x + (a - b)}{p},$$

$$u \mp w = p.$$

Thus $u^2 = \alpha x + a = \frac{1}{4}\left\{\dfrac{(\alpha - \beta)x + (a - b)}{p} + p\right\}^2;$

therefore $\{(\alpha - \beta)x + a - b + p^2\}^2 = 4p^2(\alpha x + a).$

This reduces to

$$(\alpha - \beta)^2 x^2 + 2x\{(\alpha - \beta)(a - b) - p^2(\alpha + \beta)\}$$
$$+ (a - b)^2 - 2p^2(a + b) + p^4 = 0.$$

In order that this equation may reduce to a simple equation, either

(1) the coefficient of x^2 must vanish, or $\alpha - \beta = 0$,

or (2) the absolute term must vanish, that is,

$$p^4 - 2p^2(a+b) + (a-b)^2 = 0,$$

or

$$\{p^2 - (a+b)\}^2 = 4ab,$$

so that ab must be a square number.

As regards condition (1) we observe that it is really sufficient if $\alpha n^2 = \beta m^2$, since, if $\alpha x + a$ is a square, $(\alpha x + a)n^2$ is equally a square, and, if $\beta x + b$ is a square, so is $(\beta x + b)m^2$, and vice versa.

That is, (1) we can solve any pair of equations of the form

$$\left.\begin{aligned} \alpha m^2 x + a &= u^2 \\ \alpha n^2 x + b &= w^2 \end{aligned}\right\}.$$

Multiply by n^2, m^2 respectively, and we have to solve the equations

$$\left.\begin{aligned} \alpha m^2 n^2 x + a n^2 &= u'^2 \\ \alpha m^2 n^2 x + b m^2 &= w'^2 \end{aligned}\right\}.$$

Separate the difference, $an^2 - bm^2$, into two factors p, q and put

$$u' \pm w' = p,$$

$$u' \mp w' = q;$$

therefore $$u'^2 = \tfrac{1}{4}(p+q)^2, \quad w'^2 = \tfrac{1}{4}(p-q)^2,$$

and $$\alpha m^2 n^2 x + an^2 = \tfrac{1}{4}(p+q)^2,$$

$$\alpha m^2 n^2 x + bm^2 = \tfrac{1}{4}(p-q)^2;$$

and from either of these equations we get

$$x = \frac{\tfrac{1}{4}(p^2+q^2) - \tfrac{1}{2}(an^2+bm^2)}{\alpha m^2 n^2},$$

since $$pq = an^2 - bm^2.$$

Any factors p, q can be chosen provided that the resulting value of x is *positive*.

Ex. from Diophantus:

$$\left.\begin{aligned} 65-\ 6x &= u^2 \\ 65-24x &= w^2 \end{aligned}\right\};\qquad \text{(IV. 32)}$$

therefore
$$\left.\begin{aligned} 260-24x &= u'^2 \\ 65-24x &= w'^2 \end{aligned}\right\}.$$

The difference $= 195 = 15.13$, say;

therefore $\frac{1}{4}(15-13)^2 = 65-24x$; that is, $24x = 64$, and $x = \frac{8}{3}$.

Taking now the condition (2) that ab is a square, we see that the equations can be solved in the cases where either a and b are both squares, or the ratio of a to b is the ratio of a square to a square. If the equations are

$$\alpha x + c^2 = u^2,$$
$$\beta x + d^2 = w^2,$$

and factors are taken of the difference between the expressions as they stand, then, since one factor p, as we saw, satisfies the

equation $\{p^2-(c^2+d^2)\}^2 = 4c^2d^2$,

we must have $p = c \pm d$.

Ex. from Diophantus:

$$\left.\begin{aligned} 10x+9 &= u^2 \\ 5x+4 &= w^2 \end{aligned}\right\}.\qquad \text{(III. 15)}$$

The difference is $5x+5 = 5(x+1)$; the solution is given by $(\frac{1}{2}x+3)^2 = 10x+9$, and $x = 28$.

Another method is to multiply the equations by squares such that, when the expressions are subtracted, the absolute term vanishes. The case can be worked out generally, thus.

Multiply by d^2 and c^2 respectively, and we have to solve

$$\left.\begin{aligned} \alpha d^2x + c^2d^2 &= u^2 \\ \beta c^2x + c^2d^2 &= w^2 \end{aligned}\right\}.$$

Difference $= (\alpha d^2 - \beta c^2)x = px.q$ say.

Then x is found from the equation

$$\alpha d^2x + c^2d^2 = \tfrac{1}{4}(px+q)^2,$$

which gives $p^2x^2 + 2x(pq - 2\alpha d^2) + q^2 - 4c^2d^2 = 0$,

or, since $$pq = \alpha d^2 - \beta c^2,$$

$$p^2x^2 - 2x(\alpha d^2 + \beta c^2) + q^2 - 4c^2d^2 = 0.$$

In order that this may reduce to a simple equation, as Diophantus requires, the absolute term must vanish, so that $q = 2cd$. The method therefore only gives one solution, since q is restricted to the value $2cd$.

Ex. from Diophantus:

$$\left.\begin{aligned} 8x+4 &= u^2 \\ 6x+4 &= w^2 \end{aligned}\right\}. \qquad \text{(IV. 39)}$$

Difference $2x$; q necessarily taken to be $2\sqrt{4}$ or 4; factors therefore $\frac{1}{2}x$, 4. Therefore $8x+4 = \frac{1}{4}(\frac{1}{2}x+4)^2$, and $x = 112$.

(β) Second method of solution of a double equation of the first degree.

There is only one case of this in Diophantus, the equations being of the form

$$\left.\begin{aligned} hx + n^2 &= u^2 \\ (h+f)x + n^2 &= w^2 \end{aligned}\right\}.$$

Suppose $hx + n^2 = (y+n)^2$; therefore $hx = y^2 + 2ny$,

and $$(h+f)x + n^2 = (y+n)^2 + \frac{f}{h}(y^2 + 2ny).$$

It only remains to make the latter expression a square, which is done by equating it to $(py-n)^2$.

The case in Diophantus is the same as that last mentioned (IV. 39). Where I have used y, Diophantus as usual contrives to use his one unknown a second time.

2. *Double equations of the second degree.*

The general form is

$$\left.\begin{aligned} Ax^2 + Bx + C &= u^2 \\ A'x^2 + B'x + C' &= w^2 \end{aligned}\right\};$$

but only three types appear in Diophantus, namely

(1) $\left.\begin{aligned} p^2x^2 + \alpha x + a &= u^2 \\ p^2x^2 + \beta x + b &= w^2 \end{aligned}\right\}$, where, except in one case, $a = b$.

$$(2)\quad \left.\begin{aligned} x^2+\alpha x+a &= u^2 \\ \beta x^2+a &= w^2 \end{aligned}\right\}.$$

(The case where the absolute terms are in the ratio of a square to a square reduces to this.)

In all examples of these cases the usual method of solution applies.

$$(3)\quad \left.\begin{aligned} \alpha x^2+ax &= u^2 \\ \beta x^2+bx &= w^2 \end{aligned}\right\}.$$

The usual method does not here serve, and a special artifice is required.

Diophantus assumes $u^2 = m^2x^2$.

Then $x = a/(m^2-\alpha)$ and, by substitution in the second equation, we have

$$\beta\left(\frac{a}{m^2-\alpha}\right)^2+\frac{ba}{m^2-\alpha}, \text{ which must be made a square,}$$

or $$a^2\beta+ba(m^2-\alpha) \text{ must be a square.}$$

We have therefore to solve the equation

$$abm^2+a(a\beta-\alpha b) = y^2,$$

which can or cannot be solved by Diophantus's methods according to the nature of the coefficients. Thus it can be solved if $(a\beta-\alpha b)/a$ is a square, or if a/b is a square. Examples in VI. 12, 14.

(*b*) *Indeterminate equations of a degree higher than the second.*

(1) *Single equations.*

There are two classes, namely those in which expressions in x have to be made squares or cubes respectively. The general form is therefore

$$Ax^n+Bx^{n-1}+\ldots+Kx-L = y^2 \text{ or } y^3.$$

In Diophantus n does not exceed 6, and in the second class of cases, where the expression has to be made a cube, n does not generally exceed 3.

The species of the first class found in the *Arithmetica* are as follows.

1. Equation $Ax^3 + Bx^2 + Cx + d^2 = y^2$.

As the absolute term is a square, we can assume

$$y = \frac{C}{2d}x + d,$$

or we might assume $y = m^2x^2 + nx + d$ and determine m, n so that the coefficients of x, x^2 in the resulting equation both vanish.

Diophantus has only one case, $x^3 - 3x^2 + 3x + 1 = y^2$ (VI. 18), and uses the first method.

2. Equation $Ax^4 + Bx^3 + Cx^2 + Dx + E = y^2$, where either A or E is a square.

If A is a square ($= a^2$), we may assume $y = ax^2 + \frac{B}{2a}x + n$, determining n so that the term in x^2 in the resulting equation may vanish. If E is a square ($= e^2$), we may assume $y = mx^2 + \frac{D}{2e}x + e$, determining m so that the term in x^2 in the resulting equation may vanish. We shall then, in either case, obtain a simple equation in x.

3. Equation $Ax^4 + Cx^2 + E = y^2$, but in special cases only where all the coefficients are squares.

4. Equation $Ax^4 + E = y^2$.

The case occurring in Diophantus is $x^4 + 97 = y^2$ (V. 29). Diophantus tries one assumption, $y = x^2 - 10$, and finds that this gives $x^2 = \frac{3}{20}$, which leads to no rational result. He therefore goes back and alters his assumptions so that he is able to replace the refractory equation by $x^4 + 337 = y^2$, and at the same time to find a suitable value for y, namely $y = x^2 - 25$, which produces a rational result, $x = \frac{12}{5}$.

5. Equation of sixth degree in the special form

$$x^6 - Ax^3 + Bx + c^2 = y^2.$$

Putting $y = x^3 + c$, we have $-Ax^2 + B = 2cx^2$, and $x^2 = \frac{B}{A + 2c}$, which gives a rational solution if $\frac{B}{A + 2c}$ is

a square. Where this does not hold (in IV. 18) Diophantus harks back and replaces the equation $x^6 - 16x^3 + x + 64 = y^2$ by another, $x^6 - 128x^3 + x + 4096 = y^2$.

Of expressions which have to be made *cubes*, we have the following cases.

1. $Ax^2 + Bx + C = y^3$.

There are only two cases of this. First, in VI. 1, $x^2 - 4x + 4$ has to be made a cube, being already a square. Diophantus naturally makes $x - 2$ a cube.

Secondly, a peculiar case occurs in VI. 17, where a cube has to be found exceeding a square by 2. Diophantus assumes $(x-1)^3$ for the cube and $(x+1)^2$ for the square. This gives

$$x^3 - 3x^2 + 3x - 1 = x^2 + 2x + 3,$$

or $x^3 + x = 4x^2 + 4$. We divide out by $x^2 + 1$, and $x = 4$. It seems evident that the assumptions were made with knowledge and intention. That is, Diophantus knew of the solution 27 and 25 and deliberately led up to it. It is unlikely that he was aware of the fact, observed by Fermat, that 27 and 25 are the only integral numbers satisfying the condition.

2. $Ax^3 + Bx^2 + Cx + D = y^3$, where either A or D is a cube number, or both are cube numbers. Where A is a cube (a^3), we have only to assume $y = ax + \frac{B}{3a^2}$, and where D is a cube (d^3), $y = \frac{C}{3d^2}x + d$. Where $A = a^3$ and $D = d^3$, we can use either assumption, or put $y = ax + d$. Apparently Diophantus used the last assumption only in this case, for in IV. 27 he rejects as impossible the equation $8x^3 - x^2 + 8x - 1 = y^3$, because the assumption $y = 2x - 1$ gives a negative value $x = -\frac{2}{11}$, whereas either of the above assumptions gives a rational value.

(2) *Double equations.*

Here one expression has to be made a square and another a cube. The cases are mostly very simple, e.g. (VI. 19)

$$\left.\begin{aligned} 4x + 2 &= y^3 \\ 2x + 1 &= z^2 \end{aligned}\right\};$$

thus $y^3 = 2z^2$, and $z = 2$.

More complicated is the case in VI. 21:

$$\left.\begin{array}{r} 2x^2+2x=y^2 \\ x^3+2x^2+x=z^3 \end{array}\right\}.$$

Diophantus assumes $y = mx$, whence $x = 2/(m^2-2)$, and

$$\left(\frac{2}{m^2-2}\right)^3 + 2\left(\frac{2}{m^2-2}\right)^2 + \frac{2}{m^2-2} = z^3,$$

or
$$\frac{2m^4}{(m^2-2)^3} = z^3.$$

We have only to make $2m^4$, or $2m$, a cube.

II. Method of Limits.

As Diophantus often has to find a series of numbers in order of magnitude, and as he does not admit negative solutions, it is often necessary for him to reject a solution found in the usual course because it does not satisfy the necessary conditions; he is then obliged, in many cases, to find solutions lying *within certain limits* in place of those rejected. For example:

1. It is required to find a value of x such that some power of it, x^n, shall lie between two given numbers, say a and b.

Diophantus multiplies both a and b by 2^n, 3^n, and so on, successively, until some nth power is seen which lies between the two products. Suppose that c^n lies between ap^n and bp^n; then we can put $x = c/p$, for $(c/p)^n$ lies between a and b.

Ex. To find a square between $1\frac{1}{4}$ and 2. Diophantus multiplies by a square 64; this gives 80 and 128, between which lies 100. Therefore $(\frac{10}{8})^2$ or $\frac{25}{16}$ solves the problem (IV. 31 (2)).

To find a sixth power between 8 and 16. The sixth powers of 1, 2, 3, 4 are 1, 64, 729, 4096. Multiply 8 and 16 by 64 and we have 512 and 1024, between which 729 lies; $\frac{729}{64}$ is therefore a solution (VI. 21).

2. Sometimes a value of x has to be found which will give

some function of x a value intermediate between the values of two other functions of x.

Ex. 1. In IV. 25 a value of x is required such that $8/(x^2+x)$ shall lie between x and $x+1$.

One part of the condition gives $8 > x^3+x^2$. Diophantus accordingly assumes $8 = (x+\frac{1}{3})^3 = x^3+x^2+\frac{1}{3}x+\frac{1}{27}$, which is $> x^3+x^2$. Thus $x+\frac{1}{3} = 2$ or $x = \frac{5}{3}$ satisfies one part of the condition. Incidentally it satisfies the other, namely $8/(x^2+x) < x+1$. This is a piece of luck, and Diophantus is satisfied with it, saying nothing more.

Ex. 2. We have seen how Diophantus concludes that, if

$$\tfrac{1}{5}(x^2-60) > x > \tfrac{1}{8}(x^2-60),$$

then x is not less than 11 and not greater than 12 (V. 30).

The problem further requires that x^2-60 shall be a square. Assuming $x^2-60 = (x-m)^2$, we find $x = (m^2+60)/2m$.

Since $x > 11$ and < 12, says Diophantus, it follows that

$$24m > m^2+60 > 22m;$$

from which he concludes that m lies between 19 and 21. Putting $m = 20$, he finds $x = 11\frac{1}{2}$.

III. Method of approximation to Limits.

Here we have a very distinctive method called by Diophantus παρισότης or παρισότητος ἀγωγή. The object is to solve such problems as that of finding two or three square numbers the sum of which is a given number, while each of them either approximates to one and the same number, or is subject to limits which may be the same or different.

Two examples will best show the method.

Ex. 1. Divide 13 into two squares each of which > 6 (V. 9).

Take half of 13, i.e. $6\frac{1}{2}$, and find what *small* fraction $1/x^2$ added to it will give a square;

thus $$6\tfrac{1}{2}+\frac{1}{x^2}, \text{ or } 26+\frac{1}{y^2}, \text{ must be a square.}$$

Diophantus assumes

$$26 + \frac{1}{y^2} = \left(5 + \frac{1}{y}\right)^2, \text{ or } 26y^2 + 1 = (5y+1)^2,$$

whence

$y = 10$, and $1/y^2 = \frac{1}{100}$, i.e. $1/x^2 = \frac{1}{400}$; and $6\frac{1}{2} + \frac{1}{400} = (\frac{51}{20})^2$.

[The assumption of $5 + \frac{1}{y}$ as the side is not haphazard: 5 is chosen because it is the most suitable as giving the largest rational value for y.]

We have now, says Diophantus, to divide 13 into two squares each of which is as nearly as possible equal to $(\frac{51}{20})^2$.

Now $13 = 3^2 + 2^2$ [it is necessary that the original number shall be capable of being expressed as the sum of two squares];

and $\quad 3 > \frac{51}{20}$ by $\frac{9}{20}$,

while $\quad 2 < \frac{51}{20}$ by $\frac{11}{20}$.

But if we took $3 - \frac{9}{20}$, $2 + \frac{11}{20}$ as the sides of two squares, their sum would be $2(\frac{51}{20})^2 = \frac{5202}{400}$, which is > 13.

Accordingly we assume $3 - 9x$, $2 + 11x$ as the sides of the required squares (so that x is not exactly $\frac{1}{20}$ but near it).

Thus $\quad (3-9x)^2 + (2+11x)^2 = 13$,

and we find $\quad x = \frac{5}{101}$.

The sides of the required squares are $\frac{257}{101}$, $\frac{258}{101}$.

Ex. 2. Divide 10 into three squares each of which > 3 (V. 11).

[The original number, here 10, must of course be expressible as the sum of three squares.]

Take one-third of 10, i.e. $3\frac{1}{3}$, and find what small fraction $1/x^2$ added to it will make a square; i.e. we have to make $3\frac{1}{3} + \frac{1}{x^2}$ a square, i.e. $30 + \frac{9}{x^2}$ must be a square, or $30 + \frac{1}{y^2}$ = a square, where $3/x = 1/y$.

Diophantus assumes

$$30y^2 + 1 = (5y+1)^2,$$

the coefficient of y, i.e. 5, being so chosen as to make $1/y$ as small as possible;

therefore $y = 2$, and $1/x^2 = \frac{1}{36}$; and $3\frac{1}{3} + \frac{1}{36} = \frac{121}{36}$, a square.

We have now, says Diophantus, to divide 10 into three squares with sides as near as may be to $\frac{11}{6}$.

Now $$10 = 9 + 1 = 3^2 + (\tfrac{3}{5})^2 + (\tfrac{4}{5})^2.$$

Bringing $3, \frac{3}{5}, \frac{4}{5}$ and $\frac{11}{6}$ to a common denominator, we have $\frac{90}{30}, \frac{18}{30}, \frac{24}{30}$ and $\frac{55}{30}$,

and
$$3 > \tfrac{55}{30} \text{ by } \tfrac{35}{30},$$
$$\tfrac{3}{5} < \tfrac{55}{30} \text{ by } \tfrac{37}{30},$$
$$\tfrac{4}{5} < \tfrac{55}{30} \text{ by } \tfrac{31}{30}.$$

If now we took $3 - \frac{35}{30}, \frac{3}{5} + \frac{37}{30}, \frac{4}{5} + \frac{31}{30}$ as the sides of squares, the sum of the squares would be $3(\frac{11}{6})^2$ or $\frac{363}{36}$, which is > 10.

Accordingly we assume as the sides $3 - 35x$, $\frac{3}{5} + 37x$, $\frac{4}{5} + 31x$, where x must therefore be not exactly $\frac{1}{30}$ but near it.

Solving $$(3 - 35x)^2 + (\tfrac{3}{5} + 37x)^2 + (\tfrac{4}{5} + 31x)^2 = 10,$$
or $$10 - 116x + 3555x^2 = 10,$$
we find $x = \frac{116}{3555}$;

thus the sides of the required squares are $\frac{1321}{711}, \frac{1285}{711}, \frac{1288}{711}$;

the squares themselves are $\frac{1745041}{505521}, \frac{1651225}{505521}, \frac{1658944}{505521}$.

Other instances of the application of the method will be found in V. 10, 12, 13, 14.

Porisms and propositions in the Theory of Numbers.

I. Three propositions are quoted as occurring in the *Porisms* ('We have it in the Porisms that...'); and some other propositions assumed without proof may very likely have come from the same collection. The three propositions from the *Porisms* are to the following effect.

1. If a is a given number and x, y numbers such that $x + a = m^2$, $y + a = n^2$, then, if $xy + a$ is also a square, m and n differ by unity (V. 3).

[From the first two equations we obtain easily
$$xy + a = m^2n^2 - a(m^2 + n^2 - 1) + a^2,$$
and this is obviously a square if $m^2 + n^2 - 1 = 2mn$, or $m - n = \pm 1$.]

2. If m^2, $(m+1)^2$ be consecutive squares and a third number be taken equal to $2\{m^2+(m+1)^2\}+2$, or $4(m^2+m+1)$, the three numbers have the property that the product of any two *plus* either the sum of those two or the remaining number gives a square (V. 5).

[In fact, if X, Y, Z denote the numbers respectively,

$$XY+X+Y=(m^2+m+1)^2, \quad XY+Z=(m^2+m+2)^2,$$

$$YZ+Y+Z=(2m^2+3m+3)^2, \quad YZ+X=(2m^2+3m+2)^2,$$

$$ZX+Z+X=(2m^2+m+2)^2, \quad ZX+Y=(2m^2+m+1)^2.]$$

3. The difference of any two cubes is also the sum of two cubes, i.e. can be transformed into the sum of two cubes (V. 16).

[Diophantus merely states this without proving it or showing how to make the transformation. The subject of the transformation of sums and differences of cubes was investigated by Vieta, Bachet and Fermat.]

II. Of the many other propositions assumed or implied by Diophantus which are not referred to the *Porisms* we may distinguish two classes.

1. The first class are of two sorts; some are more or less of the nature of identical formulae, e.g. the facts that the expressions $\{\frac{1}{2}(a+b)\}^2-ab$ and $a^2(a+1)^2+a^2+(a+1)^2$ are respectively squares, that $a(a^2-a)+a+(a^2-a)$ is always a cube, and that 8 times a triangular number *plus* 1 gives a square, i.e. $8\cdot\frac{1}{2}x(x+1)+1=(2x+1)^2$. Others are of the same kind as the first two propositions quoted from the *Porisms*, e.g.

(1) If $X=a^2x+2a$, $Y=(a+1)^2x+2(a+1)$ or, in other words, if $xX+1=(ax+1)^2$ and $xY+1=\{(a+1)x+1\}^2$, then $XY+1$ is a square (IV. 20). In fact

$$XY+1=\{a(a+1)x+(2a+1)\}^2.$$

(2) If $X\pm a=m^2$, $Y\pm a=(m+1)^2$, and $Z=2(X+Y)-1$,

then $\quad YZ\pm a,\ ZX\pm a,\ XY\pm a$ are all squares (V. 3, 4).

In fact $YZ \pm a = \{(m+1)(2m+1) \mp 2a\}^2$,

$ZX \pm a = \{m(2m+1) \mp 2a\}^2$,

$XY \pm a = \{m(m+1) \mp a\}^2$.

(3) If

$$X = m^2 + 2,\ Y = (m+1)^2 + 2,\ Z = 2\{m^2 + (m+1)^2 + 1\} + 2,$$

then the six expressions

$$YZ - (Y+Z),\quad ZX - (Z+X),\quad XY - (X+Y),$$

$$YZ - X,\quad ZX - Y,\quad XY - Z$$

are all squares (V. 6).

In fact

$$YZ - (Y+Z) = (2m^2 + 3m + 3)^2,\ YZ - X = (2m^2 + 3m + 4)^2,\ \&c.$$

2. The second class is much more important, consisting of propositions in the Theory of Numbers which we find first stated or assumed in the *Arithmetica*. It was in explanation or extension of these that Fermat's most famous notes were written. How far Diophantus possessed scientific proofs of the theorems which he assumes must remain largely a matter of speculation.

(*a*) *Theorems on the composition of numbers as the sum of two squares.*

(1) Any square number can be resolved into two squares in any number of ways (II. 8).

(2) Any number which is the sum of two squares can be resolved into two other squares in any number of ways (II. 9).

(It is implied throughout that the squares may be fractional as well as integral.)

(3) If there are two whole numbers each of which is the sum of two squares, the product of the numbers can be resolved into the sum of two squares in two ways.

In fact $(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2$.

This proposition is used in III. 19, where the problem is to find four rational right-angled triangles with the same

hypotenuse. The method is this. Form two right-angled triangles from (a, b) and (c, d) respectively, by which Diophantus means, form the right-angled triangles

$$(a^2+b^2,\ a^2-b^2,\ 2ab) \text{ and } (c^2+d^2,\ c^2-d^2,\ 2cd).$$

Multiply all the sides in each triangle by the hypotenuse of the other; we have then two rational right-angled triangles with the same hypotenuse $(a^2+b^2)(c^2+d^2)$.

Two others are furnished by the formula above; for we have only to 'form two right-angled triangles' from $(ac+bd, ad-bc)$ and from $(ac-bd, ad+bc)$ respectively. The method fails if certain relations hold between a, b, c, d. They must not be such that one number of either pair vanishes, i.e. such that $ad = bc$ or $ac = bd$, or such that the numbers in either pair are equal to one another, for then the triangles are illusory.

In the case taken by Diophantus $a^2+b^2 = 2^2+1^2 = 5$, $c^2+d^2 = 3^2+2^2 = 13$, and the four right-angled triangles are

$$(65, 52, 39), (65, 60, 25), (65, 63, 16) \text{ and } (65, 56, 33).$$

On this proposition Fermat has a long and interesting note as to the number of ways in which a prime number of the form $4n+1$ and its powers can be (*a*) the hypotenuse of a rational right-angled triangle, (*b*) the sum of two squares. He also extends theorem (3) above: 'If a prime number which is the sum of two squares be multiplied by another prime number which is also the sum of two squares, the product will be the sum of two squares in two ways; if the first prime be multiplied by the square of the second, the product will be the sum of two squares in three ways; the product of the first and the cube of the second will be the sum of two squares in four ways, and so on *ad infinitum*.'

Although the hypotenuses selected by Diophantus, 5 and 13, are prime numbers of the form $4n+1$, it is unlikely that he was aware that prime numbers of the form $4n+1$ and numbers arising from the multiplication of such numbers are the only classes of numbers which are always the sum of two squares; this was first proved by Euler.

(4) More remarkable is a condition of possibility of solution prefixed to V. 9, 'To divide 1 into two parts such that, if

a given number is added to either part, the result will be a square.' The condition is in two parts. There is no doubt as to the first, 'The given number must not be odd' [i.e. no number of the form $4n+3$ or $4n-1$ can be the sum of two squares]; the text of the second part is corrupt, but the words actually found in the text make it quite likely that corrections made by Hankel and Tannery give the real meaning of the original, 'nor must the double of the given number *plus* 1 be measured by any prime number which is less by 1 than a multiple of 4'. This is tolerably near the true condition stated by Fermat, 'The given number must not be odd, and the double of it increased by 1, when divided by the greatest square which measures it, must not be divisible by a prime number of the form $4n-1$.'

(β) *On numbers which are the sum of three squares.*

In V. 11 the number $3a+1$ has to be divisible into three squares. Diophantus says that a 'must not be 2 or any multiple of 8 increased by 2'. That is, '*a number of the form $24n+7$ cannot be the sum of three squares*'. As a matter of fact, the factor 3 in the 24 is irrelevant here, and Diophantus might have said that a number of the form $8n+7$ cannot be the sum of three squares. The latter condition is true, but does not include *all* the numbers which cannot be the sum of three squares. Fermat gives the conditions to which a must be subject, proving that $3a+1$ cannot be of the form $4^n(24k+7)$ or $4^n(8k+7)$, where $k = 0$ or any integer.

(γ) *Composition of numbers as the sum of four squares.*

There are three problems, IV. 29, 30 and V. 14, in which it is required to divide a number into four squares. Diophantus states no necessary condition in this case, as he does when it is a question of dividing a number into *three* or *two* squares. Now *every number is either a square or the sum of two, three or four squares* (a theorem enunciated by Fermat and proved by Lagrange who followed up results obtained by Euler), and this shows that any number can be divided into four squares (admitting fractional as well as integral squares), since any square number can be divided into two other squares, integral

or fractional. It is possible, therefore, that Diophantus was *empirically* aware of the truth of the theorem of Fermat, but we cannot be sure of this.

Conspectus of the *Arithmetica*, with typical solutions.

There seems to be no means of conveying an idea of the extent of the problems solved by Diophantus except by giving a conspectus of the whole of the six Books. Fortunately this can be done by the help of modern notation without occupying too many pages.

It will be best to classify the propositions according to their character rather than to give them in Diophantus's order. It should be premised that x, y, z ... indicating the first, second and third ... numbers required do not mean that Diophantus indicates any of them by his unknown (ς); he gives his unknown in each case the signification which is most convenient, his object being to express all his required numbers at once in terms of the one unknown (where possible), thereby avoiding the necessity for eliminations. Where I have occasion to specify Diophantus's unknown, I shall as a rule call it ξ, except when a problem includes a subsidiary problem and it is convenient to use different letters for the unknown in the original and subsidiary problems respectively, in order to mark clearly the distinction between them. When in the equations expressions are said to be $= u^2, v^2, w^2, t^2$ ... this means simply that they are to be made squares. Given numbers will be indicated by a, b, c ... m, n ... and will take the place of the numbers used by Diophantus, which are always specific numbers.

Where the solutions, or particular devices employed, are specially ingenious or interesting, the methods of solution will be shortly indicated. The character of the book will be best appreciated by means of such illustrations.

[The problems marked with an asterisk are probably spurious.]

(i) Equations of the first degree with one unknown.

I. 7. $x-a = m(x-b)$.

I. 8. $x+a = m(x+b)$.

I. 9. $a-x = m(b-x)$.

I. 10. $x+b = m(a-x)$.

I. 11. $x+b = m(x-a)$.

I. 39. $\left.\begin{aligned}(a+x)b+(b+x)a &= 2(a+b)x,\\ \text{or } (a+b)x+(b+x)a &= 2(a+x)b,\\ \text{or } (a+b)x+(a+x)b &= 2(b+x)a.\end{aligned}\right\} (a > b)$

Diophantus states this problem in this form, 'Given two numbers (a, b), to find a third number (x) such that the numbers

$$(a+x)b,\ (b+x)a,\ (a+b)x$$

are in arithmetical progression.'

The result is of course different according to the order of magnitude of the three expressions. If $a>b$ (5 and 3 are the numbers in Diophantus), then $(a+x)b < (b+x)a$; there are consequently three alternatives, since $(a+x)b$ must be either the least or the middle, and $(b+x)a$ either the middle or the greatest of the three products. We may have

$$(a+x)b < (a+b)x < (b+x)a,$$

$$\text{or}\quad (a+b)x < (a+x)b < (b+x)a,$$

$$\text{or}\quad (a+x)b < (b+x)a < (a+b)x,$$

and the corresponding equations are as set out above.

(ii) Determinate systems of equations of the first degree.

I. 1. $x+y = a$, $x-y = b$.

$\begin{cases}\text{I. 2. } x+y = a,\ x = my,\\ \text{I. 4. } x-y = a,\ x = my.\end{cases}$

I. 3. $x+y = a$, $x = my+b$.

$\begin{cases}\text{I. 5. } x+y = a,\ \dfrac{1}{m}x+\dfrac{1}{n}y=b, \text{ subject to necessary condition.}\\ \text{I. 6. } x+y = a,\ \dfrac{1}{m}x-\dfrac{1}{n}y=b, \quad \text{,,} \quad \text{,,} \quad \text{,,}\end{cases}$

I. 12. $x_1 + x_2 = y_1 + y_2 = a,\ x_1 = my_2,\ y_1 = nx_2\ (x_1 > x_2,\ y_1 > y_2)$.

I. 13. $\left.\begin{array}{l} x_1 + x_2 = y_1 + y_2 = z_1 + z_2 = a \\ x_1 = my_2,\ y_1 = nz_2,\ z_1 = px_2 \end{array}\right\}\ (x_1 > x_2,\ y_1 > y_2,\ z_1 > z_2)$.

I. 15. $x + a = m(y - a),\ y + b = n(x - b)$.

[Diophantus puts $y = \xi + a$, where ξ is his unknown.]

I. 16. $y + z = a,\ z + x = b,\ x + y = c$. [Dioph. puts $\xi = x + y + z$.]

I. 17. $y + z + w = a,\ z + w + x = b,\ w + x + y = c,\ x + y + z = d$.

[$x + y + z + w = \xi$.]

I. 18. $y + z - x = a,\ z + x - y = b,\ x + y - z = c$.

[Dioph. puts $2\xi = x + y + z$.]

I. 19. $y + z + w - x = a,\ z + w + x - y = b,\ w + x + y - z = c,$
$x + y + z - w = d$.

[$2\xi = x + y + z + w$.]

I. 20. $x + y + z = a,\ x + y = mz,\ y + z = nx$.

I. 21. $x = y + \frac{1}{m}z,\ y = z + \frac{1}{n}x,\ z = a + \frac{1}{p}y$ (where $x > y > z$), with necessary condition.

II. 18*. $x - \left(\frac{1}{m}x + a\right) + \left(\frac{1}{p}z + c\right) = y - \left(\frac{1}{n}y + b\right) + \left(\frac{1}{m}x + a\right)$
$= z - \left(\frac{1}{p}z + c\right) + \left(\frac{1}{n}y + b\right),\ x + y + z = a$.

[Solution wanting.]

(iii) Determinate systems of equations reducible to the first degree.

I. 26. $ax = \alpha^2,\ bx = \alpha$.

I. 29. $x + y = a,\ x^2 - y^2 = b$. [Dioph. puts $2\xi = x - y$.]

I. 31. $x = my,\ x^2 + y^2 = n(x + y)$.

I. 32. $x = my,\ x^2 + y^2 = n(x - y)$.

I. 33. $x = my,\ x^2 - y^2 = n(x + y)$.

I. 34. $x = my,\ x^2 - y^2 = n(x - y)$.

I. 34. Cor. 1. $x = my,\ xy = n(x + y)$.

Cor. 2. $x = my,\ xy = n(x - y)$.

I. 35. $x = my,\ y^2 = nx$.

I. 36. $x = my,\ y^2 = ny$.

I. 37. $x = my,\ y^2 = n(x+y)$.

I. 38. $x = my,\ y^2 = n(x-y)$.

I. 38. Cor. $x = my,\ x^2 = ny$.

" $x = my,\ x^2 = nx$.

" $x = my,\ x^2 = n(x+y)$.

" $x = my,\ x^2 = n(x-y)$.

II. 6*. $x-y = a,\ x^2-y^2 = x-y+b$.

IV. 36. $yz = m(y+z),\ zx = n(z+x),\ xy = p(x+y)$.

[Solved by means of Lemma: see under (vi) Indeterminate equations of the first degree.]

(iv) Determinate systems reducible to equations of second degree.

I. 27. $x+y = a,\ xy = b$.

[Dioph. states the necessary condition, namely that $\frac{1}{4}a^2-b$ must be a square, with the words ἔστι δὲ τοῦτο πλασματικόν, which no doubt means 'this is of the nature of a formula (easily obtained)'. He puts $x-y = 2\xi$.]

I. 30. $x-y = a,\ xy = b$.

[Necessary condition (with the same words) $4b+a^2 =$ a square. $x+y$ is put $= 2\xi$.]

I. 28. $x+y = a,\ x^2+y^2 = b$.

[Necessary condition $2b-a^2 =$ a square. $x-y = 2\xi$.]

IV. 1. $x^3+y^3 = a,\ x+y = b$.

[Dioph. puts $x-y=2\xi$, whence $x=\frac{1}{2}b+\xi,\ y=\frac{1}{2}b-\xi$. The numbers a, b are so chosen that $(a-\frac{1}{4}b^3)/3b$ is a square.]

IV. 2. $x^3-y^3 = a,\ x-y = b$.

[$x+y = 2\xi$.]

IV. 15. $(y+z)x = a,\ (z+x)y = b,\ (x+y)z = c.$

[Dioph. takes the third number z as his unknown;

thus $$x+y = c/z.$$

Assume $x = p/z,\ y = q/z.$ Then

$$\frac{pq}{z^2}+p = a,$$

$$\frac{pq}{z^2}+q = b.$$

These equations are inconsistent unless $p-q = a-b$. We have therefore to determine p, q by dividing c into two parts such that their difference $= a-b$ (cf. I. 1).

A very interesting use of the 'false hypothesis' (Diophantus first takes two *arbitrary* numbers for p, q such that $p+q = c$, and finds that the values taken have to be corrected).

The final equation being $\frac{pq}{z^2}+p = a$, where p, q are determined in the way described, $z^2 = pq/(a-p)$ or $pq/(b-q)$, and the numbers a, b, c have to be such that either of these expressions gives a square.]

IV. 34. $yz+(y+z) = a^2-1,\ zx+(z+x) = b^2-1,$

$$xy+(x+y) = c^2-1.$$

[Dioph. states as the necessary condition for a rational solution that each of the three constants to which the three expressions are to be equal must be some square diminished by 1. The true condition is seen in our notation by transforming the equations $yz+(y+z) = \alpha$, $zx+(z+x) = \beta$, $xy+(x+y) = \gamma$ into

$$(y+1)(z+1) = \alpha+1,$$

$$(z+1)(x+1) = \beta+1,$$

$$(x+1)(y+1) = \gamma+1,$$

whence $$x+1 = \sqrt{\left\{\frac{(\beta+1)(\gamma+1)}{\alpha+1}\right\}} \text{ \&c.};$$

and it is only necessary that $(\alpha+1)(\beta+1)(\gamma+1)$ should be a square, not that *each* of the expressions $\alpha+1$, $\beta+1$, $\gamma+1$ should be a square.

Dioph. finds in a Lemma (see under (vi) below) a solution ἐν ἀορίστῳ (indeterminately) of $xy+(x+y)=k$, which practically means finding y in terms of x.]

IV. 35. $$yz-(y+z) = a^2-1,\ zx-(z+x) = b^2-1,$$
$$xy-(x+y) = c^2-1.$$

[The remarks on the last proposition apply *mutatis mutandis*. The lemma in this case is the indeterminate solution of $xy-(x+y)=k$.]

IV. 37. $yz = a(x+y+z)$, $zx = b(x+y+z)$, $xy = c(x+y+z)$.

[Another interesting case of 'false hypothesis'. Dioph. first gives $x+y+z$ an *arbitrary* value, then finds that the result is not rational, and proceeds to solve the new problem of finding a value of $x+y+z$ to take the place of the first value.

If $w = x+y+z$, we have $x = cw/y$, $z = aw/y$, so that $zx = acw^2/y^2 = bw$ by hypothesis; therefore $y^2 = \frac{ac}{b}w$.

For a rational solution this last expression must be a square. Suppose, therefore, that $w = \frac{ac}{b}\xi^2$, and we have

$$x+y+z = \frac{ac}{b}\xi^2,\quad y = \frac{ac}{b}\xi,\quad z = a\xi,\quad x = c\xi.$$

Eliminating x, y, z, we obtain $\xi = (bc+ca+ab)/ac$, and

$$x = (bc+ca+ab)/a,\ y = (bc+ca+ab)/b,$$
$$z = (bc+ca+ab)/c.]$$

Lemma to V. 8. $yz = a^2$, $zx = b^2$, $xy = c^2$.

(v) Systems of equations apparently indeterminate but really reduced, by arbitrary assumptions, to determinate equations of the first degree.

I. 14. $xy = m(x+y)$. [Value of y arbitrarily assumed.]

II. 3*. $xy = m(x+y)$, and $xy = m(x-y)$.
II. 1*. (cf. I. 31). $x^2+y^2 = m(x+y)$.
II. 2*. (cf. I. 34). $x^2-y^2 = m(x-y)$.
II. 4*. (cf. I. 32). $x^2+y^2 = m(x-y)$.
II. 5*. (cf. I. 33). $x^2-y^2 = m(x+y)$.
[x assumed $= 2y$.]

II. 7*. $x^2-y^2 = m(x-y)+a$. [Dioph. assumes $x-y = 2$.]

I. 22. $x - \frac{1}{m}x + \frac{1}{p}z = y - \frac{1}{n}y + \frac{1}{m}x = z - \frac{1}{p}z + \frac{1}{n}y$.

[Value of y assumed.]

I. 23. $x - \frac{1}{m}x + \frac{1}{q}w = y - \frac{1}{n}y + \frac{1}{m}x = z - \frac{1}{p}z + \frac{1}{n}y$

$= w - \frac{1}{q}w + \frac{1}{p}z$. [Value of y assumed.]

I. 24. $x + \frac{1}{m}(y+z) = y + \frac{1}{n}(z+x) = z + \frac{1}{p}(x+y)$.

[Value of $y+z$ assumed.]

I. 25. $x + \frac{1}{m}(y+z+w) = y + \frac{1}{n}(z+w+x)$

$= z + \frac{1}{p}(w+x+y) = w + \frac{1}{q}(x+y+z)$.

[Value of $y+z+w$ assumed.]

II. 17*. (cf. I. 22). $x - \left(\frac{1}{m}x + a\right) + \left(\frac{1}{p}z + c\right)$

$= y - \left(\frac{1}{n}y + b\right) + \left(\frac{1}{m}x + a\right) = z - \left(\frac{1}{p}z + c\right) + \left(\frac{1}{n}y + b\right)$.

[Ratio of x to y assumed.]

IV. 33. $x+\frac{1}{z}y = m\left(y-\frac{1}{z}y\right), \quad y+\frac{1}{z}x = n\left(x-\frac{1}{z}x\right)$.

[Dioph. assumes $\frac{1}{z}y = 1$.]

(vi) Indeterminate equations of the first degree.

Lemma to IV. 34. $xy+(x+y) = a$.
" " IV. 35. $xy-(x+y) = a$.
" " IV. 36. $xy = m(x+y)$.
[Solutions ἐν ἀορίστῳ. y practically found in terms of x.]

(vii) Indeterminate analysis of the second degree.

II. 8. $x^2+y^2 = a^2$.

[$y^2 = a^2-x^2$ must be a square $= (mx-a)^2$, say.]

II. 9. $x^2+y^2 = a^2+b^2$. [Put $x = \xi+a$, $y = m\xi-b$.]

II. 10. $x^2-y^2 = a$.

[Put $x = y+m$, choosing m such that $m^2 < a$.]

II. 11. $x+a = u^2, \quad x+b = v^2$.

II. 12. $a-x = u^2, \quad b-x = v^2$.

II. 13. $x-a = u^2, \quad x-b = v^2$.

[Dioph. solves II. 11 and 13, (1) by means of the 'double equation' (see p. 469 above), (2) without a double equation by putting $x = \xi^2 \pm a$ and equating $(\xi^2 \pm a) \pm b$ to $(\xi-m)^2$. In II. 12 he puts $x = a-\xi^2$.]

II. 14 = III. 21. $x+y = a$, $x+z^2 = u^2$, $y+z^2 = v^2$.

[Diophantus takes z as the unknown, and puts $u^2 = (z+m)^2$, $v^2 = (z+n)^2$. Therefore $x = 2mz+m^2$, $y = 2nz+n^2$, and z is found, by substitution in the first equation, to be $\frac{a-(m^2+n^2)}{2(m+n)}$. In order that the solution may be rational, m, n must satisfy a certain condition. Dioph. takes them such that $m^2+n^2 < a$, but it is sufficient, if $m > n$, that $a+mn$ should be $> n^2$.]

II. 15 = III. 20. $x+y = a$, $z^2-x = u^2$, $z^2-y = v^2$.

[The solution is similar, and a similar remark applies to Diophantus's implied condition.]

II. 16. $x = my$, $a^2 + x = u^2$, $a^2 + y = v^2$.

II. 19. $x^2 - y^2 = m(y^2 - z^2)$.

II. 20. $x^2 + y = u^2$, $y^2 + x = v^2$.

[Assume $y = 2mx + m^2$, and one condition is satisfied.]

II. 21. $x^2 - y = u^2$, $y^2 - x = v^2$.

[Assume $x = \xi + m$, $y = 2m\xi + m^2$, and one condition is satisfied.]

II. 22. $x^2 + (x+y) = u^2$, $y^2 + (x+y) = v^2$.

[Put $x + y = 2mx + m^2$.]

II. 23. $x^2 - (x+y) = u^2$, $y^2 - (x+y) = v^2$.

II. 24. $(x+y)^2 + x = u^2$, $(x+y)^2 + y = v^2$.

[Assume $x = (m^2 - 1)\xi^2$, $y = (n^2 - 1)\xi^2$, $x + y = \xi$.]

II. 25. $(x+y)^2 - x = u^2$, $(x+y)^2 - y = v^2$.

II. 26. $xy + x = u^2$, $xy + y = v^2$, $u + v = a$.

[Put $y = m^2 x - 1$.]

II. 27. $xy - x = u^2$, $xy - y = v^2$, $u + v = a$.

II. 28. $x^2y^2 + x^2 = u^2$, $x^2y^2 + y^2 = v^2$.

II. 29. $x^2y^2 - x^2 = u^2$, $x^2y^2 - y^2 = v^2$.

II. 30. $xy + (x+y) = u^2$, $xy - (x+y) = v^2$.

[Since $m^2 + n^2 \pm 2mn$ is a square, assume

$$xy = (m^2 + n^2)\xi^2 \text{ and } x + y = 2mn\xi^2;$$

put $x = p\xi$, $y = q\xi$, where $pq = m^2 + n^2$; then

$$(p+q)\xi = 2mn\xi^2.]$$

II. 31. $xy + (x+y) = u^2$, $xy - (x+y) = v^2$, $x + y = w^2$.

[Suppose $w^2 = 2 \cdot 2m \cdot m$, which is a square, and use formula $(2m)^2 + m^2 \pm 2 \cdot 2m \cdot m =$ a square.]

II. 32. $y^2 + z = u^2$, $z^2 + x = v^2$, $x^2 + y = w^2$.

[$y = \xi$, $z = (2a\xi + a^2)$, $x = 2b(2a\xi + a^2) + b^2$.]

II. 33. $y^2 - z = u^2$, $z^2 - x = v^2$, $x^2 - y = w^2$.

II. 34. $x^2+(x+y+z)=u^2,\ y^2+(x+y+z)=v^2,$

$$z^2+(x+y+z)=w^2.$$

[Since $\{\frac{1}{2}(m-n)\}^2+mn$ is a square, take any number separable into two factors (m, n) in three ways. This gives three values, say, p, q, r for $\frac{1}{2}(m-n)$. Put $x=p\xi, y=q\xi, z=r\xi$, and $x+y+z=mn\xi^2$; therefore $(p+q+r)\xi=mn\xi^2$, and ξ is found.]

II. 35. $x^2-(x+y+z)=u^2,\ y^2-(x+y+z)=v^2,$

$$z^2-(x+y+z)=w^2.$$

[Use the formula $\{\frac{1}{2}(m+n)\}^2-mn=$ a square and proceed similarly.]

III. 1*. $(x+y+z)-x^2=u^2,\ (x+y+z)-y^2=v^2,$

$$(x+y+z)-z^2=w^2.$$

III. 2*. $(x+y+z)^2+x=u^2,\ (x+y+z)^2+y=v^2,$

$$(x+y+z)^2+z=w^2.$$

III. 3*. $(x+y+z)^2-x=u^2,\ (x+y+z)^2-y=v^2,$

$$(x+y+z)^2-z=w^2.$$

III. 1*. $x-(x+y+z)^2=u^2,\ y-(x+y+z)^2=v^2,$

$$z-(x+y+z)^2=w^2.$$

III. 5. $x+y+z=t^2,\ y+z-x=u^2,\ z+x-y=v^2,$

$$x+y-z=w^2.$$

[The first solution of this problem assumes

$$t^2=x+y+z=(\xi+1)^2,\ w^2=1,\ u^2=\xi^2,$$

whence x, y, z are found in terms of ξ, and $z+x-y$ is then made a square.

The alternative solution, however, is much more elegant, and can be generalized thus.

We have to find x, y, z so that

$$\left.\begin{array}{r} -x+y+z = \text{a square} \\ x-y+z = \text{a square} \\ x+y-z = \text{a square} \\ x+y+z = \text{a square} \end{array}\right\}.$$

Equate the first three expressions to a^2, b^2, c^2, being squares such that their sum is also a square $=k^2$, say.

Then, since the sum of the first three expressions is itself equal to $x+y+z$, we have

$$x = \tfrac{1}{2}(b^2+c^2),\ y = \tfrac{1}{2}(c^2+a^2),\ z = \tfrac{1}{2}(a^2+b^2).]$$

III. 6. $x+y+z = t^2,\ y+z = u^2,\ z+x = v^2,\ x+y = w^2.$

III. 7. $x-y = y-z,\ y+z = u^2,\ z+x = v^2,\ x+y = w^2.$

III. 8. $x+y+z+a = t^2,\ y+z+a = u^2,\ z+x+a = v^2,$
$x+y+a = w^2.$

III. 9. $x+y+z-a = t^2,\ y+z-a = u^2,\ z+x-a = v^2,$
$x+y-a = w^2.$

III. 10. $yz+a = u^2,\ zx+a = v^2,\ xy+a = w^2.$

[Suppose $yz+a = m^2$, and let $y = (m^2-a)\xi$, $z = 1/\xi$: also let $zx+a = n^2$; therefore $x = (n^2-a)\xi$.
We have therefore to make

$$(m^2-a)(n^2-a)\xi^2+a \text{ a square.}$$

Diophantus takes $m^2 = 25$, $a = 12$, $n^2 = 16$, and arrives at $52\xi^2+12$, which is to be made a square. Although $52 \cdot 1^2+12$ is a square, and it follows that any number of other solutions giving a square are possible by substituting $1+\eta$ for ξ in the expression, and so on, Diophantus says that the equation could easily be solved if 52 was a square, and proceeds to solve the problem of finding two squares such that each increased by 12 will give a square, in which case their product also will be a square. In other words, we have to find m^2 and n^2 such that m^2-a, n^2-a are both squares, which, as he says, is easy. We have to find two pairs of squares differing by a. If

$$a = pq = p'q',\ \{\tfrac{1}{2}(p-q)\}^2+a = \{\tfrac{1}{2}(p+q)\}^2,$$

and $$\{\tfrac{1}{2}(p'-q')\}^2+a = \{\tfrac{1}{2}(p'+q')\}^2;$$

let, then, $m^2 = \{\tfrac{1}{2}(p+q)\}^2,\ n^2 = \{\tfrac{1}{2}(p'+q')\}^2$.]

III. 11. $yz-a = u^2,\ zx-a = v^2,\ xy-a = w^2.$

[The solution is like that of III. 10 *mutatis mutandis*.]

III. 12. $yz+x = u^2,\ zx+y = v^2,\ xy+z = w^2.$

III. 13. $yz-x = u^2,\ zx-y = v^2,\ xy-z = w^2.$

III. 14. $yz+x^2 = u^2,\ zx+y^2 = v^2,\ xy+z^2 = w^2.$

III. 15. $yz+(y+z) = u^2,\ zx+(z+x) = v^2,\ xy+(x+y) = w^2.$

[*Lemma.* If a, $a+1$ be two consecutive numbers, $a^2(a+1)^2+a^2+(a+1)^2$ is a square. Let

$$y = m^2,\ z = (m+1)^2.$$

Therefore
$$\left.\begin{array}{l}(m^2+2m+2)x+(m+1)^2 \\ (m^2+1)x+m^2\end{array}\right\}$$
and

have to be made squares. This is solved as a double-equation; in Diophantus's problem $m = 2$.

Second solution. Let x be the first number, m the second; then $(m+1)x+m$ is a square $= n^2$, say; therefore $x = (n^2-m)/(m+1)$, while $y = m$. We have then

$$\left.\begin{array}{l}(m+1)z+m = \text{a square} \\ \text{and}\quad \left(\dfrac{n^2+1}{m+1}\right)z+\dfrac{n^2-m}{m+1} = \text{a square}\end{array}\right\}.$$

Diophantus has $m = 3$, $n = 5$, so that the expressions to be made squares are with him

$$\left.\begin{array}{l}4z+3 \\ 6\frac{1}{2}z+5\frac{1}{2}\end{array}\right\}.$$

This is not possible because, of the corresponding coefficients, neither pair are in the ratio of squares. In order to substitute, for $6\frac{1}{2}$, 4, coefficients which are in the ratio of a square to a square he then finds two numbers, say, p, q to replace $5\frac{1}{2}$, 3 such that $pq+p+q =$ a square, and $(p+1)/(q+1) =$ a square. He assumes ξ and $4\xi+3$, which satisfies the second condition, and then solves for ξ, which must satisfy

$$4\xi^2+8\xi+3 = \text{a square} = (2\xi-3)^2, \text{ say},$$

which gives $\xi = \frac{3}{10}$, $4\xi+3 = 4\frac{1}{5}$.

He then solves, for z, the third number, the double-equation

$$\left.\begin{array}{l}5\frac{1}{5}z+4\frac{1}{5} = \text{square} \\ \frac{13}{10}z+\frac{3}{10} = \text{square}\end{array}\right\}$$

after multiplying by 25 and 100 respectively, making expressions

$$\left.\begin{array}{l} 130x+105 \\ 130x+\ \ 30 \end{array}\right\}.$$

In the above equations we should only have to make n^2+1 a square, and then multiply the first by n^2+1 and the second by $(m+1)^2$.

Diophantus, with his notation, was hardly in a position to solve, as we should, by writing

$$(y+1)(z+1) = a^2+1,$$
$$(z+1)(x+1) = b^2+1,$$
$$(x+1)(y+1) = c^2+1,$$

which gives $x+1 = \sqrt{\{(b^2+1)(c^2+1)/(a^2+1)\}}$, &c.]

III. 16. $yz-(y+z) = u^2,\ zx-(z+x) = v^2,\ xy-(x+y) = w^2.$

[The method is the same *mutatis mutandis* as the second of the above solutions.]

III. 17. $xy+(x+y) = u^2,\ xy+x = v^2,\ xy+y = w^2.$

III. 18. $xy-(x+y) = u^2,\ xy-x = v^2,\ xy-y = w^2.$

III. 19. $(x_1+x_2+x_3+x_4)^2 \pm x_1 = \begin{cases} t^2 \\ t'^2 \end{cases}$

$$(x_1+x_2+x_3+x_4)^2 \pm x_2 = \begin{cases} u^2 \\ u'^2 \end{cases}$$

$$(x_1+x_2+x_3+x_4)^2 \pm x_3 = \begin{cases} v^2 \\ v'^2 \end{cases}$$

$$(x_1+x_2+x_3+x_4)^2 \pm x_4 = \begin{cases} w^2 \\ w'^2 \end{cases}.$$

[Diophantus finds, in the way we have seen (p. 482), four different rational right-angled triangles with the same hypotenuse, namely (65, 52, 39), (65, 60, 25), (65, 56, 33), (65, 63, 16), or, what is the same thing, a square which is divisible into two squares in four different ways; this will solve the problem, since, if h, p, b be the three sides of a right-angled triangle, $h^2 \pm 2pb$ are both squares.

Put therefore $x_1+x_2+x_3+x_4 = 65\xi$.

and $x_1 = 2.39.52\xi^2$, $x_2 = 2.25.60\xi^2$, $x_3 = 2.33.56\xi^2$,

$$x_4 = 2.16.63\xi^2;$$

this gives $12768\xi^2 = 65\xi$, and $\xi = \frac{65}{12768}$.]

$\left\{\begin{array}{l}\text{IV. 4. } x^2+y = u^2,\ x+y = u. \\ \text{IV. 5. } x^2+y = u,\ x+y = u^2.\end{array}\right.$

IV. 13. $x+1 = t^2$, $y+1 = u^2$, $x+y+1 = v^2$, $y-x+1 = w^2$.

[Put $x = (m\xi+1)^2-1 = m^2\xi^2+2m\xi$; the second and third conditions require us to find two squares with x as difference. The difference $m^2\xi^2+2m\xi$ is separated into the factors $m^2\xi+2m$, ξ; the square of half the difference $= \{\frac{1}{2}(m^2-1)\xi+m\}^2$. Put this equal to $y+1$, so that $y = \frac{1}{4}(m^2-1)^2\xi^2+m(m^2-1)\xi+m^2-1$, and the first three conditions are satisfied. The fourth gives $\frac{1}{4}(m^4-6m^2+1)\xi^2+(m^3-3m)\xi+m^2 =$ a square, which we can equate to $(n\xi-m)^2$.]

IV. 14. $x^2+y^2+z^2 = (x^2-y^2)+(y^2-z^2)+(x^2-z^2)$. $(x>y>z.)$

IV. 16. $x+y+z = t^2$, $x^2+y = u^2$, $y^2+z = v^2$, $z^2+x = w^2$.

[Put $4m\xi$ for y, and by means of the factors $2m\xi$, 2 we can satisfy the second condition by making x equal to half the difference, or $m\xi-1$. The third condition is satisfied by subtracting $(4m\xi)^2$ from some square, say $(4m\xi+1)^2$; therefore $z = 8m\xi+1$. By the first condition $13m\xi$ must be a square. Let it be $169\eta^2$; the numbers are therefore $13\eta^2-1$, $52\eta^2$, $104\eta^2+1$, and the last condition gives $10816\eta^4+221\eta^2 =$ a square, i.e. $10816\eta^2+221 =$ a square $= (104\eta+1)^2$, say. This gives the value of η, and solves the problem.]

IV. 17. $x+y+z = t^2$, $x^2-y = u^2$, $y^2-z = v^2$, $z^2-x = w^2$.

IV. 19. $yz+1 = u^2$, $zx+1 = v^2$, $xy+1 = w^2$.

[We are asked to solve this indeterminately (ἐν τῷ ἀορίστῳ). Put for yz some square minus 1, say $m^2\xi^2+2m\xi$; one condition is now satisfied. Put $z = \xi$, so that $y = m^2\xi+2m$.

Similarly we satisfy the second condition by assuming $zx = n^2\xi^2 + 2n\xi$; therefore $x = n^2\xi + 2n$. To satisfy the third condition, we must have

$$(m^2n^2\xi^2 + 2mn.\overline{m+n}\xi + 4mn) + 1 \text{ a square.}$$

We must therefore have $4mn+1$ a square and also $mn(m+n) = mn\sqrt{(4mn+1)}$. The first condition is satisfied by $n = m+1$, which incidentally satisfies the second condition also. We put therefore $yz = (m\xi+1)^2 - 1$ and $zx = \{(m+1)\xi+1\}^2 - 1$, and assume that $z = \xi$, so that $y = m^2\xi + 2m$, $x = (m+1)^2\xi + 2(m+1)$, and we have shown that the third condition is also satisfied. Thus we have a solution in terms of the undetermined unknown ξ. The above is only slightly generalized from Diophantus.]

IV. 20. $x_2x_3 + 1 = r^2$, $x_3x_1 + 1 = s^2$, $x_1x_2 + 1 = t^2$,

$x_1x_4 + 1 = u^2$, $x_2x_4 + 1 = v^2$, $x_3x_4 + 1 = w^2$.

[This proposition depends on the last, x_1, x_2, x_3 being determined as in that proposition. If x_3 corresponds to z in that proposition, we satisfy the condition $x_3x_4 + 1 = w^2$ by putting $x_3x_4 = \{(m+2)\xi+1\}^2 - 1$, and so find x_4 in terms of ξ, after which we have only two conditions more to satisfy. The condition $x_1x_4 + 1 = \text{square}$ is automatically satisfied, since

$$\{(m+1)^2\xi + 2(m+1)\}\{(m+2)^2\xi + 2(m+2)\} + 1$$

is a square, and it only remains to satisfy $x_2x_4 + 1 = \text{square}$ That is,

$$(m^2\xi + 2m)\{(m+2)^2\xi + 2(m+2)\} + 1$$
$$= m^2(m+2)^2\xi^2 + 2m(m+2)(2m+2)\xi + 4m(m+2) + 1$$

has to be made a square, which is easy, since the coefficient of ξ^2 is a square.

With Diophantus $m = 1$, so that $x_1 = 4\xi + 4$, $x_2 = \xi + 2$ $x_3 = \xi$, $x_4 = 9\xi + 6$, and $9\xi^2 + 24\xi + 13$ has to be made a square. He equates this to $(3\xi - 4)^2$, giving $\xi = \frac{1}{16}$.]

IV. 21. $xz = y^2$, $x - y = u^2$, $x - z = v^2$, $y - z = w^2$. $(x > y > z$

IV. 22. $xyz + x = u^2$, $xyz + y = v^2$, $xyz + z = w^2$.

IV. 23. $xyz - x = u^2$, $xyz - y = v^2$, $xyz - z = w^2$.

IV. 29. $x^2+y^2+z^2+w^2+x+y+z+w = a$.

[Since $x^2+x+\frac{1}{4}$ is a square,

$$(x^2+x)+(y^2+y)+(z^2+z)+(w^2+w)+1$$

is the sum of four squares, and we only have to separate $a+1$ into four squares.]

IV. 30. $x^2+y^2+z^2+w^2-(x+y+z+w) = a$.

IV. 31. $x+y = 1$, $(x+a)(y+b) = u^2$.

IV. 32. $x+y+z = a$, $xy+z = u^2$, $xy-z = v^2$.

IV. 39. $x-y = m(y-z)$, $y+z = u^2$, $z+x = v^2$, $x+y = w^2$.

IV. 40. $x^2-y^2 = m(y-z)$, $y+z = u^2$, $z+x = v^2$, $x+y = w^2$.

V. 1. $xz = y^2$, $x-a = u^2$, $y-a = v^2$, $z-a = w^2$.

V. 2. $xz = y^2$, $x+a = u^2$, $y+a = v^2$, $z+a = w^2$.

V. 3. $x+a = r^2$, $y+a = s^2$, $z+a = t^2$,

$yz+a = u^2$, $zx+a = v^2$, $xy+a = w^2$.

V. 4. $x-a = r^2$, $y-a = s^2$, $z-a = t^2$,

$yz-a = u^2$, $zx-a = v^2$, $xy-a = w^2$.

[Solved by means of the *Porisms* that, if a be the given number, the numbers m^2-a, $(m+1)^2-a$ satisfy the conditions of V. 3, and the numbers m^2+a, $(m+1)^2+a$ the conditions of V. 4 (see p. 479 above). The third number is taken to be $2\{m^2\mp a+(m+1)^2\mp a\}-1$, and the three numbers automatically satisfy two more conditions (see p. 480 above). It only remains to make

$$2\{m^2\mp a+(m+1)^2\mp a\}-1\pm a \text{ a square,}$$

or $$4m^2+4m\mp 3a+1 = \text{a square,}$$

which is easily solved.

With Diophantus $\xi+3$ takes the place of m in V. 3 and ξ takes its place in V. 4, while a is 5 in V. 3 and 6 in V. 4.]

V. 5. $y^2z^2+x^2 = r^2$, $z^2x^2+y^2 = s^2$, $x^2y^2+z^2 = t^2$,

$y^2z^2+y^2+z^2=u^2$, $z^2x^2+z^2+x^2=v^2$, $x^2y^2+x^2+y^2=w^2$

[Solved by means of the Porism numbered 2 on p. 480.

V. 6. $x-2=r^2$, $y-2=s^2$, $z-2=t^2$,

$yz-y-z=u^2$, $zx-z-x=v^2$, $xy-x-y=w^2$,

$yz-x=u'^2$, $zx-y=v'^2$, $xy-z=w'^2$.

[Solved by means of the proposition numbered (3) on p. 481.]

Lemma 1 to V. 7. $xy+x^2+y^2=u^2$.

V. 7. $x^2\pm(x+y+z)=\begin{cases}u^2\\u'^2\end{cases}$, $y^2\pm(x+y+z)=\begin{cases}v^2\\v'^2\end{cases}$,

$$z^2\pm(x+y+z)=\begin{cases}w^2\\w'^2\end{cases}.$$

[Solved by means of the subsidiary problem (Lemma 2) of finding three rational right-angled triangles with equal area. If m, n satisfy the condition in Lemma 1, i.e. if $mn+m^2+n^2=p^2$, the triangles are 'formed' from the pairs of numbers (p, m), (p, n), $(p, m+n)$ respectively. Diophantus assumes this, but it is easy to prove. In his case $m=3$, $n=5$, so that $p=7$. Now, in a right-angled triangle, (hypotenuse)$^2\pm$ four times area is a square. We equate, therefore, $x+y+z$ to four times the common area multiplied by ξ^2, and the several numbers x, y, z to the three hypotenuses multiplied by ξ, and equate the two values. In Diophantus's case the triangles are (40, 42, 58), (24, 70, 74) and (15, 112, 113), and $245\xi=3360\xi^2$.]

V. 8. $yz\pm(x+y+z)=\begin{cases}u^2\\u'^2\end{cases}$, $zx\pm(x+y+z)=\begin{cases}v^2\\v'^2\end{cases}$,

$$xy\pm(x+y+z)=\begin{cases}w^2\\w'^2\end{cases}.$$

[Solved by means of the same three rational right-angled triangles found in the Lemma to V. 7, together with the Lemma that we can solve the equations $yz=a^2$, $zx=b^2$, $xy=c^2$.]

{V. 9. (Cf. II. 11). $x+y=1$, $x+a=u^2$, $y+a=v^2$.
{V. 11. $x+y+z=1$, $x+a=u^2$, $y+a=v^2$, $z+a=w^2$.

[These are the problems of παρισότητος ἀγωγή

described above (pp. 477–9). The problem is 'to divide unity into two (or three) parts such that, if one and the same given number be added to each part, the results are all squares'.]

V. 10. $x+y = 1,\ x+a = u^2,\ y+b = v^2.$

V. 12. $x+y+z = 1,\ x+a = u^2,\ y+b = v^2,\ z+c = w^2.$

[These problems are like the preceding except that *different* given numbers are added. The second of the two problems is not worked out, but the first is worth reproducing. We must take the particular figures used by Diophantus, namely $a = 2$, $b = 6$. We have then to divide 9 into two squares such that one of them lies between 2 and 3. Take two squares lying between 2 and 3, say $\frac{289}{144}$, $\frac{361}{144}$. We have then to find a square ξ^2 lying between them; if we can do this, we can make $9-\xi^2$ a square, and so solve the problem.

Put $9-\xi^2 = (3-m\xi)^2$, say, so that $\xi = 6m/(m^2+1)$; and m has to be determined so that ξ lies between $\frac{17}{12}$ and $\frac{19}{12}$.

Therefore $$\frac{17}{12} < \frac{6m}{m^2+1} < \frac{19}{12}.$$

Diophantus, as we have seen, finds *a fortiori* integral limits for m by solving these inequalities, making m not greater than $\frac{67}{17}$ and not less than $\frac{66}{19}$ (see pp. 463–5 above). He then takes $m = 3\frac{1}{2}$ and puts $9-\xi^2 = (3-3\frac{1}{2}\xi)^2$, which gives $\xi = \frac{84}{53}$.]

V. 13. $x+y+z = a,\ y+z = u^2,\ z+x = v^2,\ x+y = w^2.$

V. 14. $x+y+z+w = a,\ x+y+z = s^2,\ y+z+w = t^2,$
$z+w+x = u^2,\ w+x+y = v^2.$

[The method is the same.]

V. 21. $x^2y^2z^2+x^2 = u^2,\ x^2y^2z^2+y^2 = v^2,\ x^2y^2z^2+z^2 = w^2.$

V. 22. $x^2y^2z^2-x^2 = u^2,\ x^2y^2z^2-y^2 = v^2,\ x^2y^2z^2-z^2 = w^2.$

V. 23. $x^2-x^2y^2z^2 = u^2,\ y^2-x^2y^2z^2 = v^2,\ z^2-x^2y^2z^2 = w^2.$

[Solved by means of right-angled triangles in rational numbers.]

V. 24. $y^2z^2+1 = u^2$, $z^2x^2+1 = v^2$, $x^2y^2+1 = w^2$.

V. 25. $y^2z^2-1 = u^2$, $z^2x^2-1 = v^2$, $x^2y^2-1 = w^2$.

V. 26. $1-y^2z^2 = u^2$, $1-z^2x^2 = v^2$, $1-x^2y^2 = w^2$.

[These reduce to the preceding set of three problems.]

V. 27. $y^2+z^2+a = u^2$, $z^2+x^2+a = v^2$, $x^2+y^2+a = w^2$.

V. 28. $y^2+z^2-a = u^2$, $z^2+x^2-a = v^2$, $x^2+y^2-a = w^2$.

V. 30. $mx+ny = u^2$, $u^2+a = (x+y)^2$.

[This problem is enunciated thus. 'A man buys a certain number of measures of wine, some at 8 drachmas, some at 5 drachmas each. He pays for them a *square* number of drachmas; and if 60 is added to this number, the result is a square, the side of which is equal to the whole number of measures. Find the number bought at each price.'

Let ξ = the whole number of measures; therefore ξ^2-60 was the number of drachmas paid, and ξ^2-60 = a square, say $(\xi-m)^2$; hence $\xi = (m^2+60)/2m$.

Now $\frac{1}{5}$ of the price of the five-drachma measures $+\frac{1}{8}$ of that of the eight-drachma measures $= \xi$; therefore ξ^2-60, the total price, has to be divided into two parts such that $\frac{1}{5}$ of one $+\frac{1}{8}$ of the other $= \xi$.

We cannot have a real solution of this unless

$$\xi > \tfrac{1}{8}(\xi^2-60) \text{ and } < \tfrac{1}{5}(\xi^2-60);$$

therefore $$5\xi < \xi^2-60 < 8\xi.$$

Diophantus concludes, as we have seen (p. 464 above), that ξ is not less than 11 and not greater than 12.

Therefore, from above, since $\xi = (m^2+60)/2m$,

$$22m < m^2+60 < 24m;$$

and Diophantus concludes that m is not less than 19 and not greater than 21. He therefore puts $m = 20$.

Therefore $\xi = (m^2+60)/2m = 11\frac{1}{2}$, $\xi^2 = 132\frac{1}{4}$, and $\xi^2-60 = 72\frac{1}{4}$.

We have now to divide $72\frac{1}{4}$ into two parts such that $\frac{1}{5}$ of one part $+\frac{1}{8}$ of the other $= 11\frac{1}{2}$.

Let the first part $= 5z$; therefore $\frac{1}{8}$ (second part) $= 11\frac{1}{2} - z$, or second part $= 92 - 8z$.

Therefore $5z + 92 - 8z = 72\frac{1}{4}$, and $z = \frac{79}{12}$;

therefore the number of five-drachma measures is $\frac{79}{12}$ and the number of eight-drachma measures $\frac{59}{12}$.]

Lemma 2 to VI. 12. $ax^2 + b = u^2$ (where $a + b = c^2$). (see p. 467 above.)
Lemma to VI. 15. $ax^2 - b = u^2$ (where $ad^2 - b = c^2$).

[III. 15]. $xy + x + y = u^2$, $x + 1 = \frac{v^2}{w^2}(y + 1)$.

[III. 16]. $xy - (x + y) = u^2$, $x - 1 = \frac{v^2}{w^2}(y - 1)$.

[IV. 32]. $x + 1 = \frac{u^2}{v^2}(x - 1)$.

[V. 21]. $x^2 + 1 = u^2$, $y^2 + 1 = v^2$, $z^2 + 1 = w^2$.

(viii) Indeterminate analysis of the third degree.

IV. 3. $x^2 y = u$, $xy = u^3$.

IV. 6. $x^3 + y^2 = u^3$, $z^2 + y^2 = v^2$.
IV. 7. $x^3 + y^2 = u^2$, $z^2 + y^2 = v^3$.

IV. 8. $x + y^3 = u^3$, $x + y = u$.
IV. 9. $x + y^3 = u$, $x + y = u^3$.
IV. 10. $x^3 + y^3 = x + y$.
IV. 11. $x^3 - y^3 = x - y$. } the same problem.
IV. 12. $x^3 + y = y^3 + x$. }
(IV. 8–12 really reducible to the second degree.)

[We may give as examples the solutions of IV. 7, IV. 8, IV. 11.

IV. 7. Since $z^2 + y^2 =$ a cube, suppose $z^2 + y^2 = x^3$. To make $x^3 + y^2$ a square, put $x^3 = a^2 + b^2$, $y^2 = 2ab$, which also satisfies $x^3 - y^2 = z^2$. We have then to make $2ab$ a square. Let $a = \xi$, $b = 2\xi$; therefore $a^2 + b^2 = 5\xi^2$, $2ab = 4\xi^2$, $y = 2\xi$, $z = \xi$, and we have only to make $5\xi^2$ a cube. $\xi = 5$, and $x^3 = 125$, $y^2 = 100$, $z^2 = 25$.

IV. 8. Suppose $x = \xi$, $y^3 = m^3\xi^3$; therefore $u = (m+1)\xi$ must be the side of the cube $m^3\xi^3 + \xi$, and

$$m^3\xi^2 + 1 = (m^3 + 3m^2 + 3m + 1)\xi^2.$$

To solve this, we must have $3m^2 + 3m + 1$ (the difference between consecutive cubes) a square. Put

$3m^2 + 3m + 1 = (1 - nm)^2$, and $m = (3 + 2n)/(n^2 - 3)$.

IV. 11. Assume $x = (m+1)\xi$, $y = m\xi$, and we have to make $(3m^3 + 3m^2 + 1)\xi^2$ equal to 1, i.e. we have only to make $3m^2 + 3m + 1$ a square.]

IV. 18. $x^3 + y = u^3$, $y^2 + x = v^2$.

IV. 24. $x + y = a$, $xy = u^3 - u$.

[$y = a - x$; therefore $ax - x^2$ has to be made a cube *minus* its side, say $(mx-1)^3 - (mx-1)$.

Therefore $ax - x^2 = m^3x^3 - 3m^2x^2 + 2mx$.

To reduce this to a simple equation, we have only to put $m = \frac{1}{2}a$.]

IV. 25. $x + y + z = a$, $xyz = \{(x-y) + (x-z) + (y-z)\}^3$.

$(x > y > z)$

[The cube $= 8(x-z)^3$. Let $x = (m+1)\xi$, $z = m\xi$, so that $y = 8\xi/(m^2+m)$, and we have only to contrive that $8/(m^2+m)$ lies between m and $m+1$. Dioph. takes the first limit $8 > m^3 + m^2$, and puts

$$8 = (m + \tfrac{1}{3})^3 \text{ or } m^3 + m^2 + \tfrac{1}{3}m + \tfrac{1}{27},$$

whence $m = \frac{5}{3}$; therefore $x = \frac{8}{3}\xi$, $y = \frac{9}{5}\xi$, $z = \frac{5}{3}\xi$. Or, multiplying by 15, we have $x = 40\xi$, $y = 27\xi$, $z = 25\xi$. The first equation then gives ξ.]

{ IV. 26. $xy + x = u^3$, $xy + y = v^3$.
{ IV. 27. $xy - x = u^3$, $xy - y = v^3$.

IV. 28. $xy + (x+y) = u^3$, $xy - (x+y) = v^3$.

[$x + y = \frac{1}{2}(u^3 - v^3)$, $xy = \frac{1}{2}(u^3 + v^3)$; therefore

$$(x-y)^2 = \tfrac{1}{4}(u^3 - v^3)^2 - 2(u^3 + v^3),$$

which latter expression has to be made a square.

Diophantus assumes $u = \xi + 1$, $v = \xi - 1$, whence

$$\tfrac{1}{4}(6\xi^2+2)^2 - 2(2\xi^3+6\xi)$$

must be a square, or

$$9\xi^4 - 4\xi^3 + 6\xi^2 - 12\xi + 1 = \text{a square} = (3\xi^2 - 6\xi + 1)^2, \text{ say};$$

therefore $32\xi^3 = 36\xi^2$, and $\xi = \frac{9}{8}$. Thus u, v are found, and then x, y.

The second (alternative) solution uses the formula that $\xi(\xi^2-\xi) + (\xi^2-\xi) + \xi = \text{a cube}$. Put $x = \xi$, $y = \xi^2 - \xi$, and one condition is satisfied. We then only have to make $\xi(\xi^2-\xi) - \xi - (\xi^2-\xi)$ or $\xi^3 - 2\xi^2$ a cube (less than ξ^3), i.e. $\xi^3 - 2\xi^2 = (\frac{1}{2}\xi)^3$, say.]

IV. 38. $(x+y+z)x = \frac{1}{2}u(u+1)$, $(x+y+z)y = v^2$,

$$(x+y+z)z = w^3, \ [x+y+z = t^2].$$

[Suppose $x+y+z = \xi^2$; then

$$x = \frac{u(u+1)}{2\xi^2}, \quad y = \frac{v^2}{\xi^2}, \quad z = \frac{w^3}{\xi^2};$$

therefore $\quad \xi^4 = \frac{1}{2}u(u+1) + v^2 + w^3.$

Diophantus puts 8 for w^3, but we may take any cube, as m^3; and he assumes $v^2 = (\xi^2-1)^2$, for which we might substitute $(\xi^2 - n^2)^2$. We then have the triangular number $\frac{1}{2}u(u+1) = 2n^2\xi^2 - n^4 - m^3$. Since 8 times a triangular number *plus* 1 gives a square,

$$16n^2\xi^2 - 8n^4 - 8m^3 + 1 = \text{a square} = (4n\xi - k)^2, \text{ say},$$

and the problem is solved.]

V. 15. $(x+y+z)^3 + x = u^3$, $(x+y+z)^3 + y = v^3$,

$$(x+y+z)^3 + z = w^3.$$

[Let $x+y+z = \xi$, $u^3 = m^3\xi^3$, $v^3 = n^3\xi^3$, $w^3 = p^3\xi^3$;

therefore $\quad \xi = \{(m^3-1) + (n^3-1) + (p^3-1)\}\xi^3;$

and we have to find three cubes m^3, n^3, p^3 such that $m^3+n^3+p^3-3 = \text{a square}$. Diophantus assumes as the sides of the cubes $(k+1)$, $(2-k)$, 2; this gives

$9k^2 - 9k + 14 =$ a square $= (3k - l)^2$, say; and k is found. Retracing our steps, we find ξ and therefore x, y, z.]

V. 16. $(x+y+z)^3 - x = u^3$, $(x+y+z)^3 - y = v^3$,

$$(x+y+z)^3 - z = w^3.$$

V. 17. $x - (x+y+z)^3 = u^3$, $y - (x+y+z)^3 = v^3$,

$$z - (x+y+z)^3 = w^3.$$

V. 18. $x+y+z = t^2$, $(x+y+z)^3 + x = u^2$, $(x+y+z)^3 + y = v^2$,

$$(x+y+z)^3 + z = w^2.$$

[Put $x+y+z = \xi^2$, $x = (p^2-1)\xi^6$, $y = (q^2-1)\xi^6$, $z = (r^2-1)\xi^6$, whence $\xi^2 = (p^2 - 1 + q^2 - 1 + r^2 - 1)\xi^6$, so that $p^2 - 1 + q^2 - 1 + r^2 - 1$ must be made a fourth power. Diophantus assumes $p^2 = (m^2-1)^2$, $q^2 = (m+1)^2$, $r^2 = (m-1)^2$, since $m^4 - 2m^2 + m^2 + 2m + m^2 - 2m = m^4$.]

V. 19. $x+y+z = t^2$, $(x+y+z)^3 - x = u^2$,

$$(x+y+z)^3 - y = v^2,\ (x+y+z)^3 - z = w^2.$$

V. 19a. $x+y+z = t^2$, $x - (x+y+z)^3 = u^2$,

$$y - (x+y+z)^3 = v^2,\ z - (x+y+z)^3 = w^2.$$

V. 19. b, c. $x+y+z = a$, $(x+y+z)^3 \pm x = u^2$,

$$(x+y+z)^3 \pm y = v^2,\ (x+y+z)^3 \pm z = w^2.$$

V. 20. $x+y+z = \dfrac{1}{m}$, $x - (x+y+z)^3 = u^2$,

$$y - (x+y+z)^3 = v^2,\ z - (x+y+z)^3 = w^2.$$

[IV. 8]. $x - y = 1$, $x^3 - y^3 = u^2$.

[IV. 9, 10]. $x^3 + y^3 = \dfrac{u^2}{v^2}(x+y)$.

[IV. 11]. $x^3 - y^3 = \dfrac{u^2}{v^2}(x-y)$.

[V. 15]. $x^3 + y^3 + z^3 - 3 = u^2$.

[V. 16]. $3 - (x^3 + y^3 + z^3) = u^2$.

[V. 17]. $x^3 + y^3 + z^3 + 3 = u^2$.

(ix) Indeterminate analysis of the fourth degree.

V. 29. $x^4 + y^4 + z^4 = u^2$.

['Why', says Fermat, 'did not Diophantus seek *two* fourth powers such that their sum is a square. This problem is, in fact, impossible, as by my method I am able to prove with all rigour.' No doubt Diophantus knew this truth empirically. Let $x^2 = \xi^2$, $y^2 = p^2$, $z^2 = q^2$. Therefore $\xi^4 + p^4 + q^4 =$ a square $= (\xi^2 - r)^2$, say; therefore $\xi^2 = (r^2 - p^4 - q^4)/2r$, and we have to make this expression a square.

Diophantus puts $r = p^2 + 4$, $q^2 = 4$, so that the expression reduces to $8p^2/(2p^2+8)$ or $4p^2/(p^2+4)$. To make this a square, let $p^2 + 4 = (p+1)^2$, say; therefore $p = 1\frac{1}{2}$, and $p^2 = 2\frac{1}{4}$, $q^2 = 4$, $r = 6\frac{1}{4}$; or (multiplying by 4) $p^2 = 9$, $q^2 = 16$, $r = 25$, which solves the problem.]

[V. 18]. $x^2 + y^2 + z^2 - 3 = u^4$.

(See above under V. 18.)

(x) Problems of constructing right-angled triangles with sides in rational numbers and satisfying various other conditions.

[I shall in all cases call the hypotenuse z, and the other two sides x, y, so that the condition $x^2 + y^2 = z^2$ applies in all cases, in addition to the other conditions specified.]

[Lemma to V. 7]. $xy = x_1 y_1 = x_2 y_2$.

VI. 1. $z - x = u^3$, $z - y = v^3$.

[Form a right-angled triangle from ξ, m, so that $z = \xi^2 + m^2$, $x = 2m\xi$, $y = \xi^2 - m^2$; thus $z - y = 2m^2$, and, as this must be a cube, we put $m = 2$; therefore $z - x = \xi^2 - 4\xi + 4$ must be a cube, or $\xi - 2 =$ a cube, say n^3, and $\xi = n^3 + 2$.]

VI. 2. $z + x = u^3$, $z + y = v^3$.

VI. 3. $\frac{1}{2}xy + a = u^2$.

[Suppose the required triangle to be $h\xi, p\xi, b\xi$; therefore $\frac{1}{2}pb\xi^2 + a =$ a square $= n^2\xi^2$, say, and the ratio of a to $n^2 - \frac{1}{2}pb$ must be the ratio of a square to a square. To find n, p, b so as to satisfy this condition, form a right-angled triangle from $m, \dfrac{1}{m}$,

i.e. $$\left(m^2 + \frac{1}{m^2},\ 2,\ m^2 - \frac{1}{m^2}\right);$$

therefore $\frac{1}{2}pb = m^2 - \dfrac{1}{m^2}$. Assume $n^2 = \left(m + \dfrac{2a}{m}\right)^2$;

therefore $n^2 - \frac{1}{2}pb = 4a + \dfrac{4a^2+1}{m^2}$; and $\left(4a + \dfrac{4a^2+1}{m^2}\right)\Big/a$, or $4a^2 + \dfrac{a(4a^2+1)}{m^2}$, has to be made a square. Put $4a^2m^2 + a(4a^2+1) = (2am+k)^2$, and we have a solution. Diophantus has $a = 5$, leading to $100m^2 + 505 =$ a square $= (10m+5)^2$, say, which gives $m = \frac{24}{5}$ and $n = \frac{413}{60}$. h, p, b are thus determined in such a way that $\frac{1}{2}pb\xi^2 + a = n^2\xi^2$ gives a rational solution.]

VI. 4. $\frac{1}{2}xy - a = u^2$.

VI. 5. $a - \frac{1}{2}xy = u^2$.

VI. 6. $\frac{1}{2}xy + x = a$.

[Assume the triangle to be $h\xi, p\xi, b\xi$, so that $\frac{1}{2}pb\xi^2 + p\xi = a$, and for a rational solution of this equation we must have $(\frac{1}{2}p)^2 + a(\frac{1}{2}pb)$ a square. Diophantus assumes $p = 1$, $b = m$, whence $\frac{1}{2}am + \frac{1}{4}$ or $2am + 1$ $=$ a square.

But, since the triangle is rational, $m^2 + 1 =$ a square.

That is, we have a double equation. Difference $= m^2 - 2am = m(m-2a)$. Put

$$2am + 1 = \{\tfrac{1}{2}(m - \overline{m-2a})\}^2 = a^2, \text{ and } m = (a^2-1)/2a.$$

The sides of the auxiliary triangle are thus determined in such a way that the original equation in ξ is solved rationally.]

VI. 7. $\frac{1}{2}xy - x = a$.

{VI. 8. $\frac{1}{2}xy+(x+y)=a$.
{VI. 9. $\frac{1}{2}xy-(x+y)=a$.

[With the same assumptions we have in these cases to make $\{\frac{1}{2}(p+b)\}^2+a(\frac{1}{2}pb)$ a square. Diophantus assumes as before $1, m$ for the values of p, b, and obtains the double equation

$$\left.\begin{aligned}\tfrac{1}{4}(m+1)^2+\tfrac{1}{2}am &= \text{square}\\ m^2+1 &= \text{square}\end{aligned}\right\},$$

or
$$\left.\begin{aligned}m^2+(2a+2)m+1 &= \text{square}\\ m^2+1 &= \text{square}\end{aligned}\right\},$$

solving in the usual way.]

{VI. 10. $\frac{1}{2}xy+x+z=a$.
{VI. 11. $\frac{1}{2}xy-(x+z)=a$.

[In these cases the auxiliary right-angled triangle has to be found such that

$$\{\tfrac{1}{2}(h+p)\}^2+a(\tfrac{1}{2}pb) = \text{a square}.$$

Diophantus assumes it formed from $1, m+1$; thus

$$\tfrac{1}{4}(h+p)^2=\tfrac{1}{4}\{m^2+2m+2+m^2+2m\}^2=(m^2+2m+1)^2,$$

and $$a(\tfrac{1}{2}pb)=a(m+1)(m^2+2m).$$

Therefore

$$\begin{aligned}m^4+(a+4)m^3+(3a+6)m^2+(2a+4)m+1&\\ &= \text{a square}\\ &= \{1+(a+2)m-m^2\}^2, \text{ say};\end{aligned}$$

and m is found.]

Lemma 1 to VI. 12. $x=u^2$, $x-y=v^2$, $\frac{1}{2}xy+y=w^2$.

{VI. 12. $\frac{1}{2}xy+x=u^2$, $\frac{1}{2}xy+y=v^2$.
{VI. 13. $\frac{1}{2}xy-x=u^2$, $\frac{1}{2}xy-y=v^2$.

[These problems and the two following are interesting, but their solutions run to some length; therefore only one case can here be given. We will take VI. 12 with its Lemma 1.

Lemma 1. If a rational right-angled triangle be formed from m, n, the perpendicular sides are $2mn$, m^2-n^2. We will suppose the greater of the two to be $2mn$. The first two relations are satisfied by making $m = 2n$. Form, therefore, a triangle from ξ, 2ξ. The third condition then gives $6\xi^4+3\xi^2 =$ a square or $6\xi^2+3 =$ a square. One solution is $\xi = 1$ (and there are an infinite number of others to be found by means of it). If $\xi = 1$, the triangle is formed from 1, 2.

VI. 12. Suppose the triangle to be $(h\xi, b\xi, p\xi)$. Then $(\frac{1}{2}pb)\xi^2+p\xi =$ a square $=(k\xi)^2$, say, and $\xi = p/(k^2-\frac{1}{2}pb)$. This value must be such as to make $(\frac{1}{2}pb)\xi^2+b\xi$ a square also. By substitution of the value of ξ we get

$$\{bpk^2+\tfrac{1}{2}p^2b(p-b)\}/(k^2-\tfrac{1}{2}pb)^2;$$

so that $bpk^2+\frac{1}{2}p^2b(p-b)$ must be a square; or, if p, the greater perpendicular, is made a square number, $bk^2+\frac{1}{2}pb(p-b)$ has to be made a square. This by Lemma 2 (see p. 467 above) can be made a square if $b+\frac{1}{2}pb(p-b)$ is a square. *How to solve these problems*, says Diophantus, *is shown in the Lemmas*. It is not clear how they were applied, but, in fact, his solution is such as to make p, $p-b$, and $b+\frac{1}{2}pb$ all squares, namely $b = 3$, $p = 4$, $h = 5$.

Accordingly, putting for the original triangle 3ξ, 4ξ, 5ξ, we have

$$\left.\begin{aligned} 6\xi^2+4\xi &= \text{a square} \\ 6\xi^2+3\xi &= \text{a square} \end{aligned}\right\}.$$

Assuming $6\xi^2+4\xi = m^2\xi^2$, we have $\xi = 4/(m^2-6)$, and the second condition gives

$$\frac{96}{m^4-12m^2+36}+\frac{12}{m^2-6} = \text{a square},$$

or $$12m^2+24 = \text{a square}.$$

This can be solved, since $m = 1$ satisfies it (Lemma 2). A solution is $m^2 = 25$, whence $\xi = \frac{4}{19}$.]

$\left\{\begin{array}{l}\text{VI. 14. } \frac{1}{2}xy-z = u^2,\ \frac{1}{2}xy-x = v^2. \\ \text{VI. 15. } \frac{1}{2}xy+z = u^2,\ \frac{1}{2}xy+x = v^2.\end{array}\right.$

[The auxiliary right-angled triangle in this case must be such that

$$m^2 hp - \tfrac{1}{2}pb \,.\, p(h-p) \text{ is a square.}$$

If, says Diophantus (VI. 14), we form a triangle from the numbers X_1, X_2 and suppose that $p = 2X_1X_2$, and if we then divide out by $(X_1 - X_2)^2$, which is equal to $h-p$, we must find a square $k^2 \,[= m^2/(X_1 - X_2)^2]$ such that $k^2 hp - \tfrac{1}{2}pb \,.\, p$ is a square.

The problem, says Diophantus, can be solved if X_1, X_2 are 'similar plane numbers' (numbers such as ab, $\frac{m^2}{n^2}ab$). This is stated without proof, but it can easily be verified that, if $k^2 = X_1X_2$, the expression is a square. Dioph. takes 4, 1 as the numbers, so that $k^2 = 4$. The equation for m becomes

$$8 \,.\, 17m^2 - 4 \,.\, 15 \,.\, 8 \,.\, 9 = \text{a square,}$$

or $$136m^2 - 4320 = \text{a square.}$$

The solution $m^2 = 36$ (derived from the fact that

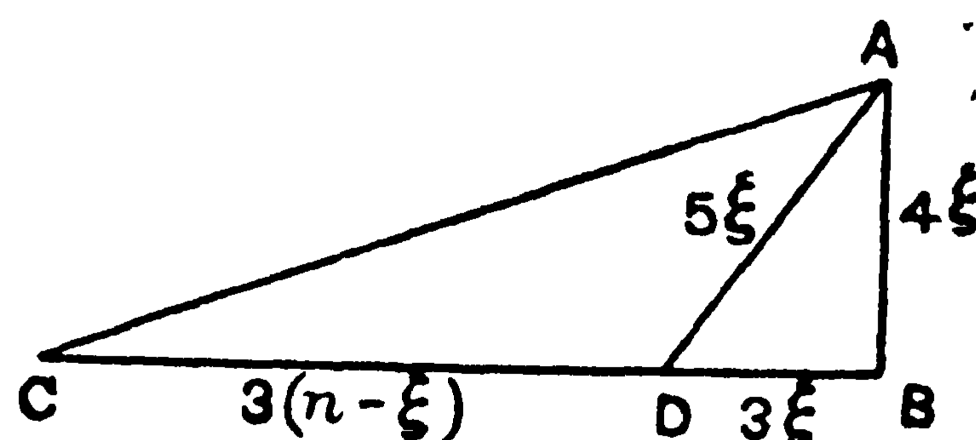
$$k^2 = m^2/(X_1 - X_2)^2, \text{ or } 4 = m^2/3^2)$$

satisfies the condition that

$$m^2 hp - \tfrac{1}{2}pb \,.\, p(h-p) \text{ is a square.}]$$

VI. 16. $\xi + \eta = x$, $\xi/\eta = y/z$.

[To find a rational right-angled triangle such that the number representing the (portion intercepted within the triangle of the) bisector of an acute angle is rational.



Let the bisector be 5ξ, the segment BD of the base 3ξ, so that the perpendicular is 4ξ.

Let $CB = 3n$. Then $AC : AB = CD : DB$,

so that $AC = 4(n-\xi)$. Therefore (Eucl. I. 47)

$$16(n^2-2n\xi+\xi^2) = 16\xi^2+9n^2,$$

so that $\xi = 7n^2/32n = \frac{7}{32}n$. [Dioph. has $n = 1$.]

VI. 17. $\frac{1}{2}xy+z = u^2$, $x+y+z = v^3$.

[Let ξ be the area $\frac{1}{2}xy$, and let $z = k^2-\xi$. Since $xy = 2\xi$, suppose $x = 2$, $y = \xi$. Therefore $2+k^2$ must be a cube. As we have seen (p. 475), Diophantus takes $(m-1)^3$ for the cube and $(m+1)^2$ for k^2, giving $m^3-3m^2+3m-1 = m^2+2m+3$, whence $m=4$. Therefore $k = 5$, and we assume $\frac{1}{2}xy = \xi$, $z = 25-\xi$, with $x = 2$, $y = \xi$ as before. Then we have to make $(25-\xi)^2 = 4+\xi^2$, and $\xi = \frac{621}{50}$.]

VI. 18. $\frac{1}{2}xy+z = u^3$, $x+y+z = v^2$.

VI. 19. $\frac{1}{2}xy+x = u^2$, $x+y+z = v^3$.

[Here a right-angled triangle is formed from one odd number, say $2\xi+1$, according to the Pythagorean formula $m^2+\{\frac{1}{2}(m^2-1)\}^2 = \{\frac{1}{2}(m^2+1)\}^2$, where m is an odd number. The sides are therefore $2\xi+1$, $2\xi^2+2\xi$, $2\xi^2+2\xi+1$. Since the perimeter = a cube,

$$4\xi^2+6\xi+2 = (4\xi+2)(\xi+1) = \text{a cube}.$$

Or, if we divide the sides by $\xi+1$, $4\xi+2$ has to be made a cube.

Again $\frac{1}{2}xy+x = \dfrac{2\xi^3+3\xi^2+\xi}{(\xi+1)^2} + \dfrac{2\xi+1}{\xi+1} = \text{a square}$,

which reduces to $2\xi+1$ = a square.

But $4\xi+2$ is a cube. We therefore put 8 for the cube, and $\xi = 1\frac{1}{2}$.]

VI. 20. $\frac{1}{2}xy+x = u^3$, $x+y+z = v^2$.

VI. 21. $x+y+z = u^2$, $\frac{1}{2}xy+(x+y+z) = v^3$.

[Form a right-angled triangle from ξ, 1, i.e. $(2\xi, \xi^2-1, \xi^2+1)$. Then $2\xi^2+2\xi$ must be a square, and $\xi^3+2\xi^2+\xi$

a cube. Put $2\xi^2+2\xi = m^2\xi^2$, so that $\xi = 2/(m^2-2)$, and we have to make

$$\frac{8}{(m^2-2)^3}+\frac{8}{(m^2-2)^2}+\frac{2}{m^2-2}, \text{ or } \frac{2m^4}{(m^2-2)^3}, \text{ a cube.}$$

Make $2m$ a cube $= n^3$, so that $2m^4 = m^3n^3$, and $m = \frac{1}{2}n^3$; therefore $\xi = \frac{8}{n^6-8}$, and ξ must be made greater than 1, in order that ξ^2-1 may be positive.

Therefore $$8 < n^6 < 16;$$

this is satisfied by $n^6 = \frac{729}{64}$ or $n^3 = \frac{27}{8}$, and $m = \frac{27}{16}$.]

VI. 22. $x+y+z = u^3$, $\frac{1}{2}xy+(x+y+z) = v^2$.

[(1) First seek a rational right-angled triangle such that its perimeter and its area are given numbers, say p, m.

Let the perpendiculars be $\frac{1}{\xi}$, $2m\xi$; therefore the hypotenuse $= p-\frac{1}{\xi}-2m\xi$, and (Eucl. I. 47)

$$\frac{1}{\xi^2}+4m^2\xi^2+(p^2+4m)-\frac{2p}{\xi}-4mp\xi = \frac{1}{\xi^2}+4m^2\xi^2,$$

or $$p^2+4m = 4mp\xi+\frac{2p}{\xi},$$

that is, $$(p^2+4m)\xi = 4mp\xi^2+2p.$$

(2) In order that this may have a rational solution,

$$\{\tfrac{1}{2}(p^2+4m)\}^2-8p^2m \text{ must be a square,}$$

i.e. $$4m^2-6p^2m+\tfrac{1}{4}p^4 = \text{a square,}$$

or $$m^2-\tfrac{3}{2}p^2m+\tfrac{1}{16}p^4 = \text{a square}$$

Also, by the second condition, $m+p$ = a square

To solve this, we must take for p some number which is both a square and a cube (in order that it may be possible, by multiplying the second equation by some square, to make the constant term equal to the constant

term in the first). Diophantus takes $p = 64$, making the equations

$$\left.\begin{aligned} m^2 - 6144m + 1048576 &= \text{a square} \\ m + 64 &= \text{a square} \end{aligned}\right\}$$

Multiplying the second by 16384, and subtracting the two expressions, we have as the difference $m^2 - 22528m$.

Diophantus observes that, if we take m, $m - 22528$ as the factors, we obtain $m = 7680$, an impossible value for the area of a right-angled triangle of perimeter $p = 64$.

We therefore take as factors $11m$, $\frac{1}{11}m - 2048$, and, equating the square of half the difference $(= \frac{60}{11}m + 1024)$ to $16384m + 1048576$, we have $m = \frac{39424}{225}$.

(3) Returning to the original problem, we have to substitute this value for m in

$$\left(64 - \frac{1}{\xi} - 2m\xi\right)^2 = \frac{1}{\xi^2} + 4m^2\xi^2,$$

and we obtain

$$78848\xi^2 - 8432\xi + 225 = 0,$$

the solution of which is rational, namely $\xi = \frac{25}{448}$ (or $\frac{9}{176}$). Diophantus naturally takes the first value, though the second gives the same triangle.]

VI. 23. $z^2 = u^2 + u$, $z^2/x = v^3 + v$.

VI. 24. $z = u^3 + u$, $x = v^3 - v$, $y = w^3$.

[VI. 6, 7]. $(\frac{1}{2}x)^2 + \frac{1}{2}mxy = u^2$.

[VI. 8, 9]. $\{\frac{1}{2}(x+y)\}^2 + \frac{1}{2}mxy = u^2$.

[VI. 10, 11]. $\{\frac{1}{2}(z+x)\}^2 + \frac{1}{2}mxy = u^2$.

[VI. 12.] $y + (x-y) \cdot \frac{1}{2}xy = u^2$, $x = v^2$. $(x > y.)$

[VI. 14, 15]. $u^2zx - \frac{1}{2}xy \cdot x(z-x) = v^2$. $(u^2 <$ or $> \frac{1}{2}xy.)$

The treatise on Polygonal Numbers.

The subject of Polygonal Numbers on which Diophantus also wrote is, as we have seen, an old one, going back to the

Pythagoreans, while Philippus of Opus and Speusippus carried on the tradition. Hypsicles (about 170 B.C.) is twice mentioned by Diophantus as the author of a 'definition' of a polygonal number which, although it does not in terms mention any polygonal number beyond the pentagonal, amounts to saying that the nth a-gon (1 counting as the first) is

$$\tfrac{1}{2}n\{2+(n-1)(a-2)\}.$$

Theon of Smyrna, Nicomachus and Iamblichus all devote some space to polygonal numbers. Nicomachus in particular gives various rules for transforming triangles into squares, squares into pentagons, &c.

1. If we put two consecutive triangles together, we get a square. In fact

$$\tfrac{1}{2}(n-1)n+\tfrac{1}{2}n(n+1)=n^2.$$

2. A pentagon is obtained from a square by adding to it a triangle the side of which is 1 less than that of the square; similarly a hexagon from a pentagon by adding a triangle the side of which is 1 less than that of the pentagon, and so on. In fact

$$\tfrac{1}{2}n\{2+(n-1)(a-2)\}+\tfrac{1}{2}(n-1)n$$
$$=\tfrac{1}{2}n[2+(n-1)\{(a+1)-2\}].$$

3. Nicomachus sets out the first triangles, squares, pentagons, hexagons and heptagons in a diagram thus:

Triangles	1	3	6	10	15	21	28	36	45	55,
Squares	1	4	9	16	25	36	49	64	81	100,
Pentagons	1	5	12	22	35	51	70	92	117	145,
Hexagons	1	6	15	28	45	66	91	120	153	190,
Heptagons	1	7	18	34	55	81	112	148	189	235,

and observes that:

Each polygon is equal to the polygon immediately above it in the diagram *plus* the triangle with 1 less in its side, i.e. the triangle in the preceding column.

4. The vertical columns are in arithmetical progression, the common difference being the triangle in the preceding column.

Plutarch, a contemporary of Nicomachus, mentions another method of transforming triangles into squares. *Every triangular number taken eight times and then increased by 1 gives a square.*

In fact, $8 \cdot \frac{1}{2}n(n+1)+1 = (2n+1)^2$.

Only a fragment of Diophantus's treatise *On Polygonal Numbers* survives. Its character is entirely different from that of the *Arithmetica.* The method of proof is strictly geometrical, and has the disadvantage, therefore, of being long and involved. He begins with some preliminary propositions of which two may be mentioned. Prop. 3 proves that, if a be the first and l the last term in an arithmetical progression of n terms, and if s is the sum of the terms, $2s = n(l+a)$. Prop. 4 proves that, if $1, 1+b, 1+2b, \ldots 1+(n-1)b$ be an A. P., and s the sum of the terms,

$$2s = n\{2+(n-1)b\}.$$

The main result obtained in the fragment as we have it is a generalization of the formula $8 \cdot \frac{1}{2}n(n+1)+1 = (2n+1)^2$. Prop. 5 proves the fact stated in Hypsicles's definition and also (the generalization referred to) that

$$8P(a-2)+(a-4)^2 = \text{a square},$$

where P is any polygonal number with a angles.

It is also proved that, if P be the nth a-gonal number (1 being the first),

$$8P(a-2)+(a-4)^2 = \{2+(2n-1)(a-2)\}^2.$$

Diophantus deduces rules as follows.

1. *To find the number from its side.*

$$P = \frac{\{2+(2n-1)(a-2)\}^2-(a-4)^2}{8(a-2)}.$$

2. *To find the side from the number.*

$$n = \tfrac{1}{2}\left(\frac{\sqrt{\{8P(a-2)+(a-4)^2\}}-2}{a-2}+1\right).$$

The last proposition, which breaks off in the middle, is:

Given a number, to find in how many ways it can be polygonal.

The proposition begins in a way which suggests that Diophantus first proved geometrically that, if

$$8P(a-2)+(a-4)^2 = \{2+(2n-1)(a-2)\}^2,$$

then $$2P = n\{2+(n-1)(a-2)\}.$$

Wertheim (in his edition of Diophantus) has suggested a restoration of the complete proof of this proposition, and I have shown (in my edition) how the proof can be made shorter. Wertheim adds an investigation of the main problem, but no doubt opinions will continue to differ as to whether Diophantus actually solved it.

XXI

COMMENTATORS AND BYZANTINES

We have come to the last stage of Greek mathematics; it only remains to include in a last chapter references to commentators of more or less note who contributed nothing original but have preserved, among observations and explanations obvious or trivial from a mathematical point of view, valuable extracts from works which have perished, or historical allusions which, in the absence of original documents, are precious in proportion to their rarity. Nor must it be forgotten that in several cases we probably owe to the commentators the fact that the masterpieces of the great mathematicians have survived, wholly or partly, in the original Greek or at all. This may have been the case even with the works of Archimedes on which Eutocius wrote commentaries. It was no doubt these commentaries which aroused in the school of Isidorus of Miletus (the colleague of Anthemius as architect of Saint Sophia at Constantinople) a new interest in the works of Archimedes and caused them to be sought out in the various libraries or wherever they had lain hid. This revived interest apparently had the effect of evoking new versions of the famous works commented upon in a form more convenient for the student, with the Doric dialect of the original eliminated; this translation of the Doric into the more familiar dialect was systematically carried out in those books only which Eutocius commented on, and it is these versions which alone survive. Again, Eutocius's commentary on Apollonius's *Conics* is extant for the first four Books, and it is probably owing to their having been commented on by Eutocius, as well as to their being more elementary than the rest, that these four Books alone

survive in Greek. Tannery, as we have seen, conjectured that, in like manner, the first six of the thirteen Books of Diophantus's *Arithmetica* survive because Hypatia wrote commentaries on these Books only and did not reach the others.

The first writer who calls for notice in this chapter is one who was rather more than a commentator in so far as he wrote a couple of treatises to supplement the *Conics* of Apollonius, I mean SERENUS. Serenus came from Antinoeia or Antinoupolis, a city in Egypt founded by Hadrian (A.D. 117–38). His date is uncertain, but he most probably belonged to the fourth century A.D., and came between Pappus and Theon of Alexandria. He tells us himself that he wrote a commentary on the *Conics* of Apollonius.[1] This has perished and, apart from a certain proposition 'of Serenus the philosopher, from the Lemmas' preserved in certain manuscripts of Theon of Smyrna (to the effect that, if a number of rectilineal angles be subtended at a point on a diameter of a circle which is not the centre, by equal arcs of that circle, the angle nearer to the centre is always less than the angle more remote), we have only the two small treatises by him entitled *On the Section of a Cylinder* and *On the Section of a Cone*. These works came to be connected, from the seventh century onwards, with the *Conics* of Apollonius, on account of the affinity of the subjects, and this no doubt accounts for their survival. They were translated into Latin by Commandinus in 1566; the first Greek text was brought out by Halley along with his Apollonius (Oxford 1710), and we now have the definitive text edited by Heiberg (Teubner 1896).

(α) *On the Section of a Cylinder.*

The occasion and the object of the tract *On the Section of a Cylinder* are stated in the preface. Serenus observes that many persons who were students of geometry were under the erroneous impression that the oblique section of a cylinder was different from the oblique section of a cone known as an ellipse, whereas it is of course the same curve. Hence he thinks it necessary to establish, by a regular geometrical

[1] Serenus, *Opuscula*, ed. Heiberg, p. 52. 25-6.

proof, that the said oblique sections cutting all the generators are equally ellipses whether they are sections of a cylinder or of a cone. He begins with 'a more general definition' of a cylinder to include any oblique circular cylinder. 'If in two equal and parallel circles which remain fixed the diameters, while remaining parallel to one another throughout, are moved round in the planes of the circles about the centres, which remain fixed, and if they carry round with them the straight line joining their extremities on the same side until they bring it back again to the same place, let the surface described by the straight line so carried round be called a *cylindrical surface*.' The *cylinder* is the figure contained by the parallel circles and the cylindrical surface intercepted by them; the parallel circles are the *bases*, the *axis* is the straight line drawn through their centres; the generating straight line in any position is a *side*. Thirty-three propositions follow. Of these Prop. 6 proves the existence in an oblique cylinder of the parallel circular sections subcontrary to the series of which the bases are two, Prop. 9 that the section by any plane not parallel to that of the bases or of one of the subcontrary sections but cutting all the generators is not a circle; the next propositions lead up to the main results, namely those in Props. 14 and 16, where the said section is proved to have the property of the ellipse which we write in the form

$$QV^2 : PV . P'V = CD^2 : CP^2,$$

and in Prop. 17, where the property is put in the Apollonian form involving the *latus rectum*, $QV^2 = PV . VR$ (see figure on p. 137 above), which is expressed by saying that the square on the semi-ordinate is equal to the rectangle applied to the *latus rectum* PL, having the abscissa PV as breadth and falling short by a rectangle similar to the rectangle contained by the diameter PP' and the *latus rectum* PL (which is determined by the condition $PL . PP' = DD'^2$ and is drawn at right angles to PV). Prop. 18 proves the corresponding property with reference to the conjugate diameter DD' and the corresponding *latus rectum*, and Prop. 19 gives the main property in the form $QV^2 : PV . P'V = Q'V'^2 : PV' . P'V'$. Then comes the proposition that 'it is possible to exhibit a cone and a cylinder which are alike cut in one and the same ellipse' (Prop. 20).

Serenus then solves such problems as these: Given a cone (or cylinder) and an ellipse on it, to find the cylinder (cone) which is cut in the same ellipse as the cone (cylinder) (Props. 21, 22); given a cone (cylinder), to find a cylinder (cone) and to cut both by one and the same plane so that the sections thus made shall be similar ellipses (Props. 23, 24). Props. 27, 28 deal with similar elliptic sections of a scalene cylinder and cone; there are two pairs of infinite sets of these similar to any one given section, the first pair being those which are parallel and subcontrary respectively to the given section, the other pair subcontrary to one another but not to either of the other sets and having the conjugate diameter occupying the corresponding place to the transverse in the other sets, and vice versa.

In the propositions (29–33) from this point to the end of the book Serenus deals with what is really an optical problem. It is introduced by a remark about a certain geometer, Peithon by name, who wrote a tract on the subject of parallels. Peithon, not being satisfied with Euclid's treatment of parallels, thought to define parallels by means of an illustration, observing that parallels are such lines as are shown on a wall or a roof by the shadow of a pillar with a light behind it. This definition, it appears, was generally ridiculed; and Serenus seeks to rehabilitate Peithon, who was his friend, by showing that his statement is after all mathematically sound. He therefore proves, with regard to the cylinder, that, if any number of rays from a point outside the cylinder are drawn touching it on both sides, all the rays pass through the sides of a parallelogram (a section of the cylinder parallel to the axis)—Prop. 29—and if they are produced farther to meet any other plane parallel to that of the parallelogram the points in which they meet the plane will lie on two parallel lines (Prop. 30); he adds that the lines will not *seem* parallel (*vide* Euclid's *Optics*, Prop. 6). The problem about the rays touching the surface of a cylinder suggests the similar one about any number of rays from an external point touching the surface of a *cone*; these meet the surface in points on a triangular section of the cone (Prop. 32) and, if produced to meet a plane parallel to that of the triangle, meet that plane in points forming a similar triangle

(Prop. 33). Prop. 31 preceding these propositions is a particular case of the constancy of the anharmonic ratio of a pencil of four rays. If two sides AB, AC of a triangle meet a transversal through D, an external point, in E, F and another ray AG between AB and AC cuts DEF in a point G such that $ED:DF = EG:GF$, then any other transversal through D meeting AB, AG, AC in K, L, M is also divided harmonically, i.e. $KD:DM = KL:LM$. To prove the succeeding propositions, 32 and 33, Serenus uses this proposition and a reciprocal of it combined with the harmonic property of the pole and polar with reference to an ellipse.

(β) *On the Section of a Cone.*

The treatise *On the Section of a Cone* is even less important, although Serenus claims originality for it. It deals mainly with the areas of triangular sections of right or scalene cones made by planes passing through the vertex and either through the axis or not through the axis, showing when the area of a certain triangle of a particular class is a maximum, under what conditions two triangles of a class may be equal in area, and so on, and solving in some easy cases the problem of finding triangular sections of given area. This sort of investigation occupies Props. 1–57 of the work, these propositions including various lemmas required for the proofs of the substantive theorems. Props. 58–69 constitute a separate section of the book dealing with the volumes of right cones in relation to their heights, their bases and the areas of the triangular sections through the axis.

The essence of the first portion of the book up to Prop. 57 is best shown by means of modern notation. We will call h the height of a right cone, r the radius of the base; in the case of an oblique cone, let p be the perpendicular from the vertex to the plane of the base, d the distance of the foot of this perpendicular from the centre of the base, r the radius of the base.

Consider first the right cone, and let $2x$ be the base of any triangular section through the vertex, while of course $2r$ is the base of the triangular section through the axis. Then, if A be the area of the triangular section with base $2x$,

$$A = x\sqrt{(r^2 - x^2 + h^2)}.$$

Observing that the sum of x^2 and $r^2-x^2+h^2$ is constant, we see that A^2, and therefore A, is a maximum when

$$x^2 = r^2-x^2+h^2, \quad \text{or} \quad x^2 = \tfrac{1}{2}(r^2+h^2);$$

and, since x is not greater than r, it follows that, for a real value of x (other than r), h is less than r, or the cone is obtuse-angled. When h is not less than r, the maximum triangle is the triangle through the axis and vice versa (Props. 5, 8); when $h = r$, the maximum triangle is also right-angled (Prop. 13).

If the triangle with base $2c$ is equal to the triangle through the axis, $h^2 r^2 = c^2 (r^2-c^2+h^2)$, or $(r^2-c^2)(c^2-h^2) = 0$, and, since $c<r$, $h = c$, so that $h<r$ (Prop. 10). If x lies between r and c in this case, $(r^2-x^2)(x^2-h^2)>0$ or $x^2(r^2-x^2+h^2)>h^2r^2$, and the triangle with base $2x$ is greater than either of the equal triangles with bases $2r$, $2c$, or $2h$ (Prop. 11).

In the case of the scalene cone Serenus compares individual triangular sections belonging to one of three classes with other sections of the same class as regards their area. The classes are:

(1) axial triangles, including all sections through the axis;

(2) isosceles sections, i.e. the sections the bases of which are perpendicular to the projection of the axis of the cone on the plane of the base;

(3) a set of triangular sections the bases of which are (a) the diameter of the circular base which passes through the foot of the perpendicular from the vertex to the plane of the base, and (b) the chords of the circular base parallel to that diameter.

After two preliminary propositions (15, 16) and some lemmas, Serenus compares the areas of the first class of triangles through the axis. If, as we said, p is the perpendicular from the vertex to the plane of the base, d the distance of the foot of this perpendicular from the centre of the base, and θ the angle which the base of any axial triangle with area A makes with the base of the axial triangle passing through p the perpendicular,

$$A = r\sqrt{(p^2+d^2\sin^2\theta)}.$$

This area is a minimum when $\theta = 0$, and increases with θ

until $\theta = \frac{1}{2}\pi$ when it is a maximum, the triangle being then isosceles (Prop. 24).

In Prop. 29 Serenus takes up the third class of sections with bases parallel to d. If the base of such a section is $2x$,

$$A = x\sqrt{(r^2 - x^2 + p^2)}$$

and, as in the case of the right cone, we must have for a real maximum value

$$x^2 = \tfrac{1}{2}(r^2 + p^2), \text{ while } x < r,$$

so that, for a real value of x other than r, p must be less than r, and, if p is not less than r, the maximum triangle is that which is perpendicular to the base of the cone and has $2r$ for its base (Prop. 29). If $p < r$, the triangle in question is not the maximum of the set of triangles (Prop. 30).

Coming now to the isosceles sections (2), we may suppose 2θ to be the angle subtended at the centre of the base by the base of the section in the direction away from the projection of the vertex. Then

$$A = r\sin\theta\sqrt{\{p^2 + (d + r\cos\theta)^2\}}.$$

If A_0 be the area of the isosceles triangle through the axis, we have

$$\begin{aligned} A_0{}^2 - A^2 &= r^2(p^2 + d^2) - r^2\sin^2\theta\,(p^2 + d^2 + r^2\cos^2\theta + 2dr\cos\theta) \\ &= r^2(p^2 + d^2)\cos^2\theta - r^4\sin^2\theta\cos^2\theta - 2dr^3\cos\theta\sin^2\theta. \end{aligned}$$

If $A = A_0$, we must have for triangles on the side of the centre of the base of the cone towards the vertex of the cone (since $\cos\theta$ is negative for such triangles)

$$p^2 + d^2 < r^2\sin^2\theta, \text{ and } \textit{a fortiori } p^2 + d^2 < r^2 \text{ (Prop. 35).}$$

If $p^2 + d^2 \geqq r^2$, A_0 is always greater than A, so that A_0 is the maximum isosceles triangle of the set (Props. 31, 32).

If A is the area of any one of the isosceles triangles with bases on the side of the centre of the base of the cone away from the projection of the vertex, $\cos\theta$ is positive and A_0 is proved to be neither the minimum nor the maximum triangle of this set of triangles (Props. 36, 40-4).

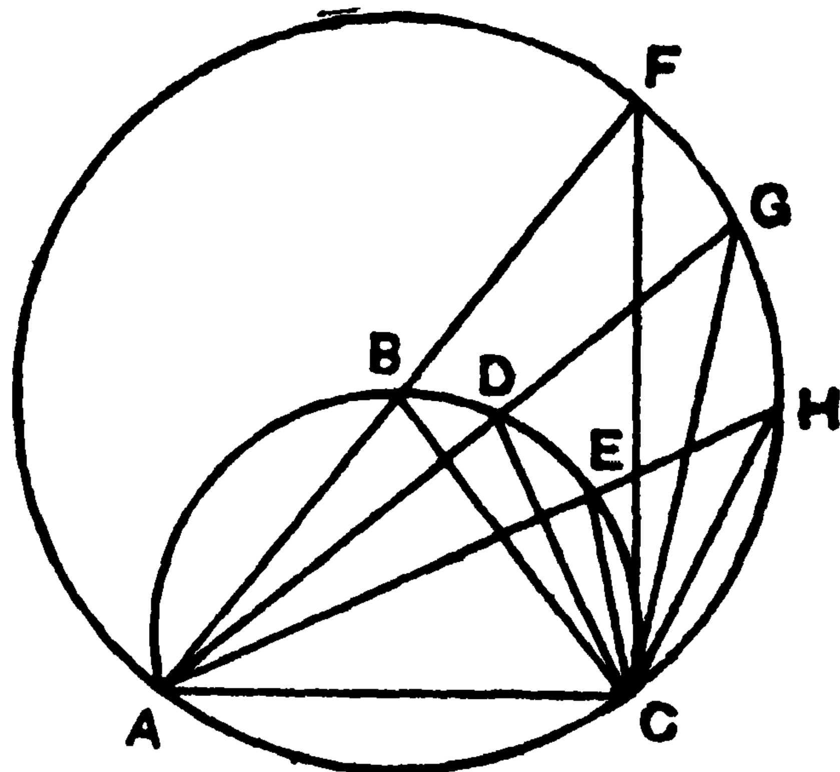
In Prop. 45 Serenus returns to the set of triangular sections through the axis, proving that the feet of the perpendiculars from the vertex of the cone on their bases all lie on a circle the diameter of which is the straight line joining the centre of

the base of the cone to the projection of the vertex on its plane; the areas of the axial triangles are therefore proportional to the generators of the cone with the said circle as base and the same vertex as the original cone. Prop. 50 is to the effect that, if the axis of the cone is equal to the radius of the base, the least axial triangle is a mean proportional between the greatest axial triangle and the isosceles triangular section perpendicular to the base; that is, with the above notation, if $r = \sqrt{(p^2+d^2)}$, then $r\sqrt{(p^2+d^2)} : rp = rp : p\sqrt{(r^2-d^2)}$, which is indeed obvious.

Prop. 57 is interesting because of the lemmas leading to it. It proves that the greater axial triangle in a scalene cone has the greater perimeter, and conversely. This is proved by means of the lemma (Prop. 54), applied to the variable sides of axial triangles, that if $a^2+d^2 = b^2+c^2$ and $a>b \geqq c>d$, then $a+d<b+c$ (a, d are the sides other than the base of one axial triangle, and b, c those of the other axial triangle compared with it; and if ABC, ADE be two axial triangles and O the centre of the base, $BA^2+AC^2 = DA^2+AE^2$ because each of these sums is equal to $2AO^2+2BO^2$, Prop. 17). This proposition again depends on the lemma (Props. 52, 53) that, if straight lines be 'inflected' from the ends of the base of a segment of a circle to the curve (i.e. if we join the ends of the base to any point on the curve) the line (i.e. the sum of the chords) is greatest when the point taken is the middle point of the arc, and diminishes as the point is taken farther and farther from that point.

Let B be the middle point of the arc of the segment ABC, D, E any other points on the curve towards C; I say that

$$AB+BC>AD+DC>AE+EC.$$



With B as centre and BA as radius describe a circle, and produce AB, AD, AE to meet this circle in F, G, H. Join FC, GC, HC.

Since $AB = BC = BF$, we have $AF = AB+BC$. Also the angles BFC, BCF are equal, and each of them is half of the angle ABC.

Again $\angle AGC = \angle AFC = \frac{1}{2}\angle ABC = \frac{1}{2}\angle ADC$;

therefore the angles DGC, DCG are equal and $DG = DC$;

therefore $AG = AD + DC$.

Similarly $EH = EC$ and $AH = AE + EC$.

But, by Eucl. III. 7 or 15, $AF > AG > AH$, and so on;

therefore $AB + BC > AD + DC > AE + EC$, and so on.

In the particular case where the segment ABC is a semi-circle $AB^2 + BC^2 = AC^2 = AD^2 + DC^2$, &c., and the result of Prop. 57 follows.

Props. 58–69 are propositions of this sort: In equal right cones the triangular sections through the axis are reciprocally proportional to their bases and conversely (Props. 58, 59); right cones of equal height have to one another the ratio duplicate of that of their axial triangles (Prop. 62); right cones which are reciprocally proportional to their bases have axial triangles which are to one another reciprocally in the triplicate ratio of their bases and conversely (Props. 66, 67); and so on.

THEON OF ALEXANDRIA lived towards the end of the fourth century A.D. Suidas places him in the reign of Theodosius I (379–95); he tells us himself that he observed a solar eclipse at Alexandria in the year 365, and his notes on the chronological tables of Ptolemy extend down to 372.

Commentary on the *Syntaxis*.

We have already seen him as the author of a commentary on Ptolemy's *Syntaxis* in eleven Books. This commentary is not calculated to give us a very high opinion of Theon's mathematical calibre, but it is valuable for several historical notices that it gives, and we are indebted to it for a useful account of the Greek method of operating with sexagesimal fractions, which is illustrated by examples of multiplication, division, and the extraction of the square root of a non-square number by way of approximation. These illustrations of numerical calculation have already been given above (vol. i,

pp. 58–63). Of the historical notices we may mention the following. (1) Theon mentions the treatise of Menelaus *On Chords in a Circle*, i.e. Menelaus's Table of Chords, which came between the similar Tables of Hipparchus and Ptolemy. (2) A quotation from Diophantus furnishes incidentally a lower limit for the date of the *Arithmetica*. (3) It is in the commentary on Ptolemy that Theon tells us that the second part of Euclid VI. 33 relating to *sectors* in equal circles was inserted by himself in his edition of the *Elements*, a notice which is of capital importance in that it enables the Theonine manuscripts of Euclid to be distinguished from the ante-Theonine, and is therefore the key to the question how far the genuine text of Euclid was altered in Theon's edition. (4) As we have seen (pp. 207 sq.), Theon, à propos of an allusion of Ptolemy to the theory of isoperimetric figures, has preserved for us several propositions from the treatise by Zenodorus on that subject.

Theon's edition of Euclid's *Elements*.

We are able to judge of the character of Theon's edition of Euclid by a comparison between the Theonine manuscripts and the famous Vatican MS. 190, which contains an earlier edition than Theon's, together with certain fragments of ancient papyri. It appears that, while Theon took some trouble to follow older manuscripts, it was not so much his object to get the most authoritative text as to make what he considered improvements of one sort or other. (1) He made alterations where he found, or thought he found, mistakes in the original; while he tried to remove some real blots, he altered other passages too hastily when a little more consideration would have shown that Euclid's words are right or could be excused, and offer no difficulty to an intelligent reader. (2) He made emendations intended to improve the form or diction of Euclid; in general they were prompted by a desire to eliminate anything which was out of the common in expression or in form, in order to reduce the language to one and the same standard or norm. (3) He bestowed, however, most attention upon additions designed to supplement or explain the original; (*a*) he interpolated whole propositions where he thought them necessary or useful, e.g. the addition to VI. 33

already referred to, a second case to VI. 27, a porism or corollary to II. 4, a second porism to III. 16, the proposition VII. 22, a lemma after X. 12, besides alternative proofs here and there; (*b*) he added words for the purpose of making smoother and clearer, or more precise, things which Euclid had expressed with unusual brevity, harshness, or carelessness; (*c*) he supplied intermediate steps where Euclid's argument seemed too difficult to follow. In short, while making only inconsiderable additions to the content of the *Elements*, he endeavoured to remove difficulties that might be felt by learners in studying the book, as a modern editor might do in editing a classical text-book for use in schools; and there is no doubt that his edition was approved by his pupils at Alexandria for whom it was written, as well as by later Greeks, who used it almost exclusively, with the result that the more ancient text is only preserved complete in one manuscript.

Edition of the *Optics* of Euclid.

In addition to the *Elements*, Theon edited the *Optics* of Euclid; Theon's recension as well as the genuine work is included by Heiberg in his edition. It is possible that the *Catoptrica* included by Heiberg in the same volume is also by Theon.

Next to Theon should be mentioned his daughter Hypatia, who is mentioned by Theon himself as having assisted in the revision of the commentary on Ptolemy. This learned lady is said to have been mistress of the whole of pagan science, especially of philosophy and medicine, and by her eloquence and authority to have attained such influence that Christianity considered itself threatened, and she was put to death by a fanatical mob in March 415. According to Suidas she wrote commentaries on Diophantus, on the Astronomical Canon (of Ptolemy) and on the Conics of Apollonius. These works have not survived, but it has been conjectured (by Tannery) that the remarks of Psellus (eleventh century) at the beginning of his letter about Diophantus, Anatolius, and the Egyptian method of arithmetical reckoning were taken bodily from some manuscript of Diophantus containing an ancient and systematic commentary which may very well have been that of Hypatia. Possibly her commentary may have extended

only to the first six Books, in which case the fact that Hypatia wrote a commentary on them may account for the survival of these Books while the rest of the thirteen were first forgotten and then lost.

It will be convenient to take next the series of Neo-Platonist commentators. It does not appear that Ammonius Saccas (about A.D. 175–250), the founder of Neo-Platonism, or his pupil Plotinus (A.D. 204–69), who first expounded the doctrines in systematic form, had any special connexion with mathematics, but PORPHYRY (about 232–304), the disciple of Plotinus and the reviser and editor of his works, appears to have written a commentary on the *Elements*. This we gather from Proclus, who quotes from Porphyry comments on Eucl. I. 14 and 26 and alternative proofs of I. 18, 20. It is possible that Porphyry's work may have been used later by Pappus in writing his own commentary, and Proclus may have got his references from Pappus, but the form of these references suggests that he had direct access to the original commentary of Porphyry.

IAMBLICHUS (died about A.D. 330) was the author of a commentary on the *Introductio arithmetica* of Nicomachus, and of other works which have already been mentioned. He was a pupil of Porphyry as well as of Anatolius, also a disciple of Porphyry.

But the most important of the Neo-Platonists to the historian of mathematics is PROCLUS (A.D. 410–85). Proclus received his early training at Alexandria, where Olympiodorus was his instructor in the works of Aristotle, and mathematics was taught him by one Heron (of course a different Heron from the '*mechanicus* Hero' of the *Metrica*, &c.). He afterwards went to Athens, where he learnt the Neo-Platonic philosophy from Plutarch, the grandson of Nestorius, and from his pupil Syrianus, and became one of its most prominent exponents. He speaks everywhere with the highest respect of his masters, and was in turn regarded with extravagant veneration by his contemporaries, as we learn from Marinus, his pupil and biographer. On the death of Syrianus he was put at the head of the Neo-Platonic school. He was a man of untiring industry, as is shown by the

number of books which he wrote, including a large number of commentaries, mostly on the dialogues of Plato (e.g. the *Timaeus*, the *Republic*, the *Parmenides*, the *Cratylus*). He was an acute dialectician and pre-eminent among his contemporaries in the range of his learning; he was a competent mathematician; he was even a poet. At the same time he was a believer in all sorts of myths and mysteries, and a devout worshipper of divinities both Greek and Oriental. He was much more a philosopher than a mathematician. In his commentary on the *Timaeus*, when referring to the question whether the sun occupies a middle place among the planets, he speaks as no real mathematician could have spoken, rejecting the view of Hipparchus and Ptolemy because *ὁ θεουργός* (*sc.* the Chaldean, says Zeller) thinks otherwise, 'whom it is not lawful to disbelieve'. Martin observes too, rather neatly, that 'for Proclus the Elements of Euclid had the good fortune not to be contradicted either by the Chaldean Oracles or by the speculations of Pythagoreans old and new'.

Commentary on Euclid, Book I.

For us the most important work of Proclus is his commentary on Euclid, Book I, because it is one of the main sources of our information as to the history of elementary geometry. Its great value arises mainly from the fact that Proclus had access to a number of historical and critical works which are now lost except for fragments preserved by Proclus and others.

(α) *Sources of the Commentary.*

The historical work the loss of which is most deeply to be deplored is the *History of Geometry* by Eudemus. There appears to be no reason to doubt that the work of Eudemus was accessible to Proclus at first hand. For the later writers Simplicius and Eutocius refer to it in terms such as leave no doubt that *they* had it before them. Simplicius, quoting Eudemus as the best authority on Hippocrates's quadratures of lunes, says he will set out what Eudemus says 'word for word', adding only a little explanation in the shape of references to Euclid's *Elements* 'owing to the *memorandum-like style of Eudemus*, who sets out his explanations in the abbre-

viated form usual with ancient writers. Now in the second book of the history of geometry he writes as follows'.[1] In like manner Eutocius speaks of the paralogisms handed down in connexion with the attempts of Hippocrates and Antiphon to square the circle, 'with which I imagine that all persons are accurately acquainted who have *examined* (ἐπεσκεμμένους) the geometrical history of Eudemus and know the *Ceria Aristotelica*'.[2]

The references by Proclus to Eudemus by name are not indeed numerous; they are five in number; but on the other hand he gives at least as many other historical data which can with great probability be attributed to Eudemus.

Proclus was even more indebted to Geminus, from whom he borrows long extracts, often mentioning him by name—there are some eighteen such references—but often omitting to do so. We are able to form a tolerably certain judgement as to the origin of the latter class of passages on the strength of the similarity of the subjects treated and the views expressed to those found in the acknowledged extracts. As we have seen, the work of Geminus mainly cited seems to have borne the title *The Doctrine* or *Theory of the Mathematics*, which was a very comprehensive work dealing, in a portion of it, with the 'classification of mathematics'.

We have already discussed the question of the authorship of the famous historical summary given by Proclus. It is divided, as every one knows, into two distinct parts between which comes the remark, 'Those who compiled histories bring the development of this science up to this point. Not much younger than these is Euclid, who', &c. The ultimate source at any rate of the early part of the summary must presumably have been the great work of Eudemus above mentioned.

It is evident that Proclus had before him the original works of Plato, Aristotle, Archimedes and Plotinus, the Συμμικτά of Porphyry and the works of his master Syrianus, as well as a group of works representing the Pythagorean tradition on its mystic, as distinct from its mathematical, side, from Philolaus downwards, and comprising the more or less apocryphal

[1] Simplicius on Arist. *Phys.*, p. 60. 28, Diels.
[2] Archimedes, ed. Heib., vol. iii, p. 228. 17–19.

ἱερὸς λόγος of Pythagoras, the *Oracles* (*λόγια*) and Orphic verses.

The following will be a convenient summary of the other works used by Proclus, and will at the same time give an indication of the historical value of his commentary on Euclid, Book I:

Eudemus: *History of Geometry.*

Geminus: *The Theory of the Mathematical Sciences.*

Heron: *Commentary on the Elements of Euclid.*

Porphyry: " " "

Pappus: " " "

Apollonius of Perga: A work relating to elementary geometry.

Ptolemy: *On the parallel-postulate.*

Posidonius: A book controverting Zeno of Sidon.

Carpus: *Astronomy.*

Syrianus: A discussion on the *angle.*

(β) *Character of the Commentary.*

We know that in the Neo-Platonic school the pupils learnt mathematics; and it is clear that Proclus taught this subject, and that this was the origin of his commentary. Many passages show him as a master speaking to scholars; in one place he speaks of 'my hearers'.[1] Further, the pupils whom he was addressing were *beginners* in mathematics; thus in one passage he says that he omits 'for the present' to speak of the discoveries of those who employed the curves of Nicomedes and Hippias for trisecting an angle, and of those who used the Archimedean spiral for dividing an angle in a given ratio, because these things would be 'too difficult for beginners'.[2] But there are signs that the commentary was revised and re-edited for a larger public; he speaks for instance in one place of 'those who will come across his work'.[3] There are also passages, e.g. passages about the cylindrical helix, conchoids and cissoids, which would not have been understood by the beginners to whom he lectured.

[1] Proclus on Eucl. I, p. 210. 19. [2] *Ib.*, p. 272. 12. [3] *Ib.*, p. 84. 9.

The commentary opens with two Prologues. The first is on mathematics in general and its relation to, and use in, philosophy, from which Proclus passes to the classification of mathematics. Prologue II deals with geometry generally and its subject-matter according to Plato, Aristotle and others. After this section comes the famous summary (pp. 64–8) ending with a eulogium of Euclid, with particular reference to the admirable discretion shown in the selection of the propositions which should constitute the *Elements* of geometry, the ordering of the whole subject-matter, the exactness and the conclusiveness of the demonstrations, and the power with which every question is handled. Generalities follow, such as the discussion of the nature of *elements*, the distinction between theorems and problems according to different authorities, and finally a division of Book I into three main sections, (1) the construction and properties of triangles and their parts and the comparison between triangles in respect of their angles and sides, (2) the properties of parallels and parallelograms and their construction from certain data, and (3) the bringing of triangles and parallelograms into relation as regards area.

Coming to the Book itself, Proclus deals historically and critically with all the definitions, postulates and axioms in order. The notes on the postulates and axioms are preceded by a general discussion of the principles of geometry, hypotheses, postulates and axioms, and their relation to one another; here as usual Proclus quotes the opinions of all the important authorities. Again, when he comes to Prop. 1, he discusses once more the difference between theorems and problems, then sets out and explains the formal divisions of a proposition, the *enunciation* (πρότασις), the *setting-out* (ἔκθεσις), the *definition* or *specification* (διορισμός), the *construction* (κατασκευή), the *proof* (ἀπόδειξις), the *conclusion* (συμπέρασμα), and finally a number of other technical terms, e.g. things said to be *given*, in the various senses of this term, the *lemma*, the *case*, the *porism* in its two senses, the *objection* (ἔνστασις), the *reduction* of a problem, *reductio ad absurdum*, *analysis* and *synthesis*.

In his comments on the separate propositions Proclus generally proceeds in this way: first he gives explanations regarding Euclid's proofs, secondly he gives a few different

cases, mainly for the sake of practice, and thirdly he addresses himself to refuting objections which cavillers had taken or might take to particular propositions or arguments. He does not seem to have had any notion of correcting or improving Euclid; only in one place does he propose anything of his own to get over a difficulty which he finds in Euclid; this is where he tries to prove the parallel-postulate, after giving Ptolemy's attempt to prove it and pointing out objections to Ptolemy's proof.

The book is evidently almost entirely a compilation, though a compilation 'in the better sense of the term'. The *onus probandi* is on any one who shall assert that anything in it is Proclus's own; very few things can with certainty be said to be so. Instances are (1) remarks on certain things which he quotes from Pappus, since Pappus was the last of the commentators whose works he seems to have used, (2) a defence of Geminus against Carpus, who criticized Geminus's view of the difference between theorems and problems, and perhaps (3) criticisms of certain attempts by Apollonius to improve on Euclid's proofs and constructions; but the only substantial example is (4) the attempted proof of the parallel-postulate, based on an 'axiom' to the effect that, 'if from one point two straight lines forming an angle be produced *ad infinitum*, the distance between them when so produced *ad infinitum* exceeds any finite magnitude (i.e. length)', an assumption which purports to be the equivalent of a statement in Aristotle.[1] Philoponus says that Proclus as well as Ptolemy wrote a whole book on the parallel-postulate.[2]

It is still not quite certain whether Proclus continued his commentaries beyond Book I. He certainly intended to do so, for, speaking of the trisection of an angle by means of certain curves, he says, 'we may perhaps more appropriately examine these things on the third Book, where the writer of the Elements bisects a given circumference', and again, after saying that of all parallelograms which have the same perimeter the square is the greatest 'and the rhomboid least of all', he adds, 'But this we will prove in another place, for it is more appropriate to the discussion of the hypotheses of the

[1] *De caelo*, i. 5, 271 b 28–30.
[2] Philoponus on *Anal. Post.* i. 10, p. 214 a 9–12, Brandis.

second Book'. But at the time when the commentary on Book I was written he was evidently uncertain whether he would be able to continue it, for at the end he says, 'For my part, if I should be able to discuss the other Books in the same way, I should give thanks to the gods; but, if other cares should draw me away, I beg those who are attracted by this subject to complete the exposition of the other Books as well, following the same method and addressing themselves throughout to the deeper and more sharply defined questions involved'.[1] Wachsmuth, finding a Vatican manuscript containing a collection of scholia on Books I, II, V, VI, X, headed *Εἰς τὰ Εὐκλείδου στοιχεῖα προλαμβανόμενα ἐκ τῶν Πρόκλου σποράδην καὶ κατ' ἐπιτομήν*, and seeing that the scholia on Book I were extracts from the extant commentary of Proclus, concluded that those on the other Books were also from Proclus; but the *προ-* in *προλαμβανόμενα* rather suggests that only the scholia to Book I are from Proclus. Heiberg found and published in 1903 a scholium to X. 9, in which Proclus is expressly quoted as the authority, but he does not regard this circumstance as conclusive. On the other hand, Heiberg has noted two facts which go against the view that Proclus wrote on the later Books: (1) the scholiast's copy of Proclus was not much better than our manuscripts; in particular, it had the same lacunae in the notes to I. 36, 37, and I. 41–3; this makes it improbable that the scholiast had further commentaries of Proclus which have vanished for us; (2) there is no trace in the scholia of the notes which Proclus promised in the passages already referred to. All, therefore, that we can say is that, while the Wachsmuth scholia *may* be extracts from Proclus, it is on the whole improbable.

Hypotyposis of Astronomical Hypotheses.

Another extant work of Proclus which should be referred to is his *Hypotyposis of Astronomical Hypotheses*, a sort of readable and easy introduction to the astronomical system of Hipparchus and Ptolemy. It has been well edited by Manitius (Teubner, 1909). Three things may be noted as

[1] Proclus on Eucl. I, p. 432. 9–15.

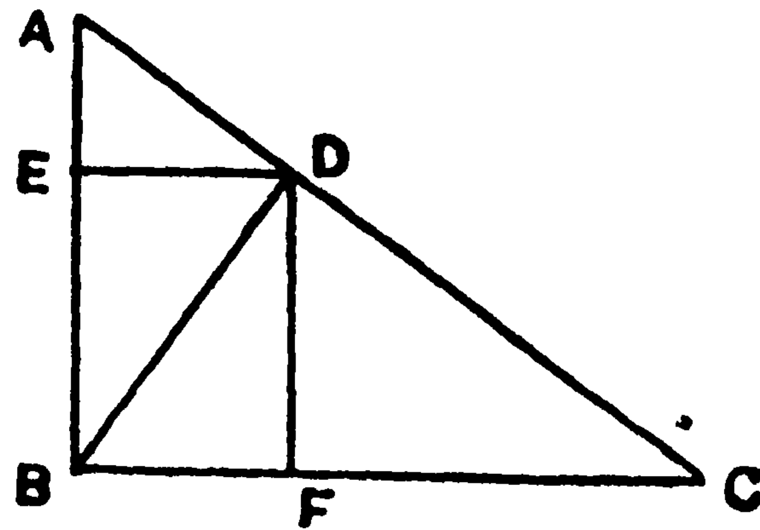
regards this work. It contains[1] a description of the method of measuring the sun's apparent diameter by means of Heron's water-clock, which, by comparison with the corresponding description in Theon's commentary to the *Syntaxis* of Ptolemy, is seen to have a common source with it. That source is Pappus, and, inasmuch as Proclus has a figure (reproduced by Manitius in his text from one set of manuscripts) corresponding to the description, while the text of Theon has no figure, it is clear that Proclus drew directly on Pappus, who doubtless gave, in his account of the procedure, a figure taken from Heron's own work on water-clocks. A simple proof of the equivalence of the epicycle and eccentric hypotheses is quoted by Proclus from one Hilarius of Antioch.[2] An interesting passage is that in chap. 4 (p. 130, 18) where Sosigenes the Peripatetic is said to have recorded in his work 'on reacting spheres' that an *annular* eclipse of the sun is sometimes observed at times of perigee; this is, so far as I know, the only allusion in ancient times to annular eclipses, and Proclus himself questions the correctness of Sosigenes's statement.

Commentary on the *Republic*.

The commentary of Proclus on the *Republic* contains some passages of great interest to the historian of mathematics. The most important is that[3] in which Proclus indicates that Props. 9, 10 of Euclid, Book II, are Pythagorean propositions invented for the purpose of proving geometrically the fundamental property of the series of 'side-' and 'diameter-' numbers, giving successive approximations to the value of $\sqrt{2}$ (see vol. i, p. 93). The explanation[4] of the passage in Plato about the Geometrical Number is defective and disappointing, but it contains an interesting reference to one Paterius, of date presumably intermediate between Nestorius and Proclus. Paterius is said to have made a calculation, in units and submultiples, of the lengths of different segments of

[1] Proclus, *Hypotyposis*, c. 4, pp. 120-22.
[2] *Ib.*, c. 3, pp. 76, 17 sq.
[3] *Procli Diadochi in Platonis Rempublicam Commentarii*, ed. Kroll, vol. ii, p. 27.
[4] *Ib.*, vol. ii, pp. 36-42.

straight lines in a figure formed by taking a triangle with sides 3, 4, 5 as ABC, then drawing BD from the right angle B perpendicular to AC, and lastly drawing perpendiculars DE, DF to AB, BC. A diagram in the text with the lengths of the segments shown alongside them in the usual numerical notation shows that Paterius obtained from the data $AB = 3$, $BC = 4$, $CA = 5$ the following:



$$DC = \gamma\ \epsilon' = 3\tfrac{1}{5}$$

$$BD = \beta\ \gamma'\ \iota\epsilon' = 2\tfrac{1}{3}\ \tfrac{1}{15}\ [= 2\tfrac{2}{5}]$$

$$AD = \alpha\ \text{S}\ \delta'\ \kappa' = 1\tfrac{1}{2}\ \tfrac{1}{4}\ \tfrac{1}{20}\ [= 1\tfrac{4}{5}]$$

$$FC = \beta\ \text{S}\ \kappa'\ \rho' = 2\tfrac{1}{2}\ \tfrac{1}{20}\ \tfrac{1}{100}\ [= 2\tfrac{14}{25}]$$

$$FB = \alpha\ \gamma'\ \iota\epsilon'\ \kappa\epsilon' = 1\tfrac{1}{3}\ \tfrac{1}{15}\ \tfrac{1}{25}\ [= 1\tfrac{11}{25}]$$

$$BE = \alpha\ \text{S}\ \gamma'\ \iota\epsilon'\ \nu' = 1\tfrac{1}{2}\ \tfrac{1}{3}\ \tfrac{1}{15}\ \tfrac{1}{50}\ [= 1\tfrac{23}{25}]$$

$$EA = \alpha\ \iota\epsilon'\ o\epsilon' = 1\tfrac{1}{15}\ \tfrac{1}{75}\ [= 1\tfrac{2}{25}].$$

This is an example of the Egyptian method of stating fractions preceding by some three or four centuries the exposition of the same method in the papyrus of Akhmīm.

MARINUS of Neapolis, the pupil and biographer of Proclus, wrote a commentary or rather introduction to the *Data* of Euclid.[1] It is mainly taken up with a discussion of the question *τί τὸ δεδομένον*, what is meant by *given*? There were apparently many different definitions of the term *given* by earlier and later authorities. Of those who tried to define it in the simplest way by means of a single *differentia*, three are mentioned by name. Apollonius in his work on *νεύσεις* and his 'general treatise' (presumably that on elementary geometry) described the *given* as *assigned* or *fixed* (*τεταγμένον*), Diodorus called it *known* (*γνώριμον*); others regarded it as *rational* (*ῥητόν*) and Ptolemy is classed with these, rather oddly, because 'he called those things given the measure of which is given either exactly or approximately'. Others

[1] See Heiberg and Menge's Euclid, vol. vi, pp. 234–56.

combined two of these ideas and called it *assigned* or *fixed* and *procurable* or capable of being found (πόριμον); others 'fixed and known', and a third class 'known and procurable'. These various views are then discussed at length.

DOMNINUS of Larissa, a pupil of Syrianus at the same time as Proclus, wrote a *Manual of Introductory Arithmetic* ἐγχειρίδιον ἀριθμητικῆς εἰσαγωγῆς, which was edited by Boissonade[1] and is the subject of two articles by Tannery,[2] who also left a translation of it, with prolegomena, which has since been published.[3] It is a sketch of the elements of the theory of numbers, very concise and well arranged, and is interesting because it indicates a serious attempt at a reaction against the *Introductio arithmetica* of Nicomachus and a return to the doctrine of Euclid. Besides Euclid, Nicomachus and Theon of Smyrna, Domninus seems to have used another source, now lost, which was also drawn upon by Iamblichus. At the end of this work Domninus foreshadows a more complete treatise on the theory of numbers under the title *Elements of Arithmetic* (ἀριθμητικὴ στοιχείωσις), but whether this was ever written or not we do not know. Another tract attributed to Domninus πῶς ἔστι λόγον ἐκ λόγου ἀφελεῖν (how a ratio can be taken out of a ratio) has been published with a translation by Ruelle[4]; if it is not by Domninus, it probably belongs to the same period.

A most honourable place in our history must be reserved for SIMPLICIUS, who has been rightly called 'the excellent Simplicius, the Aristotle-commentator, to whom the world can never be grateful enough for the preservation of the fragments of Parmenides, Empedocles, Anaxagoras, Melissus, Theophrastus and others' (v. Wilamowitz-Möllendorff). He lived in the first half of the sixth century and was a pupil, first of Ammonius of Alexandria, and then of Damascius, the last head of the Platonic school at Athens. When in the year 529 the Emperor Justinian, in his zeal to eradicate paganism, issued an edict forbidding the teaching of philo-

[1] *Anecdota Graeca*, vol. iv, pp. 413–29.
[2] *Mémoires scientifiques*, vol. ii, nos. 35, 40.
[3] *Revue des études grecques*, 1906, pp. 359–82; *Mémoires scientifiques*, vol. iii, pp. 256–81.
[4] *Revue de Philologie*, 1883, p. 83 sq.

sophy at Athens, the last members of the school, including Damascius and Simplicius, migrated to Persia, but returned about 533 to Athens, where Simplicius continued to teach for some time though the school remained closed.

Extracts from Eudemus.

To Simplicius we owe two long extracts of capital importance for the history of mathematics and astronomy. The first is his account, based upon and to a large extent quoted textually from Eudemus's *History of Geometry*, of the attempt by Antiphon to square the circle and of the quadratures of lunes by Hippocrates of Chios. It is contained in Simplicius's commentary on Aristotle's *Physics*,[1] and has been the subject of a considerable literature extending from 1870, the date when Bretschneider first called attention to it, to the latest critical edition with translation and notes by Rudio (Teubner, 1907). It has already been discussed (vol. i, pp. 183–99).

The second, and not less important, of the two passages is that containing the elaborate and detailed account of the system of concentric spheres, as first invented by Eudoxus for explaining the apparent motion of the sun, moon, and planets, and of the modifications made by Callippus and Aristotle. It is contained in the commentary on Aristotle's *De caelo*[2]; Simplicius quotes largely from Sosigenes the Peripatetic (second century A.D.), observing that he in his turn drew from Eudemus, who dealt with the subject in the second book of his *History of Astronomy*. It is this passage of Simplicius which, along with a passage in Aristotle's *Metaphysics*,[3] enabled Schiaparelli to reconstruct Eudoxus's system (see vol. i, pp. 329–34). Nor must it be forgotten that it is in Simplicius's commentary on the *Physics*[4] that the extract from Geminus's summary of the *Meteorologica* of Posidonius occurs which was used by Schiaparelli to support his view that it was Heraclides of Pontus, not Aristarchus of Samos, who first propounded the heliocentric hypothesis.

Simplicius also wrote a commentary on Euclid's *Elements*, Book I, from which an-Nairīzī, the Arabian commentator,

[1] Simpl. *in Phys.*, pp. 54–69, ed. Diels.
[2] Simpl. on Arist. *De caelo*, p. 488. 18–24 and pp. 493–506, ed. Heiberg.
[3] *Metaph.* Λ. 8, 1073 b 17–1074 a 14.
[4] Simpl. *in Phys.*, pp. 291–2, ed. Diels.

made valuable extracts, including the account of the attempt of 'Aganis' to prove the parallel-postulate (see pp. 228-30 above).

Contemporary with Simplicius, or somewhat earlier, was EUTOCIUS, the commentator on Archimedes and Apollonius. As he dedicated the commentary on Book I *On the Sphere and Cylinder* to Ammonius (a pupil of Proclus and teacher of Simplicius), who can hardly have been alive after A.D. 510, Eutocius was probably born about A.D. 480. His date used to be put some fifty years later because, at the end of the commentaries on Book II *On the Sphere and Cylinder* and on the *Measurement of a Circle*, there is a note to the effect that 'the edition was revised by Isidorus of Miletus, the mechanical engineer, *our teacher*'. But, in view of the relation to Ammonius, it is impossible that Eutocius can have been a pupil of Isidorus, who was younger than Anthemius of Tralles, the architect of Saint Sophia at Constantinople in 532, whose work was continued by Isidorus after Anthemius's death about A.D. 534. Moreover, it was to Anthemius that Eutocius dedicated, separately, the commentaries on the first four Books of Apollonius's *Conics*, addressing Anthemius as 'my dear friend'. Hence we conclude that Eutocius was an elder contemporary of Anthemius, and that the reference to Isidorus is by an editor of Eutocius's commentaries who was a pupil of Isidorus. For a like reason, the reference in the commentary on Book II *On the Sphere and Cylinder*[1] to a διαβήτης invented by Isidorus 'our teacher' for drawing a parabola must be considered to be an interpolation by the same editor.

Eutocius's commentaries on Archimedes apparently extended only to the three works, *On the Sphere and Cylinder*, *Measurement of a Circle* and *Plane Equilibriums*, and those on the *Conics* of Apollonius to the first four Books only. We are indebted to these commentaries for many valuable historical notes. Those deserving special mention here are (1) the account of the solutions of the problem of the duplication of the cube, or the finding of two mean proportionals, by 'Plato', Heron, Philon, Apollonius, Diocles, Pappus, Sporus, Menaechmus, Archytas, Eratosthenes, Nicomedes, (2) the fragment discovered by Eutocius himself containing the

[1] Archimedes, ed. Heiberg, vol. iii, p. 84. 8-11.

missing solution, promised by Archimedes in *On the Sphere and Cylinder*, II. 4, of the auxiliary problem amounting to the solution by means of conics of the cubic equation $(a-x)x^2 = bc^2$, (3) the solutions (*a*) by Diocles of the original problem of II. 4 without bringing in the cubic, (*b*) by Dionysodorus of the auxiliary cubic equation.

ANTHEMIUS of Tralles, the architect, mentioned above, was himself an able mathematician, as is seen from a fragment of a work of his, *On Burning-mirrors*. This is a document of considerable importance for the history of conic sections. Originally edited by L. Dupuy in 1777, it was reprinted in Westermann's *Παραδοξογράφοι* (*Scriptores rerum mirabilium Graeci*), 1839, pp. 149–58. The first and third portions of the fragment are those which interest us.[1] The first gives a solution of the problem, To contrive that a ray of the sun (admitted through a small hole or window) shall fall in a given spot, without moving away at any hour and season. This is contrived by constructing an elliptical mirror one focus of which is at the point where the ray of the sun is admitted while the other is at the point to which the ray is required to be reflected at all times. Let B be the hole, A the point to which reflection must always take place, BA being in the meridian and parallel to the horizon. Let BC be at right angles to BA, so that CB is an equinoctial ray; and let BD be the ray at the summer solstice, BE a winter ray.

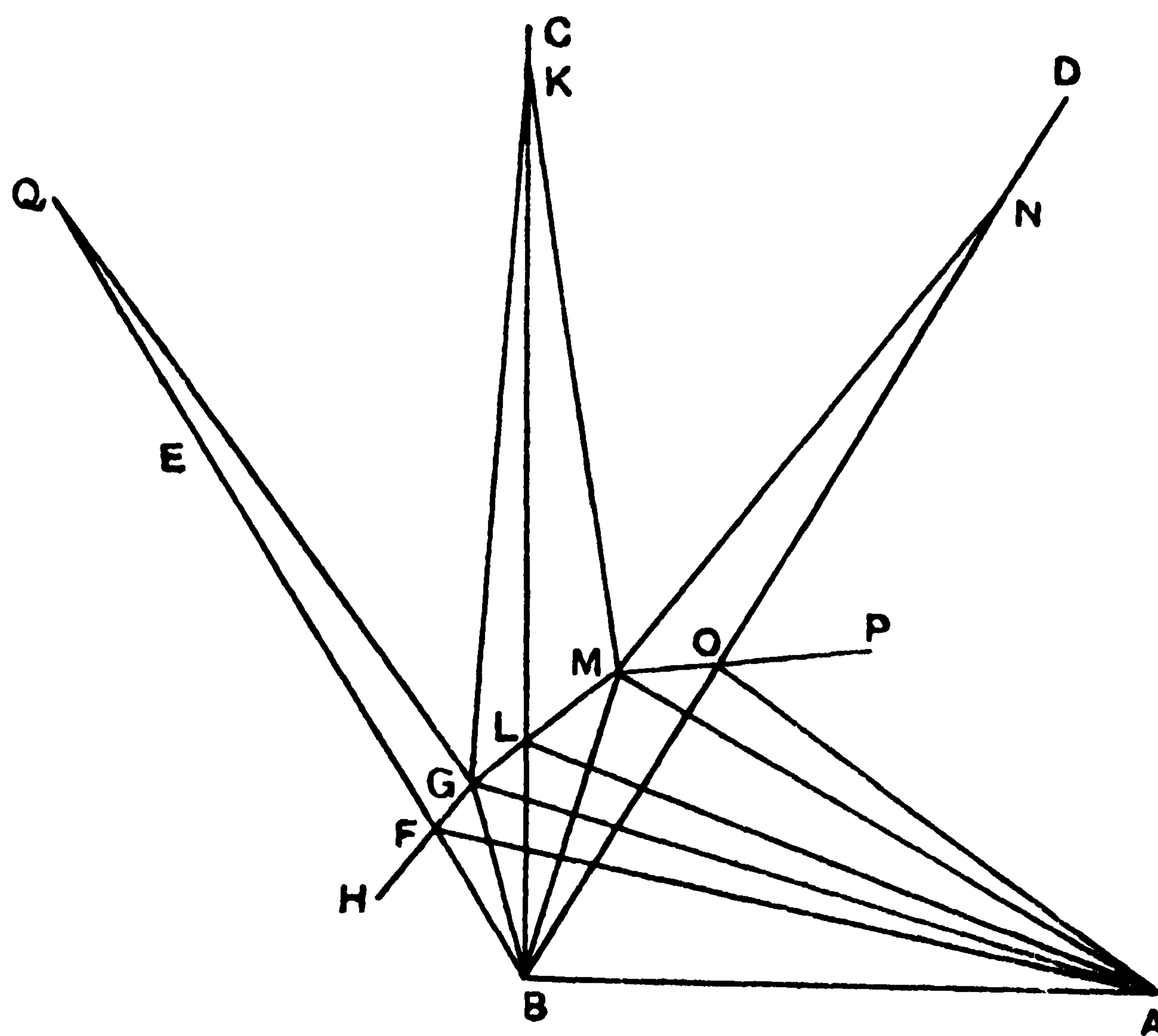
Take F at a convenient distance on BE and measure FQ equal to FA. Draw HFG through F bisecting the angle AFQ, and let BG be the straight line bisecting the angle EBC between the winter and the equinoctial rays. Then clearly, since FG bisects the angle QFA, if we have a plane mirror in the position HFG, the ray BFE entering at B will be reflected to A.

To get the equinoctial ray similarly reflected to A, join GA, and with G as centre and GA as radius draw a circle meeting BC in K. Bisect the angle KGA by the straight line GLM meeting BK in L and terminated at M, a point on the bisector of the angle CBD. Then LM bisects the angle KLA also, and $KL = LA$, and $KM = MA$. If then GLM is a plane mirror, the ray BL will be reflected to A.

[1] See *Bibliotheca mathematica*, vii$_3$, 1907, pp. 225–33.

By taking the point N on BD such that $MN = MA$, and bisecting the angle NMA by the straight line MOP meeting BD in O, we find that, if MOP is a plane mirror, the ray BO is reflected to A.

Similarly, by continually bisecting angles and making more mirrors, we can get any number of other points of impact. Making the mirrors so short as to form a continuous curve, we get the curve containing all points such that the sum of the distances of each of them from A and B is constant and equal to BQ, BK, or BN. 'If then', says Anthemius, 'we stretch a string passed



round the points A, B, and through the first point taken on the rays which are to be reflected, the said curve will be described, which is part of the so-called "ellipse", with reference to which (i.e. by the revolution of which round BA) the surface of impact of the said mirror has to be constructed.'

We have here apparently the first mention of the construction of an ellipse by means of a string stretched tight round the foci. Anthemius's construction depends upon two propositions proved by Apollonius (1) that the sum of the focal distances of any point on the ellipse is constant, (2) that the focal distances of any point make equal angles with the tangent at that point, and also (3) upon a proposition not found in Apollonius, namely that the straight line joining

the focus to the intersection of two tangents bisects the angle between the straight lines joining the focus to the two points of contact respectively.

In the third portion of the fragment Anthemius proves that parallel rays can be reflected to one single point from a parabolic mirror of which the point is the focus. The *directrix* is used in the construction, which follows, *mutatis mutandis*, the same course as the above construction in the case of the ellipse.

As to the supposition of Heiberg that Anthemius may also be the author of the *Fragmentum mathematicum Bobiense*, see above (p. 203).

The Papyrus of Akhmīm.

Next in chronological order must apparently be placed the Papyrus of Akhmīm, a manual of calculation written in Greek, which was found in the metropolis of Akhmīm, the ancient Panopolis, and is now in the Musée du Gizeh. It was edited by J. Baillet[1] in 1892. According to the editor, it was written between the sixth and ninth centuries by a Christian. It is interesting because it preserves the Egyptian method of reckoning, with proper fractions written as the sum of primary fractions or submultiples, a method which survived alongside the Greek and was employed, and even exclusively taught, in the East. The advantage of this papyrus, as compared with Ahmes's, is that we can gather the formulae used for the decomposition of ordinary proper fractions into sums of submultiples. The formulae for decomposing a proper fraction into the sum of two submultiples may be shown thus:

(1) $$\frac{a}{bc} = \frac{1}{c \cdot \frac{b+c}{a}} + \frac{1}{b \cdot \frac{b+c}{a}}.$$

Examples $\frac{2}{11} = \frac{1}{6}\frac{1}{66}$, $\frac{3}{110} = \frac{1}{70}\frac{1}{77}$, $\frac{18}{323} = \frac{1}{34}\frac{1}{38}$.

(2) $$\frac{a}{bc} = \frac{1}{c \cdot \frac{b+mc}{a}} + \frac{1}{b \cdot \frac{b+mc}{a} \cdot \frac{1}{m}}.$$

[1] *Mémoires publiés par les membres de la Mission archéologique française au Caire*, vol. ix, part 1, pp. 1–89.

$$\text{Ex. } \frac{7}{176} = \frac{1}{11\left(\frac{16+3\,.\,11}{7}\right)} + \frac{1}{16\left(\frac{16+3\,.\,11}{7}\right)\frac{1}{3}} = \frac{1}{77} + \frac{3}{112};$$

$$\text{and again } \frac{3}{112} = \frac{1}{7\left(\frac{16+2\,.\,7}{3}\right)} + \frac{1}{16\left(\frac{16+2\,.\,7}{3}\right)\frac{1}{2}} = \frac{1}{70}\,\frac{1}{80}$$

$$(3)\quad \frac{a}{cdf} = \frac{1}{c\,.\,\frac{cd+df}{a}} + \frac{1}{f\,.\,\frac{cd+df}{a}}.$$

Example.

$$\frac{28}{1320} = \frac{28}{10\,.\,12\,.\,11} = \frac{1}{10\,.\,\frac{120+132}{28}} + \frac{1}{11\,.\,\frac{120+132}{28}} = \frac{1}{90}\,\frac{1}{99}.$$

The object is, of course, to choose the factors of the denominator, and the multiplier m in (2), in such a way as to make the two denominators on the right-hand side integral.

When the fraction has to be decomposed into a sum of three or more submultiples, we take out an obvious submultiple first, then if necessary a second, until one of the formulae will separate what remains into two submultiples. Or we take out a part which is not a submultiple but which can be divided into two submultiples by one of the formulae.

For example, to decompose $\frac{31}{616}$. The factors of 616 are 8.77 or 7.88. Take out $\frac{1}{88}$, and $\frac{31}{616} = \frac{1}{88}\,\frac{24}{616} = \frac{1}{88}\,\frac{3}{77} = \frac{1}{88}\,\frac{1}{77}\,\frac{2}{77}$; and $\frac{2}{77} = \frac{1}{63}\,\frac{1}{99}$ by formula (1), so that $\frac{31}{616} = \frac{1}{63}\,\frac{1}{77}\,\frac{1}{88}\,\frac{1}{99}$.

Take $\frac{239}{6460}$. The factors of 6460 are 85.76 or 95.68. Take out $\frac{1}{85}$, and $\frac{239}{6460} = \frac{1}{85}\,\frac{163}{6460}$. Again take out $\frac{1}{95}$, and we have $\frac{1}{85}\,\frac{1}{95}\,\frac{95}{6460}$ or $\frac{1}{85}\,\frac{1}{95}\,\frac{1}{68}$. The actual problem here is to find $\frac{1}{323}$rd of $11\frac{1}{2}\,\frac{1}{3}\,\frac{1}{10}\,\frac{1}{60}$, which latter expression reduces to $\frac{1}{20}\,.\,239$.

The sort of problems solved in the book are (1) the division of a number into parts in the proportion of certain given numbers, (2) the solution of simple equations such as this: From a certain treasure we take away $\frac{1}{13}$th, then from the remainder $\frac{1}{17}$th of that remainder, and we find 150 units left; what was the treasure? $\left[\left\{x - \frac{1}{a}x - \frac{1}{b}\left(x - \frac{1}{a}x\right) - \ldots\right\} = R.\right]$

(3) subtractions such as: From $\frac{2}{3}$ subtract $\frac{1}{10}\,\frac{1}{11}\,\frac{1}{20}\,\frac{1}{22}\,\frac{1}{30}\,\frac{1}{33}\,\frac{1}{40}\,\frac{1}{44}\,\frac{1}{50}\,\frac{1}{55}\,\frac{1}{60}\,\frac{1}{66}\,\frac{1}{70}\,\frac{1}{77}\,\frac{1}{88}\,\frac{1}{90}\,\frac{1}{99}\,\frac{1}{100}\,\frac{1}{110}$. Answer, $\frac{1}{10}\,\frac{1}{50}$.

The book ends with long tables of results obtained (1) by multiplying successive numbers, tens, hundreds and thousands up to 10,000 by $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, &c., up to $\frac{1}{10}$, (2) by multiplying all the successive numbers 1, 2, 3 ... n by $\frac{1}{n}$, where n is successively 11, 12, ... and 20; the results are all arranged as the sums of integers and submultiples.

The *Geodaesia* of a Byzantine author formerly called, without any authority, 'Heron the Younger' was translated into Latin by Barocius in 1572, and the Greek text was published with a French translation by Vincent.[1] The place of the author's observations was the hippodrome at Constantinople, and the date apparently about 938. The treatise was modelled on Heron of Alexandria, especially the *Dioptra*, while some measurements of areas and volumes are taken from the *Metrica*.

MICHAEL PSELLUS lived in the latter part of the eleventh century, since his latest work bears the date 1092. Though he was called 'first of philosophers', it cannot be said that what survives of his mathematics suits this title. Xylander edited in 1556 the Greek text, with a Latin translation, of a book purporting to be by Psellus on the four mathematical sciences, arithmetic, music, geometry and astronomy, but it is evident that it cannot be entirely Psellus's own work, since the astronomical portion is dated 1008. The arithmetic contains no more than the names and classification of numbers and ratios. The geometry has the extraordinary remark that, while opinions differed as to how to find the area of a circle, the method which found most favour was to take the area as the geometric mean between the inscribed and circumscribed squares; this gives $\pi = \sqrt{8} = 2{\cdot}8284271$! The only thing of Psellus which has any value for us is the letter published by Tannery in his edition of Diophantus.[2] In this letter Psellus says that both Diophantus and Anatolius (Bishop of Laodicea about A.D. 280) wrote on the Egyptian method of reckoning,

[1] *Notices et extraits*, xix, pt. 2, Paris, 1858.
[2] Diophantus, vol. ii, pp. 37–42.

and that Anatolius's account, which was different and more succinct, was dedicated to Diophantus (this enables us to determine Diophantus's date approximately). He also notes the difference between the Diophantine and Egyptian names for the successive powers of ἀριθμός: the next power after the fourth (δυναμοδύναμις $= x^4$), i.e. x^5, the Egyptians called 'the first undescribed' (ἄλογος πρῶτος) or the 'fifth number'; the sixth, x^6, they apparently (like Diophantus) called the cube-cube; but with them the seventh, x^7, was the 'second undescribed' or the 'seventh number', the eighth (x^8) was the 'quadruple square' (τετραπλῆ δύναμις), the ninth (x^9) the 'extended cube' (κύβος ἐξελικτός). Tannery conjectures that all these remarks were taken direct from an old commentary on Diophantus now lost, probably Hypatia's.

Georgius Pachymeres (1242–1310) was the author of a work on the Quadrivium (Σύνταγμα τῶν τεσσάρων μαθημάτων or Τετράβιβλον). The arithmetical portion contains, besides excerpts from Nicomachus and Euclid, a paraphrase of Diophantus, Book I, which Tannery published in his edition of Diophantus[1]; the musical section with part of the preface was published by Vincent,[2] and some fragments from Book IV by Martin in his edition of the *Astronomy* of Theon of Smyrna.

Maximus Planudes, a monk from Nicomedia, was the envoy of the Emperor Andronicus II at Venice in the year 1297, and lived probably from about 1260 to 1310. He wrote scholia on the first two Books of Diophantus, which are extant and are included in Tannery's edition of Diophantus.[3] They contain nothing of particular interest except a number of conspectuses of the working-out of problems of Diophantus written in Diophantus's own notation but with steps in separate lines, and with abbreviations on the left of words indicating the operations (e.g. ἔκθ. = ἔκθεσις, τετρ. = τετραγωνισμός, σύνθ. = σύνθεσις, &c.); the result is to make the work almost as easy to follow as it is in our notation.

Another work of Planudes is called Ψηφοφορία κατ' Ἰνδούς, or *Arithmetic after the Indian method*, and was edited as *Das*

[1] Diophantus, vol. ii, pp. 78–122.
[2] *Notices et extraits*, xvii, 1858, pp. 362–533.
[3] Diophantus, vol. ii, pp. 125–255.

Rechenbuch des Maximus Planudes in Greek by Gerhardt (Halle, 1865) and in a German translation by H. Waeschke (Halle, 1878). There was, however, an earlier book under the similar title Ἀρχὴ τῆς μεγάλης καὶ Ἰνδικῆς ψηφιφορίας (*sic*), written in 1252, which is extant in the Paris MS. Suppl. Gr. 387; and Planudes seems to have raided this work. He begins with an account of the symbols which, he says, were

'invented by certain distinguished astronomers for the most convenient and accurate expression of numbers. There are nine of these symbols (our 1, 2, 3, 4, 5, 6, 7, 8, 9), to which is added another called *Tzifra* (cypher), written 0 and denoting zero. The nine signs as well as this one are Indian.'

But this is, of course, not the first occurrence of the Indian numerals; they were known, except the zero, to Gerbert (Pope Sylvester II) in the tenth century, and were used by Leonardo of Pisa in his *Liber abaci* (written in 1202 and rewritten in 1228). Planudes used the Persian form of the numerals, differing in this from the writer of the treatise of 1252 referred to, who used the form then current in Italy. It scarcely belongs to Greek mathematics to give an account of Planudes's methods of subtraction, multiplication, &c.

Extraction of the square root.

As regards the extraction of the square root, he claims to have invented a method different from the Indian method and from that of Theon. It does not appear, however, that there was anything new about it. Let us try to see in what the supposed new method consisted.

Planudes describes fully the method of extracting the square root of a number with several digits, a method which is essentially the same as ours. This appears to be what he refers to later on as 'the Indian method'. Then he tells us how to find a first approximation to the root when the number is not a complete square.

'Take the square root of the next lower actual square number, and double it: then, from the number the square root of which is required, subtract the next lower square number so found, and to the remainder (as numerator) give as denominator the double of the square root already found.'

The example given is $\sqrt{(18)}$. Since $4^2 = 16$ is the next lower square, the approximate square root is $4 + \frac{2}{2 \cdot 4}$ or $4\frac{1}{4}$. The formula used is, therefore, $\sqrt{(a^2+b)} = a + \frac{b}{2a}$ approximately. (An example in larger numbers is

$$\sqrt{(1690196789)} = 41112 + \tfrac{245}{82224} \text{ approximately.)}$$

Planudes multiplies $4\frac{1}{4}$ by itself and obtains $18\frac{1}{16}$, which shows that the value $4\frac{1}{4}$ is not accurate. He adds that he will explain later a method which is more exact and nearer the truth, a method 'which I claim as a discovery made by me with the help of God'. Then, coming to the method which he claims to have discovered, Planudes applies it to $\sqrt{6}$. The object is to develop this in units and sexagesimal fractions. Planudes begins by multiplying the 6 by 3600, making 21600 second-sixtieths, and finds the square root of 21600 to lie between 146 and 147. Writing the 146′ as 2 26′, he proceeds to find the rest of the approximate square root (2 26′ 58″ 9‴) by the same procedure as that used by Theon in extracting the square root of 4500 and 2 28′ respectively. The difference is that in neither of the latter cases does Theon multiply by 3600 so as to reduce the units to second-sixtieths, but he begins by taking the approximate square root of 2, viz. 1, just as he does that of 4500 (viz. 67). It is, then, the multiplication by 3600, or the reduction to second-sixtieths to start with, that constitutes the difference from Theon's method, and this must therefore be what Planudes takes credit for as a new discovery. In such a case as $\sqrt{(2\ 28')}$ or $\sqrt{3}$, Theon's method has the inconvenience that the number of *minutes* in the second term (34′ in the one case and 43′ in the other) cannot be found without some trouble, a difficulty which is avoided by Planudes's expedient. Therefore the method of Planudes had its advantage in such a case. But the discovery was not new. For it will be remembered that Ptolemy (and doubtless Hipparchus before him) expressed the chord in a circle subtending an angle of 120° at the centre (in terms of 120th parts of the diameter) as $103^p\ 55'\ 23''$, which indicates that the first step in calculating $\sqrt{3}$ was to multiply it by 3600, making 10800, the nearest square below which is 103^2 (= 10609). In

the scholia to Eucl., Book X, the same method is applied. Examples have been given above (vol. i, p. 63). The supposed new method was therefore not only already known to the scholiast, but goes back, in all probability, to Hipparchus.

Two problems.

Two problems given at the end of the Manual of Planudes are worth mention. The first is stated thus: 'A certain man finding himself at the point of death had his desk or safe brought to him and divided his money among his sons with the following words, "I wish to divide my money equally between my sons: the first shall have one piece and $\frac{1}{7}$th of the rest, the second 2 and $\frac{1}{7}$th of the remainder, the third 3 and $\frac{1}{7}$th of the remainder." At this point the father died without getting to the end either of his money or the enumeration of his sons. I wish to know how many sons he had and how much money.' The solution is given as $(n-1)^2$ for the number of coins to be divided and $(n-1)$ for the number of his sons; or rather this is how it might be stated, for Planudes takes $n = 7$ arbitrarily. Comparing the shares of the first two we must clearly have

$$1 + \frac{1}{n}(x-1) = 2 + \frac{1}{n}\{x-(1 + \frac{x-1}{n} + 2)\},$$

which gives $x = (n-1)^2$; therefore each of $(n-1)$ sons received $(n-1)$.

The other problem is one which we have already met with, that of finding two rectangles of equal perimeter such that the area of one of them is a given multiple of the area of the other. If n is the given multiple, the rectangles are $(n^2-1,\ n^3-n^2)$ and $(n-1,\ n^3-n)$ respectively. Planudes states the solution correctly, but how he obtained it is not clear.

We find also in the Manual of Planudes the 'proof by nine' (i.e. by casting out nines), with a statement that it was discovered by the Indians and transmitted to us through the Arabs.

Manuel Moschopoulos, a pupil and friend of Maximus Planudes, lived apparently under the Emperor Andronicus II (1282–1328) and perhaps under his predecessor Michael VIII (1261–82) also. A man of wide learning, he wrote (at the

instance of Nicolas Rhabdas, presently to be mentioned) a treatise on *magic squares*; he showed, that is, how the numbers 1, 2, 3 ... n^2 could be placed in the n^2 compartments of a square, divided like a chess-board into n^2 small squares, in such a way that the sum of the numbers in each horizontal and each vertical row of compartments, as well as in the rows forming the diagonals, is always the same, namely $\frac{1}{2}n\ (n^2+1)$. Moschopoulos gives rules of procedure for the cases in which $n = 2m+1$ and $n = 4m$ respectively, and these only, in the treatise as we have it; he promises to give the case where $n = 4m+2$ also, but does not seem to have done so, as the two manuscripts used by Tannery have after the first two cases the words τέλος τοῦ αὐτοῦ. The treatise was translated by De la Hire,[1] edited by S. Günther,[2] and finally edited in an improved text with translation by Tannery.[3]

The work of Moschopoulos was dedicated to Nicolas Artavasdus, called RHABDAS, a person of some importance in the history of Greek arithmetic. He edited, with some additions of his own, the Manual of Planudes; this edition exists in the Paris MS. 2428. But he is also the author of two letters which have been edited by Tannery in the Greek text with French translation.[4] The date of Rhabdas is roughly fixed by means of a calculation of the date of Easter 'in the current year' contained in one of the letters, which shows that its date was 1341. It is remarkable that each of the two letters has a preface which (except for the words τὴν δήλωσιν τῶν ἐν τοῖς ἀριθμοῖς ζητημάτων and the name or title of the person to whom it is addressed) copies word for word the first thirteen lines of the preface to Diophantus's *Arithmetica*, a piece of plagiarism which, if it does not say much for the literary resource of Rhabdas, may indicate that he had studied Diophantus. The first of the two letters has the heading 'A concise and most clear exposition of the science of calculation written at Byzantium of Constantine, by Nicolas Artavasdus

[1] *Mém. de l'Acad. Royale des Sciences*, 1705.

[2] *Vermischte Untersuchungen zur Gesch. d. Math.*, Leipzig, 1876.

[3] 'Le traité de Manuel Moschopoulos sur les carrés magiques' in *Annuaire de l'Association pour l'encouragement des études grecques*, xx, 1886, pp. 88–118.

[4] 'Notices sur les deux lettres arithmétiques de Nicolas Rhabdas' in *Notices et extraits des manuscrits de la Bibliothèque Nationale*, xxxii, pt. 1, 1886, pp. 121–252.

for 20, stretch out the forefinger straight and vertical, keep fingers 3, 4, 5 together but separate from it and inclined slightly to the palm; in this position touch the forefinger with the thumb;

„ 30, join the tips of the forefinger and thumb;

„ 40, place the thumb on the knuckle of the forefinger behind, making a figure like the letter Γ;

„ 50, make a like figure with the thumb on the knuckle of the forefinger *inside*;

„ 60, place the thumb inside the forefinger as for 50 and bring the forefinger down over the thumb, touching the ball of it;

„ 70, rest the forefinger round the tip of the thumb, making a curve like a spiral;

„ 80, fingers 3, 4, 5 being held together and inclined at an angle to the palm, put the thumb across the palm to touch the third phalanx of the middle finger (3) and in this position bend the forefinger above the first joint of the thumb;

„ 90, close the forefinger only as completely as possible.

(*d*) The same operations on the *right hand* give the *hundreds*, from 100 to 900.

The first letter also contains tables for addition and subtraction and for multiplication and division; as these are said to be the 'invention of Palamedes', we must suppose that such tables were in use from a remote antiquity. Lastly, the first letter contains a statement which, though applied to particular numbers, expresses a theorem to the effect that

$$(a_0 + 10a_1 + \dots + 10^m a_m)(b_0 + 10b_1 + \dots + 10^n b_n) \text{ is not } > 10^{m+n+2},$$

where $a_0, a_1 \dots b_0, b_1 \dots$ are any numbers from 0 to 9.

In the second letter of Rhabdas we find simple algebraical problems of the same sort as those of the *Anthologia Graeca* and the Papyrus of Akhmīm. Thus there are five problems leading to equations of the type

$$\frac{x}{m} + \frac{x}{n} + \dots = a.$$

Rhabdas solves the equation $\frac{x}{m} + \frac{x}{n} = a$, practically as we should, by multiplying up to get rid of fractions, whence he obtains $x = mna/(m+n)$. Again he solves the simultaneous equations $x+y = a$, $mx = ny$; also the pair of equations

$$x + \frac{y}{m} = y + \frac{x}{n} = a.$$

Of course, m, n, a... have particular numerical values in all cases.

Rhabdas's Rule for approximating to the square root of a non-square number.

We find in Rhabdas the equivalent of the Heronian formula for the approximation to the square root of a non-square number $A = a^2 + b$, namely

$$\alpha = a + \frac{b}{2a};$$

he further observes that, if α be an approximation by excess, then $\alpha_1 = A/\alpha$ is an approximation by defect, and $\frac{1}{2}(\alpha+\alpha_1)$ is an approximation nearer than either. This last form is of course exactly Heron's formula $\alpha = \frac{1}{2}\left(a + \frac{A}{a}\right)$. The formula was also known to Barlaam (presently to be mentioned), who also indicates that the procedure can be continued indefinitely.

It should here be added that there is interesting evidence of the Greek methods of approximating to square roots in two documents published by Heiberg in 1899.[1] The first of these documents (from a manuscript of the fifteenth century at Vienna) gives the approximate square root of certain non-square numbers from 2 to 147 in integers and proper fractions. The numerals are the Greek alphabetic numerals, but they are given place-value like our numerals: thus $\alpha\eta = 18$, $\alpha\delta\zeta = 147$, $\frac{\alpha\gamma}{\beta\eta} = \frac{13}{28}$, and so on: 0 is indicated by ϥ or, sometimes, by •. All these square roots, such as $\sqrt{(21)} = 4\frac{21}{36}$, $\sqrt{(35)} = 5\frac{11}{12}$, $\sqrt{(112)} = 10\frac{49}{84}$, and so on, can be obtained (either exactly or, in a few cases, by neglecting or adding a small fraction in the

[1] 'Byzantinische Analekten' in *Abh. zur Gesch. d. Math.* ix. Heft, 1899, pp. 163 sqq.

numerator of the fractional part of the root) in one or other of the following ways:

(1) by taking the nearest square to the given number A, say a^2, and using the Heronian formulae

$$\alpha_1 = \tfrac{1}{2}\left(a + \frac{A}{a}\right), \quad \alpha_2 = \tfrac{1}{2}\left(\alpha_1 + \frac{A}{\alpha_1}\right), \text{ \&c.};$$

(2) by using one or other of the following approximations, where

$$a^2 < A < (a+1)^2, \text{ and } A = a^2 + b = (a+1)^2 - c,$$

namely,

$$a + \frac{b}{2a}, \quad a + \cfrac{b}{2a + \cfrac{b}{2a}},$$

$$(a+1) - \frac{c}{2(a+1)}, \quad (a+1) - \cfrac{c}{2(a+1) - \cfrac{c}{2(a+1)}},$$

or a combination of two of these with

(3) the formula that, if $\frac{a}{b} < \frac{c}{d}$, then

$$\frac{a}{b} < \frac{ma + nc}{mb + nd} < \frac{c}{d}.$$

It is clear that it is impossible to deny to the Greeks the knowledge of these simple formulae.

Three more names and we have done.

IOANNES PEDIASIMUS, also called Galenus, was Keeper of the Seal to the Patriarch of Constantinople in the reign of Andronicus III (1328–41). Besides literary works of his, some notes on difficult points in arithmetic and a treatise on the duplication of the cube by him are said to exist in manuscripts. His *Geometry*, which was edited by Friedlein in 1866, follows very closely the mensuration of Heron.

BARLAAM, a monk of Calabria, was abbot at Constantinople and later Bishop of Geraci in the neighbourhood of Naples; he died in 1348. He wrote, in Greek, arithmetical demonstrations of propositions in Euclid, Book II,[1] and a *Logistic* in six Books, a laborious manual of calculation in whole numbers,

[1] Edited with Latin translation by Dasypodius in 1564, and included in Heiberg and Menge's Euclid, vol. v, *ad fin.*

ordinary fractions and sexagesimal fractions (printed at Strassburg in 1592 and at Paris in 1600). Barlaam, as we have seen, knew the Heronian formulae for finding successive approximations to square roots, and was aware that they could be indefinitely continued.

ISAAC ARGYRUS, a monk, who lived before 1368, was one of a number of Byzantine translators of Persian astronomical works. In mathematics he wrote a *Geodaesia* and scholia to the first six Books of Euclid's *Elements*. The former is contained in the Paris MS. 2428 and is called 'a method of geodesy or the measurement of surfaces, exact and shortened'; the introductory letter addressed to one Colybos is followed by a compilation of extracts from the *Geometrica* and *Stereometrica* of Heron. He is apparently the author of some further additions to Rhabdas's revision of the Manual of Planudes contained in the same manuscript. A short tract of his 'On the discovery of the square roots of non-rational square numbers' is mentioned as contained in two other manuscripts at Venice and Rome respectively (Codd. Marcianus Gr. 333 and Vaticanus Gr. 1058), where it is followed by a table of the square roots of all numbers from 1 to 102 in sexagesimal fractions (e.g. $\sqrt{2} = 1\ 24'\ 51''\ 48'''$, $\sqrt{3} = 1\ 43'\ 56''\ 0'''$).[1]

[1] Heiberg, 'Byzantinische Analekten', in *Abh. zur Gesch. d. Math.* ix, pp. 169–70.

APPENDIX

On Archimedes's proof of the subtangent-property of a spiral.

THE section of the treatise *On Spirals* from Prop. 3 to Prop. 20 is an elaborate series of propositions leading up to the proof of the fundamental property of the subtangent corresponding to the tangent at any point on any turn of the spiral. Libri, doubtless with this series of propositions in mind, remarks (*Histoire des sciences mathématiques en Italie*, i, p. 31) that 'Après vingt siècles de travaux et de découvertes, les intelligences les plus puissantes viennent encore échouer contre la synthèse difficile du *Traité des Spirales* d'Archimède.' There is no foundation for this statement, which seems to be a too hasty generalization from a dictum, apparently of Fontenelle, in the *Histoire de l'Académie des Sciences pour l'année 1704* (p. 42 of the edition of 1722), who says of the proofs of Archimedes in the work *On Spirals*: 'Elles sont si longues, et si difficiles à embrasser, que, comme on l'a pû voir dans la Préface de l'Analyse des Infiniment petits, M. Bouillaud a avoué qu'il ne les avoit jamais bien entendues, et que Viète les a injustement soupçonnées de paralogisme, parce qu'il n'avoit pû non plus parvenir à les bien entendre. Mais toutes les preuves qu'on peut donner de leur difficulté et de leur obscurité tournent à la gloire d'Archimède; car quelle vigueur d'esprit, quelle quantité de vûes différentes, quelle opiniâtreté de travail n'a-t-il pas fallu pour lier et pour disposer un raisonnement que quelques-uns de nos plus grands géomètres ne peuvent suivre, tout lié et tout disposé qu'il est?'

P. Tannery has observed[1] that, as a matter of fact, no mathematicians of real authority who have applied or extended Archimedes's methods (such men as Huygens, Pascal, Roberval and Fermat, who alone could have expressed an opinion worth having), have ever complained of the

[1] *Bulletin des sciences mathématiques*, 1895, Part i, pp. 265–71.

'obscurity' of Archimedes; while, as regards Vieta, he has shown that the statement quoted is based on an entire misapprehension, and that, so far from suspecting a fallacy in Archimedes's proofs, Vieta made a special study of the treatise *On Spirals* and had the greatest admiration for that work.

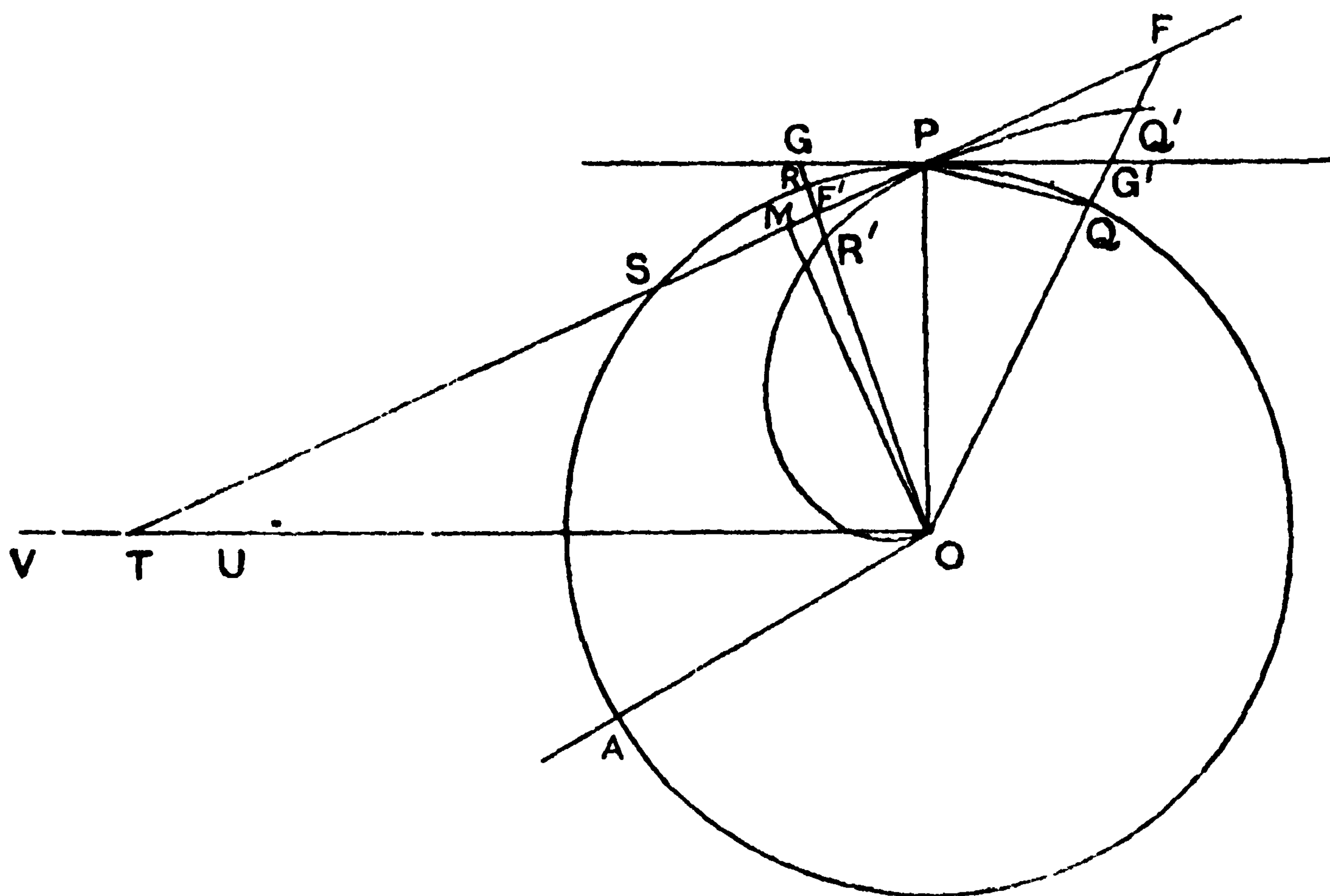
But, as in many cases in Greek geometry where the analysis is omitted or even (as Wallis was tempted to suppose) of set purpose hidden, the reading of the completed synthetical proof leaves a certain impression of mystery; for there is nothing in it to show *why* Archimedes should have taken precisely this line of argument, or how he evolved it. It is a fact that, as Pappus said, the subtangent-property can be established by purely 'plane' methods, without recourse to a 'solid' νεῦσις (whether actually solved or merely assumed capable of being solved). If, then, Archimedes chose the more difficult method which we actually find him employing, it is scarcely possible to assign any reason except his definite predilection for the form of proof by *reductio ad absurdum* based ultimately on his famous 'Lemma' or Axiom.

It seems worth while to re-examine the whole question of the discovery and proof of the property, and to see how Archimedes's argument compares with an easier 'plane' proof suggested by the figures of some of the very propositions proved by Archimedes in the treatise.

In the first place, we may be sure that the property was not discovered by the steps leading to the proof as it stands. I cannot but think that Archimedes divined the result by an argument corresponding to our use of the differential calculus for determining tangents. He must have considered the instantaneous direction of the motion of the point P describing the spiral, using for this purpose the parallelogram of velocities. The motion of P is compounded of two motions, one along OP and the other at right angles to it. Comparing the distances traversed in an instant of time in the two directions, we see that, corresponding to a small increase in the radius vector r, we have a small distance traversed perpendicularly to it, a tiny arc of a circle of radius r subtended by the angle representing the simultaneous small increase of the angle θ (AOP). Now r has a constant ratio to θ which we call a (when θ is the circular measure of the angle θ). Consequently

the small increases of r and θ are in that same ratio. Therefore what we call the tangent of the angle OPT is r/a, i.e. $OT/r = r/a$; and $OT = r^2/a$, or $r\theta$, that is, the arc of a circle of radius r subtended by the angle θ.

To *prove* this result Archimedes would doubtless begin by an *analysis* of the following sort. Having drawn OT perpendicular to OP and of length equal to the arc ASP, he had to prove that the straight line joining P to T is the tangent at P. He would evidently take the line of trying to show that, if *any* radius vector to the spiral is drawn, as OQ', on either side of OP, Q' is always on the side of TP towards O, or, if OQ' meets TP in F, OQ' is always less than OF. Suppose



that in the above figure OR' is any radius vector between OP and OS on the 'backward' side of OP, and that OR' meets the circle with radius OP in R, the tangent to it at P in G, the spiral in R', and TP in F'. We have to prove that R, R' lie on opposite sides of F', i.e. that $RR' > RF'$; and again, supposing that *any* radius vector OQ' on the 'forward' side of OP meets the circle with radius OP in Q, the spiral in Q' and TP produced in F, we have to prove that $QQ' < QF$.

Archimedes then had to prove that

(1) $$F'R : RO < RR' : RO, \text{ and}$$

(2) $$FQ : QO > QQ' : QO.$$

Now (1) is equivalent to

$$F'R : RO < (\text{arc } RP) : (\text{arc } ASP), \text{ since } RO = PO.$$

But (arc $ASP) = OT$, by hypothesis;

therefore it was necessary to prove, *alternando*, that

(3) $F'R:(\text{arc } RP) < RO:OT$, or $PO:OT$,

i.e. $< PM:MO$, where OM is perpendicular to SP.

Similarly, in order to satisfy (2), it was necessary to prove that

(4) $$FQ:(\text{arc } PQ) > PM:MO.$$

Now, as a matter of fact, (3) is *a fortiori* satisfied if

$$F'R:(\textit{chord } RP) < PM:MO;$$

but in the case of (4) we cannot substitute the *chord* PQ for the arc PQ, and we have to substitute PG', where G' is the point in which the tangent at P to the circle meets OQ produced; for of course $PG' > (\text{arc } PQ)$, so that (4) is *a fortiori* satisfied if

$$FQ:PG' > PM:MO.$$

FIG. 1.

It is remarkable that Archimedes uses for his proof of the two cases Prop. 8 and Prop. 7 respectively, and makes no use of Props 6 and 9, whereas the above argument points precisely to the use of the figures of the two latter propositions only.

For in the figure of Prop. 6 (Fig. 1), if OFP is any radius cutting AB in F, and if PB produced cuts OT, the parallel to AB through O, in H, it is obvious, by parallels, that

$$PF:(\text{chord } PB) = OP:PH.$$

Also PH becomes greater the farther P moves from B towards A, so that the ratio $PF:PB$ diminishes continually, while it is always less than $OB:BT$ (where BT is the tangent at B and meets OH in T), i.e. always less than $BM:MO$.

Hence the relation (3) is always satisfied for any point R' of the spiral on the 'backward' side of P.

But (3) is equivalent to (1), from which it follows that $F'R$ is always less than RR', so that R' always lies on the side of TP towards O.

Next, for the point Q' on the 'forward' side of the spiral from P, suppose that in the figure of Prop. 9 or Prop. 7 (Fig. 2) any radius OP of the circle meets AB *produced* in F, and

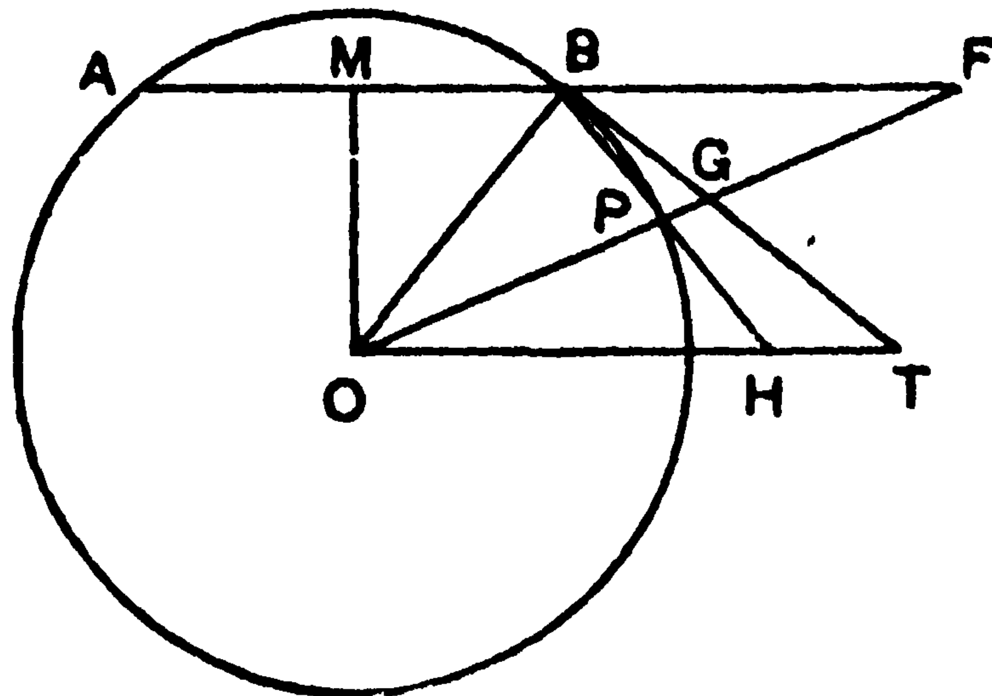


FIG. 2.

the tangent at B in G; and draw BPH, BGT meeting OT, the parallel through O to AB, in H, T.

Then $PF : BG > FG : BG$, since $PF > FG$,

$> OG : GT$, by parallels,

$> OB : BT$, *a fortiori*,

$> BM : MO$;

and obviously, as P moves away from B towards OT, i.e. as G moves away from B along BT, the ratio $OG : GT$ increases continually, while, as shown, $PF : BG$ is always $> BM : MO$, and, *a fortiori*,

$$PF : (\text{arc } PB) > BM : MO.$$

That is, (4) is always satisfied for any point Q' of the spiral 'forward' of P, so that (2) is also satisfied, and QQ' is always less than QF.

It will be observed that no νεῦσις, and nothing beyond 'plane' methods, is required in the above proof, and Pappus's criticism of Archimedes's proof is therefore justified.

Let us now consider for a moment what Archimedes actually does. In Prop. 8, which he uses to prove our proposition in the 'backward' case (R', R, F'), he shows that, if $PO : OV$ is any ratio whatever less than $PO : OT$ or $PM : MO$, we can find points F', G corresponding to any ratio $PO : OV'$ where $OT < OV' < OV$, i.e. we can find a point F' corresponding to a ratio still nearer to $PO : OT$ than $PO : OV$ is. This proves that the ratio $RF' : PG$, while it is always less than $PM : MO$,

approaches that ratio without limit as R approaches P. But the proof does not enable us to say that RF':(*chord* PR), which is $> RF' : PG$, is also always less than $PM : MO$. At first sight, therefore, it would seem that the proof must fail. Not so, however; Archimedes is nevertheless able to prove that, if PV and not PT is the tangent at P to the spiral, an absurdity follows. For his proof establishes that, if PV is the tangent and OF' is drawn as in the proposition, then

$$F'O : RO < OR' : OP,$$

or $F'O < OR'$, 'which is impossible'. Why this is impossible does not appear in Props. 18 and 20, but it follows from the argument in Prop. 13, which proves that a tangent to the spiral cannot meet the curve again, and in fact that the spiral is everywhere concave towards the origin.

Similar remarks apply to the proof by Archimedes of the impossibility of the other alternative supposition (that the tangent at P meets OT at a point U nearer to O than T is).

Archimedes's proof is therefore in both parts perfectly valid, in spite of any appearances to the contrary. The only drawback that can be urged seems to be that, if we assume the tangent to cut OT at a point *very near* to T on either side, Archimedes's construction brings us perilously near to infinitesimals, and the proof may appear to hang, as it were, on a thread, albeit a thread strong enough to carry it. But it is remarkable that he should have elaborated such a difficult proof by means of Props. 7, 8 (including the 'solid' *νεῦσις* of Prop. 8), when the figures of Props. 6 and 7 (or 9) themselves suggest the direct proof above given, which is independent of any *νεῦσις*.

P. Tannery,[1] in a paper on Pappus's criticism of the proof as unnecessarily involving 'solid' methods, has given another proof of the subtangent-property based on 'plane' methods only; but I prefer the method which I have given above because it corresponds more closely to the preliminary propositions actually given by Archimedes.

[1] Tannery, *Mémoires scientifiques*, i, 1912, pp. 300–16.

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PRINTED IN ENGLAND
AT THE OXFORD UNIVERSITY PRESS